Module #6:
Orders of Growth

Rosen 5th ed., §2.2

~22 slides, ~1 lecture

Orders of Growth (§1.8)

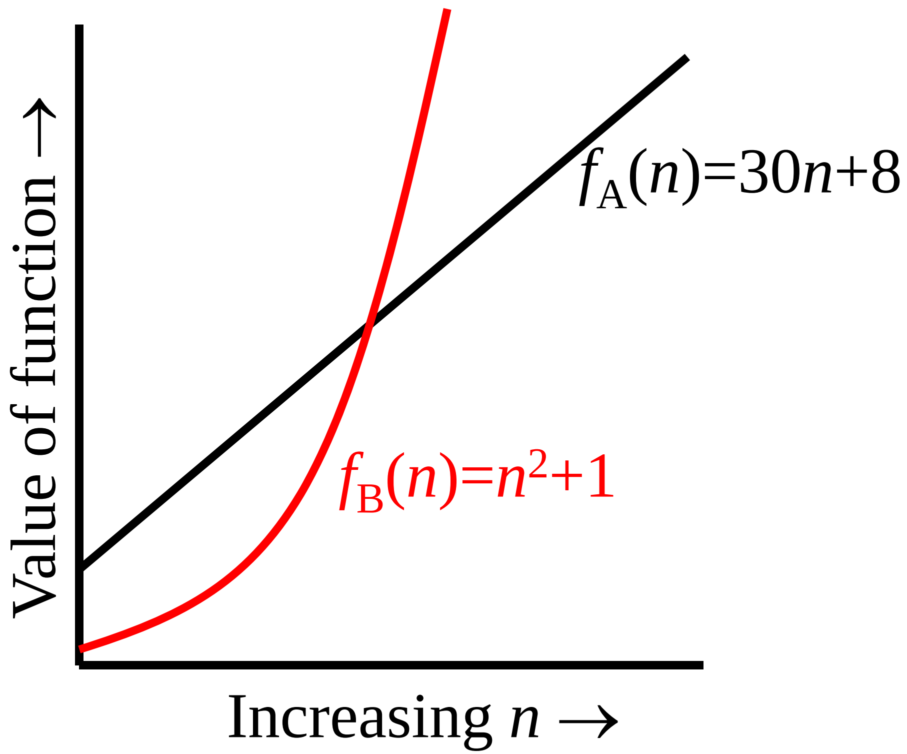
- For functions over numbers, we often need to know a rough measure of *how fast a function grows*.
- If $f(x)$ is *faster growing* than $g(x)$, then $f(x)$ always eventually becomes larger than $g(x)$ *in the limit* (for large enough values of x).
- Useful in engineering for showing that one design *scales* better or worse than another.

Orders of Growth - Motivation

- Suppose you are designing a web site to process user data (*e.g.*, financial records).
- Suppose database program A takes $f_A(n)=30n+8$ microseconds to process any n records, while program B takes $f_B(n)=n^2+1$ microseconds to process the n records.
- Which program do you choose, knowing you'll want to support millions of users?

Visualizing Orders of Growth

- On a graph, as you go to the right, a faster growing function eventually becomes larger...



Concept of order of growth

- We say $f_A(n)=30n+8$ is *order n*, or $O(n)$. It is, at most, roughly *proportional* to n.
- $f_B(n)=n^2+1$ is *order* n^2, or $O(n^2)$. It is roughly proportional to n^2.
- Any $O(n^2)$ function is faster-growing than any $O(n)$ function.
- For large numbers of user records, the $O(n^2)$ function will always take more time.

Definition: *O(g), at most order g*

Let g be any function $\mathbf{R}\rightarrow\mathbf{R}$.

- Define "*at most order g*", written O(g), to be: $\{f{:}\mathbf{R}\rightarrow\mathbf{R} \mid \exists c,k{:}\ \forall x>k{:}\ f(x) \leq cg(x)\}$.
  - "Beyond some point k, function f is at most a constant c times g (*i.e.*, proportional to g)."
- "f is *at most order g*", or "f is O(g)", or "f=O(g)" all just mean that $f\in O(g)$.
- Sometimes the phrase "at most" is omitted.

Points about the definition

- Note that f is $O(g)$ so long as *any* values of c and k exist that satisfy the definition.
- But: The particular c, k, values that make the statement true are *not* unique: **Any larger value of *c* and/or *k* will also work.**
- You are **not** required to find the smallest c and k values that work. (Indeed, in some cases, there may be no smallest values!)

