Module #8:
Basic Number Theory

Rosen 5th ed., §§2.4-2.6
~31 slides, ~2 lectures

§2.4: The Integers and Division

- Of course you already know what the integers are, and what division is…
- **But:** There are some specific notations, terminology, and theorems associated with these concepts which you may not know.
- These form the basics of *number theory*.
  - Vital in many important algorithms today (hash functions, cryptography, digital signatures).

Divides, Factor, Multiple

- Let $a,b \in \mathbf{Z}$ with $a \neq 0$.
- $a|b \equiv$ "*a divides b*" $:\equiv$ "$\exists c \in \mathbf{Z}$: $b=ac$"
  "There is an integer c such that c times a equals b."
  - Example: $3|-12 \Leftrightarrow$ **True**, but $3|7 \Leftrightarrow$ **False**.
- Iff a divides b, then we say a is a *factor* or a *divisor* of b, and b is a *multiple* of a.
- "b is even" $:\equiv 2|b$. Is 0 even? Is -4?

Facts re: the Divides Relation

- $\forall a,b,c \in \mathbf{Z}$:
  1. $a|0$
  2. $(a|b \wedge a|c) \rightarrow a \mid (b + c)$
  3. $a|b \rightarrow a|bc$
  4. $(a|b \wedge b|c) \rightarrow a|c$
- **Proof** of (2): $a|b$ means there is an s such that $b=as$, and $a|c$ means that there is a t such that $c=at$, so $b+c = as+at = a(s+t)$, so $a|(b+c)$ also.■

More Detailed Version of Proof

- Show $\forall a,b,c \in \mathbf{Z}$: $(a|b \wedge a|c) \rightarrow a \mid (b + c)$.
- Let a, b, c be any integers such that $a|b$ and $a|c$, and show that $a \mid (b + c)$.
- By defn. of |, we know $\exists s$: $b=as$, and $\exists t$: $c=at$. Let s, t, be such integers.
- Then $b+c = as + at = a(s+t)$, so $\exists u$: $b+c=au$, namely $u=s+t$. Thus $a|(b+c)$.

Prime Numbers

- An integer p>1 is *prime* iff it is not the product of any two integers greater than 1:

  $$p>1 \wedge \neg\exists a,b\in\mathbf{N}\text{:}\ a>1,\ b>1,\ ab=p.$$

- The only positive factors of a prime p are 1 and p itself. Some primes: 2,3,5,7,11,13...
- Non-prime integers greater than 1 are called *composite*, because they can be *composed* by multiplying two integers greater than 1.

Review of §2.4 So Far

- $a|b \Leftrightarrow$ "*a divides b*" $\Leftrightarrow \exists c\in\mathbf{Z}: b=ac$
- "*p* is *prime*" $\Leftrightarrow$

  $p>1 \wedge \neg\exists a\in\mathbf{N}: (1 < a < p \wedge a|p)$
- Terms *factor*, *divisor*, *multiple*, *composite*.

Fundamental Theorem of Arithmetic

Its "Prime Factorization"

- Every positive integer has a unique representation as the product of a non-decreasing series of zero or more primes.
  - 1 = (product of empty series) = 1
  - 2 = 2 (product of series with one element 2)
  - 4 = 2·2 (product of series 2,2)
  - 2000 = 2·2·2·2·5·5·5; 2001 = 3·23·29; 2002 = 2·7·11·13; 2003 = 2003

An Application of Primes

- When you visit a secure web site (https:… address, indicated by padlock icon in IE, key icon in Netscape), the browser and web site may be using a technology called *RSA encryption*.
- This *public-key cryptography* scheme involves exchanging *public keys* containing the product pq of two random large primes p and q (a *private key*) which must be kept secret by a given party.
- So, the security of your day-to-day web transactions depends critically on the fact that all known factoring algorithms are intractable!
  - **Note:** There is a tractable *quantum* algorithm for factoring; so if we can ever build big quantum computers, RSA will be insecure.

The Division "Algorithm"

- Really just a *theorem*, not an algorithm…
  - The name is used here for historical reasons.
- For any integer *dividend a* and *divisor d*≠0, there is a unique integer *quotient q* and *remainder r*∈**N** ∋ $a = dq + r$ and $0 \leq r < |d|$.
  (such that)
- $\forall a,d \in \mathbf{Z},\ d>0\colon \exists! q,r \in \mathbf{Z}\colon 0 \leq r < |d|,\ a = dq + r$.
- We can find *q* and *r* by: $q = \lfloor a/d \rfloor$, $r = a - qd$.

