Variational Principle

❖ **Vectorial Approach**

$$\vec{F} = m\vec{a}$$

❖ **Energy or Variational Approach**

- scalars
- governing principle
- correct condition is defined by

$$\delta(scalar) = 0, stationary$$

- integral

❖ **Calculus of variation**

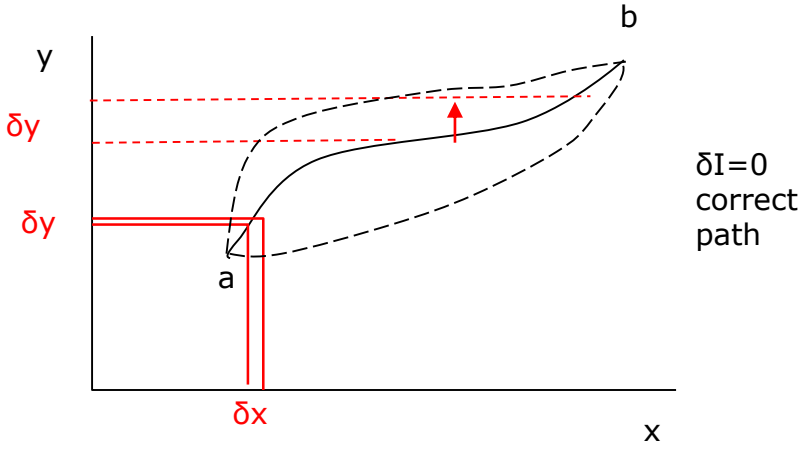
- Chap. 2. Lanczos, "Variational Principles of Mechanics"
- Chap. 2. Hildebrand, "Methods of Applied Mathematics"
- Chap. 3. Langhaar, " Energy Methods in Applied Mechanics"

Variational Principle

Calculus of variation (continued)

Behavior described by finding condition when certain meaningful quantity is stationary

$$I = \int_a^b F(y, y', x)dx$$



δy is called "variation in y" , y is a dependent variable.

$$\delta I = \int_a^b \delta F(y, y', x)dx$$

$$\delta F(y, y', x) = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial F}{\partial x}\delta \mathrm{x} = 0$$

Variational Principle

in operations

$$\delta g(y) = \frac{dg(x)}{dy}\delta y = g'(y)\delta y$$

$$\delta(y^2) = 2y\delta y$$

$$\delta \sin y = \cos y \delta y$$

$$\delta\left(\frac{dw}{dx}\right) = \frac{d\delta w}{dx}, \quad \delta\left(\frac{d^n w}{dx^n}\right) = \frac{d^n(\delta w)}{dx^n}$$

$$\delta\left(\int_a^b F dx\right) = \int_a^b \delta F dx$$

consider

$$I = \int_a^b F(y, y', x)dx$$

$$\delta I = \int_a^b \delta F dx = \int_a^b \left[\frac{\partial F}{\partial y}\delta y' + \frac{\partial F}{\partial y'}\delta y'\right]dx$$

$$= \int_a^b \left[\frac{\partial F}{\partial y}\delta y\right]dx + \underbrace{\frac{\partial F}{\partial y}\delta y'|_a^b}_{0} - \int_a^b \delta y \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)dx$$

Variational Principle

$$\delta I = -\int_a^b \left[\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}\right]\delta y dx = 0$$

$$\delta I = 0, if\ and\ only\ if$$

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0 : Euler's\ Equations$$

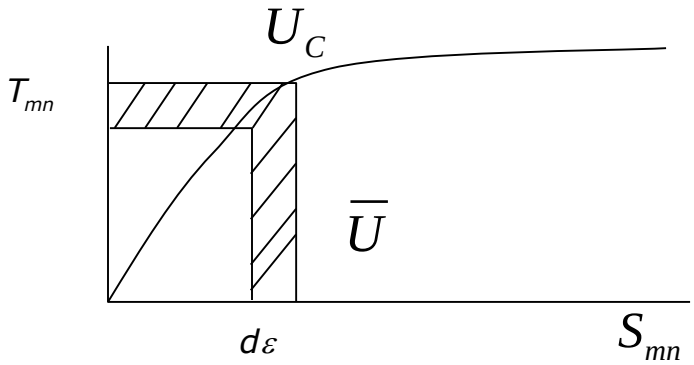
If $F = T - U + W$

$x = f$(time) ⇒ Lagrange's Egn.