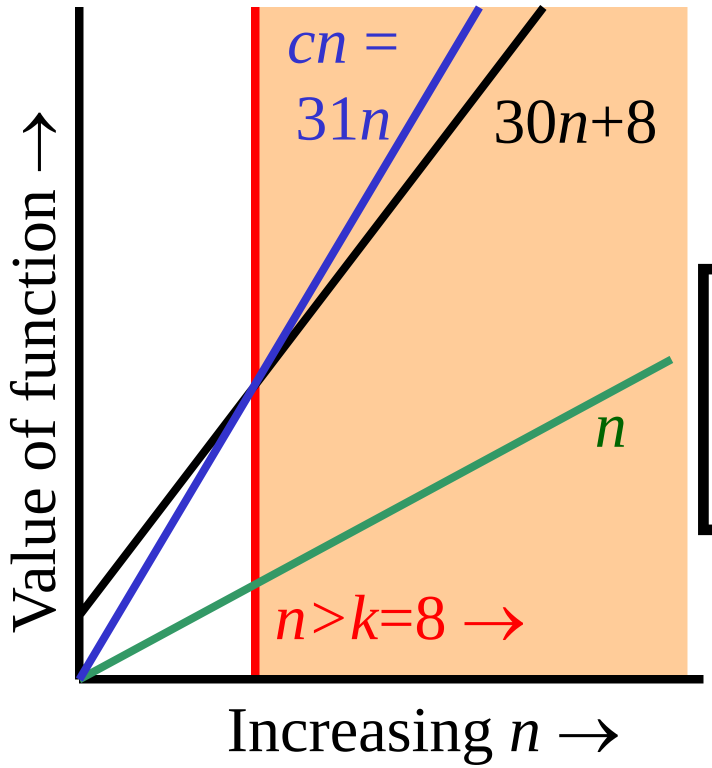
However, you should **prove** that the values you choose do work.

"Big-O" Proof Examples

- Show that $30n+8$ is $O(n)$.
  - Show $\exists c,k: \forall n>k: 30n+8 \le cn$.
    - Let $c=31$, $k=8$. Assume $n>k=8$. Then $cn = 31n = 30n + n > 30n+8$, so $30n+8 < cn$.
- Show that n^2+1 is $O(n^2)$.
  - Show $\exists c,k: \forall n>k: n^2+1 \le cn^2$.
    - Let $c=2$, $k=1$. Assume $n>1$. Then $cn^2 = 2n^2 = n^2+n^2 > n^2+1$, or $n^2+1< cn^2$.

Big-O example, graphically

- Note $30n+8$ isn't less than n *anywhere* ($n>0$).
- It isn't even less than $31n$ *everywhere*.
- But it *is* less than $31n$ <u>everywhere to the right of $n=8$</u>.



$30n+8 \in O(n)$

Useful Facts about Big O

- Big O, as a relation, is transitive:
  $f \in O(g) \wedge g \in O(h) \rightarrow f \in O(h)$
- O with constant multiples, roots, and logs...
  $\forall f$ (in $\omega(1)$) & constants $a,b \in \mathbf{R}$, with $b \geq 0$,
  af, f^{1-b}, and $(\log_b f)^a$ are all $O(f)$.
- Sums of functions:
  If $g \in O(f)$ and $h \in O(f)$, then $g+h \in O(f)$.

More Big-O facts

- $\forall c>0$, $O(cf)=O(f+c)=O(f-c)=O(f)$
- $f_1 \in O(g_1) \land f_2 \in O(g_2) \rightarrow$
  - $f_1 f_2 \in O(g_1 g_2)$
  - $f_1+f_2 \in O(g_1+g_2)$
    $= O(\max(g_1,g_2))$
    $= O(g_1)$ if $g_2 \in O(g_1)$ (Very useful!)

Orders of Growth (§1.8) - So Far

- For any g:$\mathbf{R}\rightarrow\mathbf{R}$, "*at most order g*", $O(g) \equiv \{f:\mathbf{R}\rightarrow\mathbf{R} \mid \exists c,k\ \forall x>k\ |f(x)| \leq |cg(x)|\}$.
  – Often, one deals only with positive functions and can ignore absolute value symbols.
- "$f \in O(g)$" often written "f is $O(g)$" or "$f=O(g)$".
  – The latter form is an instance of a more general convention...