Greatest Common Divisor

- The *greatest common divisor* gcd(a,b) of integers a,b (not both 0) is the largest (most positive) integer d that is a divisor both of a and of b.

$$d = \gcd(a,b) = \max(d{:}\ d|a \land d|b) \Leftrightarrow$$
$$d|a \land d|b \land \forall e{\in}\mathbf{Z}, (e|a \land e|b) \rightarrow d \geq e$$

- Example: gcd(24,36)=?
Positive common divisors: 1,2,3,4,6,12…
Greatest is 12.

GCD shortcut

- If the prime factorizations are written as

$$a = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n} \quad \text{and} \quad b = p_1^{b_1} p_2^{b_2} \ldots p_n^{b_n},$$

then the GCD is given by:

$$\gcd(a,b) = p_1^{\min(a_1,b_1)} p_2^{\min(a_2,b_2)} \ldots p_n^{\min(a_n,b_n)}.$$

- Example:
  - a=84=2·2·3·7 = $2^2 \cdot 3^1 \cdot 7^1$
  - b=96=2·2·2·2·2·3 = $2^5 \cdot 3^1 \cdot 7^0$
  - gcd(84,96) = $2^2 \cdot 3^1 \cdot 7^0$ = 2·2·3 = 12.

Relative Primality

- Integers *a* and *b* are called *relatively prime* or *coprime* iff their gcd = 1.
  - Example: Neither 21 and 10 are prime, but they are *coprime*. 21=3·7 and 10=2·5, so they have no common factors > 1, so their gcd = 1.
- A *set* of integers $\{a_1,a_2,\ldots\}$ is *(pairwise) relatively prime* if all pairs a_i, a_j, $i\neq j$, are relatively prime.

Least Common Multiple

- lcm(a,b) of positive integers a, b, is the smallest positive integer that is a multiple both of a and of b. *E.g.* lcm(6,10)=30

  $m = \text{lcm}(a,b) = \min(m{:}\ a|m \land b|m) \Leftrightarrow$
  $a|m \land b|m \land \forall n \in \mathbf{Z}{:}\ (a|n \land b|n) \rightarrow (m \leq n)$

- If the prime factorizations are written as

  $a = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$ and $b = p_1^{b_1} p_2^{b_2} \ldots p_n^{b_n}$,

  then the LCM is given by

  $$\text{lcm}(a,b) = p_1^{\max(a_1,b_1)} p_2^{\max(a_2,b_2)} \ldots p_n^{\max(a_n,b_n)}.$$

The **mod** operator

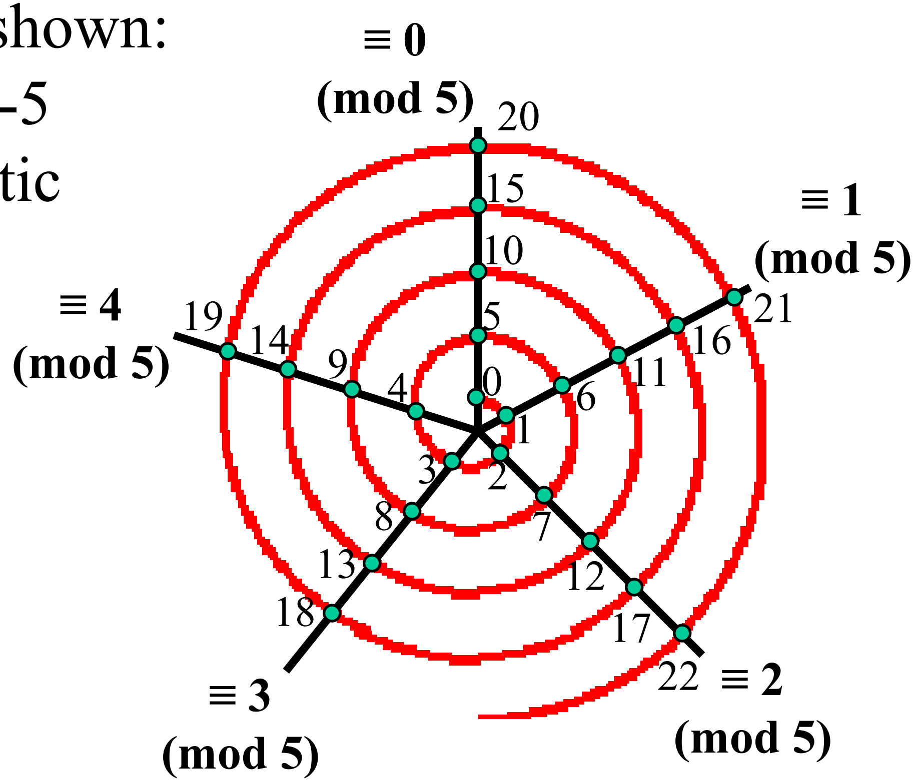
- An integer “division remainder” operator.
- Let $a,d \in \mathbf{Z}$ with $d>1$. Then a **mod** d denotes the remainder r from the division “algorithm” with dividend a and divisor d; *i.e.* the remainder when a is divided by d. (Using *e.g.* long division.)
- We can compute (a **mod** d) by: $a - d\cdot\lfloor a/d \rfloor$.
- In C programming language, “%” = mod.