Variational Principle

❖ Structural mechanics

- scalars of interest
  - Strain Energy Density $\bar{U} = \bar{U}(S_{mn})$
  - Complementary Strain Energy Density $\overline{U_c} = \bar{U}(T_{mn})$



$$\bar{U} = \int_0^{\varepsilon_{mn}} F_{mn}(S_{mn}')\, dS_{mn}'$$

- For linear system

$$U = \frac{1}{2} T_{mn} S_{mn} = \frac{1}{2} T^T S$$

Tensor Voight

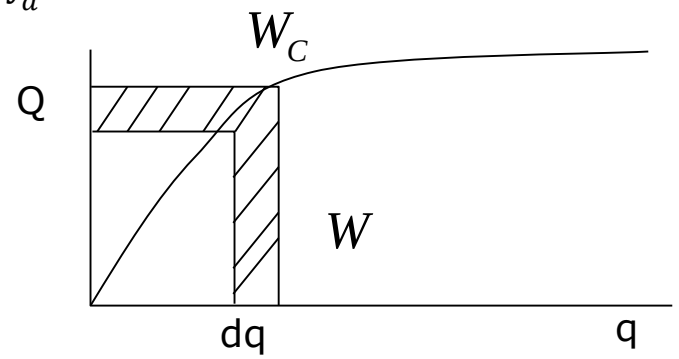
$$U = \iiint \bar{U}(S_{mn}) dV = U$$

$$U_C = \int_0^{Tmn} S_{mn}(T_{mn}') dT_{mn}$$

Variational Principle

- **Work = Force times distance**

$$\int_a^b \vec{F} \cdot d\vec{r}$$



$$W = \int_0^q Q(q')dq'$$

$$W_c = \int_0^Q q(Q')dQ'$$

- **Variation of Strain Energy**

$$U(S_{mn}) = \iiint \left\{ \int_0^{S_{mn}} T_{mn}(S_{mn}')dS_{mn}' \right\}$$

-perturbation in strain $S_{mn} + \delta S_{mn}$

consider

$$U(S_{mn} + \delta S_{mn}) - U(S_{mn})$$

$$= \iiint \left\{ \int_{S_{mn}}^{S_{mn}+\delta S_{mn}} T_{mn}(S'_{mn})\delta S'_{mn} \right\} = \Delta U\left(S_{mn}\right)$$

$$= \underbrace{\iiint \{T_{mn}(S_{mn})\delta S_{mn}\}dV}_{\delta U}$$

Variational Principle

- **Variation in Work**

$$\delta W = \iiint_A \hat{T}_{mn}\delta U_n dA + \iiint_V f_n \delta U_n dV$$

- **Derivation of Principle of Minimum Total Potential Energy**

given loaded body in equilibrium

$$\iiint_V \left\{ \frac{\partial T_{mn}}{\partial x_m} + f_n \right\} \delta U_n dV = 0$$

displacement

$$1st\ term : \iiint \frac{\partial T_{mn}}{\partial x_m}\delta U_n dV = \iiint \left\{ \frac{\partial}{\partial x_m}(T_{mn}\delta U_n) - T_{mn}\frac{\partial}{\partial x_m}(\delta U_n) \right\} \mathrm{dx}$$

$$\vec{B} = \hat{i}_n (T_{mn}\delta U_n)$$

$S_{mn}(normal\ strain)$

$$= \iiint \left\{ \nabla \vec{B} - T_{mn}\delta\left(\frac{\partial U_n}{\partial x_m}\right) \right\} dV$$

$$\iiint_V \nabla \vec{B} dV = \iint_A \vec{B} \bullet \vec{N} dA$$

$\hat{T}_{mn}$

$$\iint_A T_{mn}\cos(Nx_m)\delta U_n dA - \iiint_V T_{mn}\delta S_{mn} dV + \iiint_v f_n \delta U dV = 0$$

$$\iiint_V T_{mn}\delta S_{mn} dV = \iint_A \hat{T}_{mn}\delta U_n dA + \iiint f_n \delta U_n dV$$