Order-of-Growth Expressions

- "O(f)" when used as a term in an arithmetic expression means: "some function f such that $f \in O(f)$".
- *E.g.*: "$x^2+O(x)$" means "x^2 plus some function that is $O(x)$".
- Formally, you can think of any such expression as denoting a set of functions: "$x^2+O(x)$" $:\equiv \{g \mid \exists f \in O(x): g(x) = x^2+f(x)\}$

Order of Growth Equations

- Suppose E_1 and E_2 are order-of-growth expressions corresponding to the sets of functions S and T, respectively.
- Then the "equation" $E_1 = E_2$ really means
  $$\forall f \in S, \exists g \in T: \quad f = g$$
  or simply $S \subseteq T$.
- Example: $x^2 + O(x) = O(x^2)$ means
  $$\forall f \in O(x): \exists g \in O(x^2): \; x^2 + f(x) = g(x)$$

Useful Facts about Big O

- $\forall$ *f*,*g* & constants $a,b\in\mathbf{R}$, with $b\geq 0$,
  - $af = O(f)$; (*e.g.* $3x^2 = O(x^2)$)
  - $f+O(f) = O(f)$; (*e.g.* $x^2+x = O(x^2)$)
- Also, if $f=\Omega(1)$ (*at least* order 1), then:
  - $|f|^{1-b} = O(f)$; (*e.g.* $x^{-1} = O(x)$)
  - $(\log_b |f|)^a = O(f)$. (*e.g.* $\log x = O(x)$)
  - $g=O(fg)$ (*e.g.* $x = O(x \log x)$)
  - $fg \neq O(g)$ (*e.g.* $x \log x \neq O(x)$)
  - $a=O(f)$ (*e.g.* $3 = O(x)$)

Definition: $\Theta(g)$, *exactly order g*

- If $f \in O(g)$ and $g \in O(f)$ then we say "*g and f are of the same order*" or "*f is (exactly) order g*" and write $f \in \Theta(g)$.
- Another equivalent definition:
  $$\Theta(g) \equiv \{f:\mathbf{R}\rightarrow\mathbf{R} \mid \exists c_1 c_2 k\ \forall x>k: |c_1 g(x)| \leq |f(x)| \leq |c_2 g(x)|\ \}$$
- "Everywhere beyond some point *k*, *f*(*x*) lies in between two multiples of *g*(*x*)."

Rules for Θ

- Mostly like rules for O( ), except:
- $\forall f,g>0$ & constants $a,b\in\mathbf{R}$, with $b>0$,
  $af \in \Theta(f)$, but ← Same as with O.
  $f \notin \Theta(fg)$ unless $g=\Theta(1)$ ← Unlike O.
  $|f|^{1-b} \notin \Theta(f)$, and ← Unlike with O.
  $(\log_b |f|)^c \notin \Theta(f)$. ← Unlike with O.
- The functions in the latter two cases we say are *strictly of lower order* than $\Theta(f)$.

Θ example

- Determine whether: $\left(\sum_{i=1}^{n} i\right) \overset{?}{\in} \Theta(n^2)$
- Quick solution:

$$\left(\sum_{i=1}^{n} i\right) = n(n-1)/2$$

$$= n\,\Theta(n)/2$$

$$= n\,\Theta(n)$$

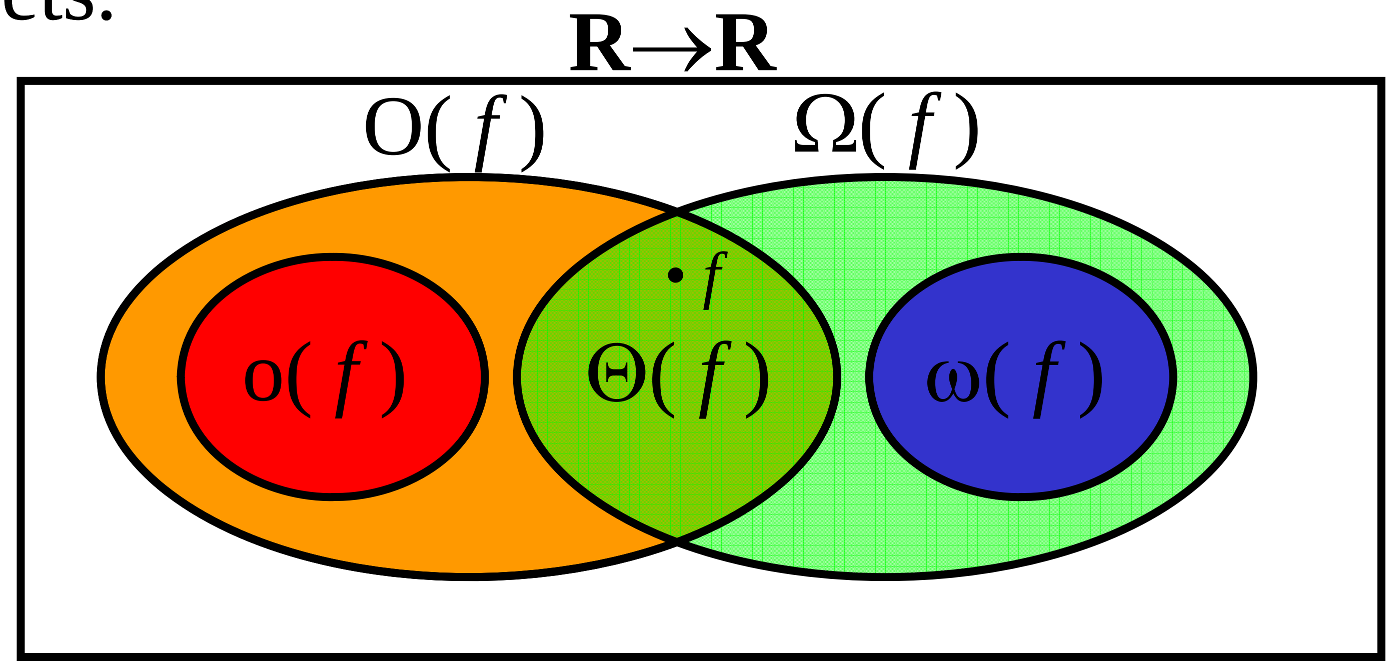
$$= \Theta(n^2)$$

Other Order-of-Growth Relations

- $\Omega(g) = \{f \mid g \in O(f)\}$
  "The functions that are *at least order g*."
- $o(g) = \{f \mid \forall c>0\ \exists k\ \forall x>k : |f(x)| < |cg(x)|\}$
  "The functions that are *strictly lower order than g*." $o(g) \subset O(g) - \Theta(g)$.
- $\omega(g) = \{f \mid \forall c>0\ \exists k\ \forall x>k : |cg(x)| < |f(x)|\}$
  "The functions that are *strictly higher order than g*." $\omega(g) \subset \Omega(g) - \Theta(g)$.

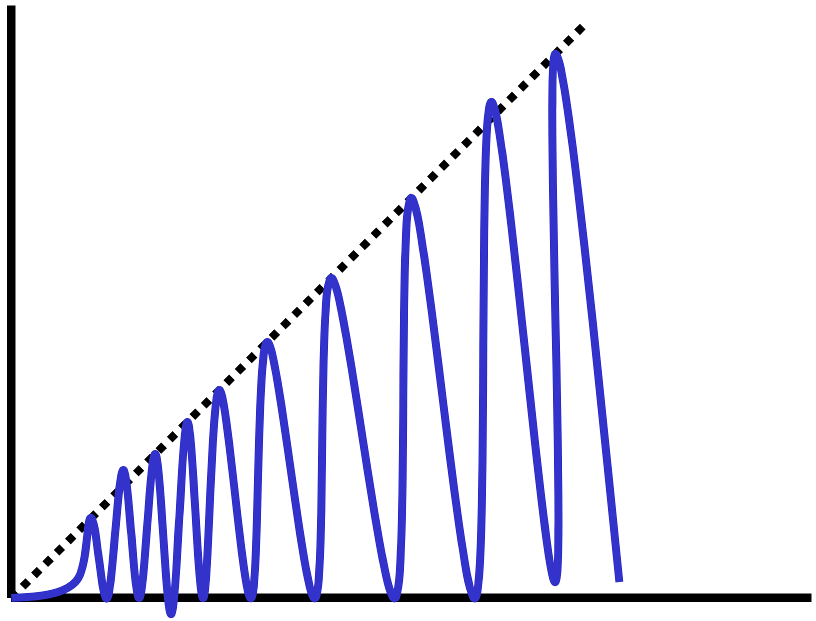
Relations Between the Relations

- Subset relations between order-of-growth sets.



Why $o(f) \subset O(x) - \Theta(x)$

- A function that is $O(x)$, but neither $o(x)$ nor $\Theta(x)$:



Strict Ordering of Functions

- Temporarily let's write $f \prec g$ to mean $f \in o(g)$,
  $f \sim g$ to mean $f \in \Theta(g)$
- Note that $$f \prec g \Leftrightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$
- Let $k>1$. Then the following are true:
  $1 \prec \log\log n \prec \log n \sim \log_k n \prec \log^k n$
  $\prec n^{1/k} \prec n \prec n \log n \prec n^k \prec k^n \prec n! \prec n^n \ldots$

Review: Orders of Growth (§1.8)

Definitions of order-of-growth sets, $\forall g:\mathbf{R}\rightarrow\mathbf{R}$

- $O(g) \equiv \{f \mid \exists\, c>0\ \exists k\ \forall x>k\ |f(x)| < |cg(x)|\}$
- $o(g) \equiv \{f \mid \forall c>0\ \exists k\ \forall x>k\ |f(x)| < |cg(x)|\}$
- $\Omega(g) \equiv \{f \mid g \in O(f)\}$
- $\omega(g) \equiv \{f \mid g \in o(f)\}$
- $\Theta(g) \equiv O(g) \cap \Omega(g)$