Modular Congruence

- Let $\mathbf{Z}^+=\{n\in\mathbf{Z} \mid n>0\}$, the positive integers.
- Let $a,b\in\mathbf{Z}$, $m\in\mathbf{Z}^+$.
- Then *a is congruent to b modulo m*, written "$a\equiv b \pmod{m}$", iff $m \mid a-b$.
- Also equivalent to: $(a-b) \bmod m = 0$.
- (Note: this is a different use of "≡" than the meaning "is defined as" I've used before.)

Spiral Visualization of **mod**

Example shown:
modulo-5
arithmetic



Useful Congruence Theorems

- Let $a,b \in \mathbf{Z}$, $m \in \mathbf{Z}^+$. Then:
  $$a \equiv b \pmod{m} \Leftrightarrow \exists k \in \mathbf{Z}\ a = b + km.$$
- Let $a,b,c,d \in \mathbf{Z}$, $m \in \mathbf{Z}^+$. Then if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then:
  - $a+c \equiv b+d \pmod{m}$, and
  - $ac \equiv bd \pmod{m}$

Rosen §2.5: Integers & Algorithms

- Topics:
  - Euclidean algorithm for finding GCD's.
  - Base-*b* representations of integers.
    - Especially: binary, hexadecimal, octal.
    - Also: Two's complement representation of negative numbers.
  - Algorithms for computer arithmetic:
    - Binary addition, multiplication, division.

Euclid's Algorithm for GCD

Euclid of Alexandria 325-265 B.C.

- Finding GCDs by comparing prime factorizations can be difficult if the prime factors are unknown.
- Euclid discovered: For all integers a, b,
  $\gcd(a, b) = \gcd((a \bmod b), b)$.
- Sort a,b so that $a>b$, and then (given $b>1$) $(a \bmod b) < a$, so problem is simplified.

Euclid's Algorithm Example

- gcd(372,164) = gcd(372 mod 164, 164).
  - 372 mod 164 = $372-164\lfloor 372/164 \rfloor = 372-164\cdot 2 = 372-328 = 44$.
- gcd(164,44) = gcd(164 mod 44, 44).
  - 164 mod 44 = $164-44\lfloor 164/44 \rfloor = 164-44\cdot 3 = 164-132 = 32$.
- gcd(44,32) = gcd(44 mod 32, 32) = gcd(12, 32) = gcd(32 mod 12, 12) = gcd(8,12) = gcd(12 mod 8, 8) = gcd(4,8) = gcd(8 mod 4, 4) = gcd(0,4) = 4.

Euclid's Algorithm Pseudocode

procedure *gcd*(a, b: positive integers)

  while $b \neq 0$

    $r := a$ **mod** b; $a := b$; $b := r$

  return a

Sorting inputs not needed b/c order will be reversed each iteration.

Fast! Number of while loop iterations turns out to be O(log(max(a,b))).

Base-*b* number systems

- Ordinarily we write *base*-10 representations of numbers (using digits 0-9).
- 10 isn't special; any base $b>1$ will work.
- For any positive integers n,b there is a unique sequence $\underbrace{a_k a_{k-1} \ldots a_1 a_0}$ of *digits* $a_i<b$ such that

$$n = \sum_{i=0}^{k} a_i b^i$$

The "*base b expansion of n*"

See module #12 for summation notation.