δU $\qquad$ δW

$$\delta(U - W) = 0$$

$I = variational\ indicator$

Variational Principle

- **Principle of stationary total potential energy of all displacement of a loaded structure satisfying geometric B.C's – the right ones(equilibrium) are those that minimize Π**

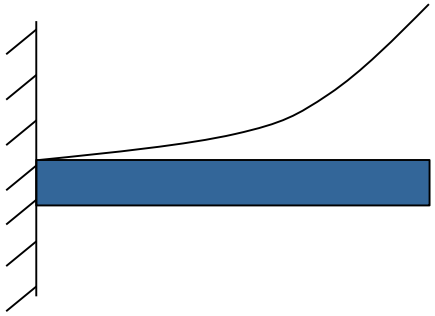
$$\Pi = V - W$$

$$\delta\Pi = \delta(U - W) = 0$$

$$U = \iiint \left\{ \int_0^{S_{mn}} T_{mn}(S_{mn})\, \mathrm{d}\, S_{mn} \right\} dV$$

$$W = \iint_A \hat{T}_{sn} u_n dA + \iiint_V f_n u_n dV$$

- Solution by minimization usually considers certain subset.

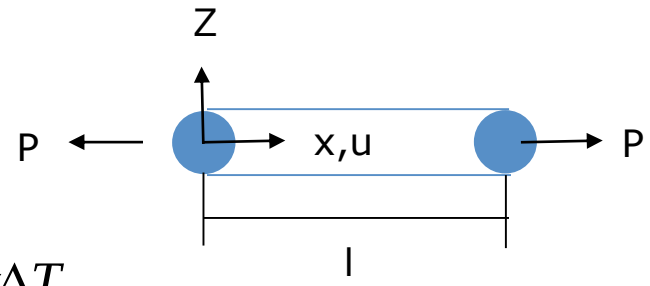


approximate solution

Variational Principle

- ***V* of a typical structure**

i) Rod



$$T_{11} = ES_{11} - E\alpha\Delta T$$

$$S_{11} = \frac{du}{dx}$$

$$\delta U = \iiint T_{11}\delta S_{11} dV$$

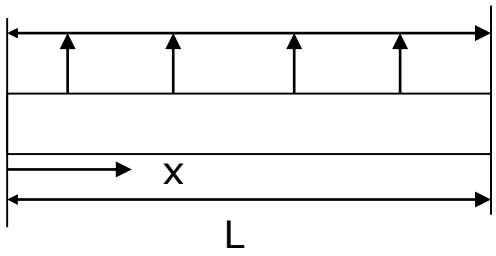
$$U = \iiint_V \left\{ \frac{1}{2} E {S_{11}}^2 - E\alpha\Delta T S_{11} \right\} dA dx$$

$$= \frac{1}{2}\int_0^l EA\left(\frac{du}{dx}\right)^2 dx - \int_0^l \left(\iint E\alpha\Delta T \frac{du}{dx} dA \right) dx$$

Variational Principle

❖ ***U* of a typical structure**

ii) Beam in Bending



$$T_{11} = ES_{11} - E\alpha\Delta T$$

$$S_{11} = -z\frac{d^2w}{dx^2}$$

$$U = \iiint \left\{\frac{1}{2}ES_{11}{}^2 - E\alpha\Delta TS_{11}\right\}dAdx$$

$$U = \frac{1}{2}\int_0^l EI\left(\frac{d^2w}{dx^2}\right)^2 dx + \int_0^l (E\alpha\Delta TzdA)\frac{d^2w}{dx^2}dx$$

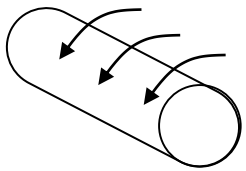
$$I = \int z^2 dA$$

$$W = \int_0^l f_w(x)w(x)\mathrm{dx}$$

Variational Principle

- ***V* of a typical structure**

iii) Torsion of Bar



$$\frac{d\alpha}{dx} = \frac{T}{GJ}\ ,\ T = GJ\frac{d\alpha}{dx}$$

$$U = \frac{1}{2}\int_0^l T\frac{d\alpha}{dx}dx = \frac{1}{2}\int_0^l GJ(\frac{d\alpha}{dx})^2 dx$$

Variational Principle

❖ Rayleigh-Ritz Approach

$$\vec{U}(x,y,z,t)=\sum_{i=1}^{N} a_i(t)\varphi_i(x,y,z)$$

$$\rightarrow \varphi \Rightarrow \text{satisfy only the geometric } B.C.\text{'s}$$

φ should be twice differentiable

$$\Pi_p = (U-W) = \Pi_p(a_1, a_2, \cdots\cdots, a_n)$$

$$\delta\Pi_p = \frac{\partial \Pi_p}{\partial a_1}\delta a_1 + \frac{\partial \Pi_p}{\partial a_2}\delta a_2 + \cdots,$$

$$\frac{\partial \Pi_p}{\partial a_1} = o, \quad \frac{\partial \Pi_p}{\partial a_2} = 0, \cdots\cdots etc.$$

For linear elastic problem,

$$[K]\{q\} = \{f\}$$

Variational Principle

❖ **Note on dynamics**

- d'Alembert's Force

$$\vec{f}_I = -\rho \ddot{\vec{u}}_n$$

$PSTPE$ (Principle of the stationarity total potential energy)

$$\int_{t1}^{t2}\Big(\underbrace{\iiint_V T_{mn}\delta S_{mn}dV}_{\delta U} + \iiint_V \rho\ddot{u}_n\delta u_n - \underbrace{\iiint_V f_{nB}\delta u_n - \iint_A \hat{T}_{sn}\delta u_n dA}_{\delta W}\Big)dt = 0$$

$$\int_{t_1}^{t_2}\iiint_V \rho\underbrace{\ddot{u}_n}_{\dot{u}}\underbrace{\delta u_n}_{\delta\dot{u}}dVdt = \iiint_V \rho\dot{u}_n\cancelto{0}{\delta u_n}\,dt\,\Big|_{t_1}^{t_2} \underbrace{\boxed{-\int_{t_1}^{t_2}\iiint_V \rho\dot{u}_n\delta\dot{u}_n\,dVdt}}_{-\delta T}$$

$$T = \iiint_V \frac{1}{2}\rho\dot{u}_n\dot{u}_n\,dV$$

$$\int_{t_1}^{t_2}(\delta U - \delta T - \delta W)\,\mathrm{dt} = 0$$

$$\boxed{\delta\int_{t_1}^{t_2}(T-U)\,\mathrm{dt} + \int_{t_1}^{t_2}\delta W dt = 0}$$

Hamilton's principle

Variational Principle

❖ **Principle of the minimum complementary energy**

- increments in stress field δT_{mn}

$$\frac{\partial T_{mn}}{\partial X_m} + \overset{0}{\cancel{\delta f_n}} = 0$$

$$\iiint_V \left\{ S_{mn} - \frac{1}{2}\left(\frac{\partial u_n}{\partial x_m} + \frac{\partial u_m}{\partial x_n} \right) \right\} \delta T_{mn} dV = 0$$

$$\iiint_V S_{mn} \delta T_{mn} dV - \iint_A U_n \delta_{sn} d\hat{T}_{sn} A = 0$$

$$\delta U^c - \delta W^c = 0$$

$$\delta \hat{T}_{sn} = \delta T_{mn} \cos(N x_n) \, \mathrm{dA}$$