Particular Bases of Interest

- Base *b*=10 (decimal):
  10 digits: 0,1,2,3,4,5,6,7,8,9.
- Base *b*=2 (binary):
  2 digits: 0,1. ("Bits"="<u>bi</u>nary digi<u>ts</u>.")
- Base *b*=8 (octal):
  8 digits: 0,1,2,3,4,5,6,7.
- Base *b*=16 (hexadecimal):
  16 digits: 0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F

Used only because we have 10 fingers

Used internally in all modern computers

Octal digits correspond to groups of 3 bits

Hex digits give groups of 4 bits

Converting to Base b

(Algorithm, informally stated)

- To convert any integer n to any base $b>1$:
- To find the value of the *rightmost* (lowest-order) digit, simply compute n mod b.
- Now replace n with the quotient $\lfloor n/b \rfloor$.
- Repeat above two steps to find subsequent digits, until n is gone (=0).

Exercise for student: Write this out in pseudocode…

Addition of Binary Numbers

procedure $add(a_{n-1}\ldots a_0, b_{n-1}\ldots b_0$: binary representations of non-negative integers a,b)

$carry := 0$

for $bitIndex := 0$ **to** $n-1$ {go through bits}

$bitSum := a_{bitIndex}+b_{bitIndex}+carry$ {2-bit sum}

$s_{bitIndex} := bitSum$ **mod** 2 {low bit of sum}

$carry := \lfloor bitSum / 2 \rfloor$ {high bit of sum}

$s_n := carry$

return $s_n\ldots s_0$: binary representation of integer s

Two's Complement

- In binary, negative numbers can be conveniently represented using *two's complement notation*.
- In this scheme, a string of n bits can represent any integer i such that $-2^{n-1} \leq i < 2^{n-1}$.
- The bit in the highest-order bit-position ($n-1$) represents a coefficient multiplying -2^{n-1};
  - The other positions $i < n-1$ just represent 2^i, as before.
- The negation of any n-bit two's complement number $a = a_{n-1}\ldots a_0$ is given by $\overline{a_{n-1}\ldots a_0} + 1$.

The bitwise logical complement of the n-bit string $a_{n-1}\ldots a_0$.

Correctness of Negation Algorithm

- **Theorem:** For an integer a represented in two's complement notation, $-a = \overline{a} + 1$.
- **Proof:** $a = -a_{n-1}2^{n-1} + a_{n-2}2^{n-2} + \ldots + a_0 2^0$, so $-a = a_{n-1}2^{n-1} - a_{n-2}2^{n-2} - \ldots - a_0 2^0$. Note $a_{n-1}2^{n-1} = (1-\overline{a}_{n-1})2^{n-1} = 2^{n-1} - \overline{a}_{n-1}2^{n-1}$. But $2^{n-1} = 2^{n-2} + \ldots + 2^0 + 1$. So we have $-a = -\overline{a}_{n-1}2^{n-1} + (1-a_{n-2})2^{n-2} + \ldots +$ $(1-a_0)2^0 + 1 = \overline{a} + 1$.

Subtraction of Binary Numbers

procedure *subtract*($a_{n-1}\ldots a_0$, $b_{n-1}\ldots b_0$: binary two's complement representations of integers a,b)

return *add*(a, *add*($\overline{b}$,1)) { $a + (-b)$ }

This fails if either of the adds causes a carry into or out of the $n-1$ position, since $2^{n-2}+2^{n-2} \neq -2^{n-1}$, and $-2^{n-1} + (-2^{n-1}) = -2^n$ isn't representable!

Multiplication of Binary Numbers

procedure *multiply*($a_{n-1}\ldots a_0$, $b_{n-1}\ldots b_0$: binary representations of $a,b\in\mathbf{N}$)

$product := 0$

for $i := 0$ to $n-1$

if $b_i = 1$ **then**

$product := add(a_{n-1}\ldots a_0 0^i, product)$

return *product*

i extra 0-bits appended after the digits of *a*

Binary Division with Remainder

procedure *div-mod*($a,d \in \mathbf{Z}^+$) {Quotient & rem. of a/d.}

  n := max(length of a in bits, length of d in bits)

  for $i := n-1$ **downto** 0

    if $a \geq d0^i$ **then** {Can we subtract at this position?}

      $q_i := 1$ {This bit of quotient is 1.}

      $a := a - d0^i$ {Subtract to get remainder.}

    else

      $q_i := 0$ {This bit of quotient is 0.}

  $r := a$

  return q,r {q = quotient, r = remainder}