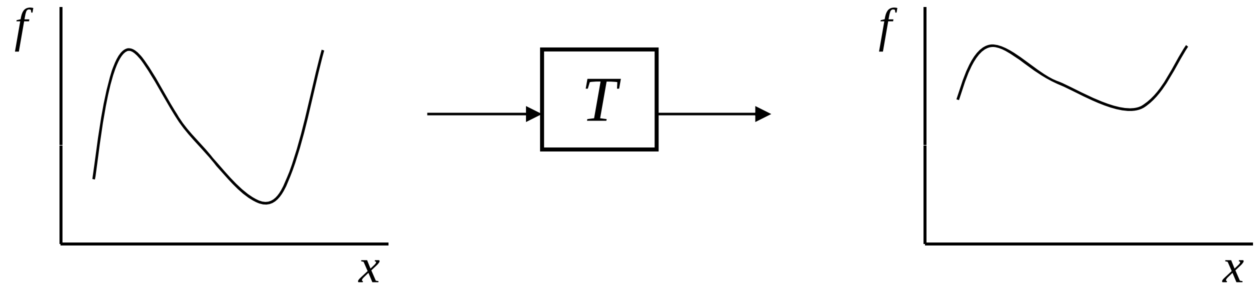
2D Geometric Transformations

Chapter 5
Intro. to Computer Graphics
Spring 2008, Y. G. Shin

Image Warping

- image filtering: change ***range*** of image

$$g(x) = T(f(x))$$



- image warping: change ***domain*** of image

$$g(x) = f(T(x))$$

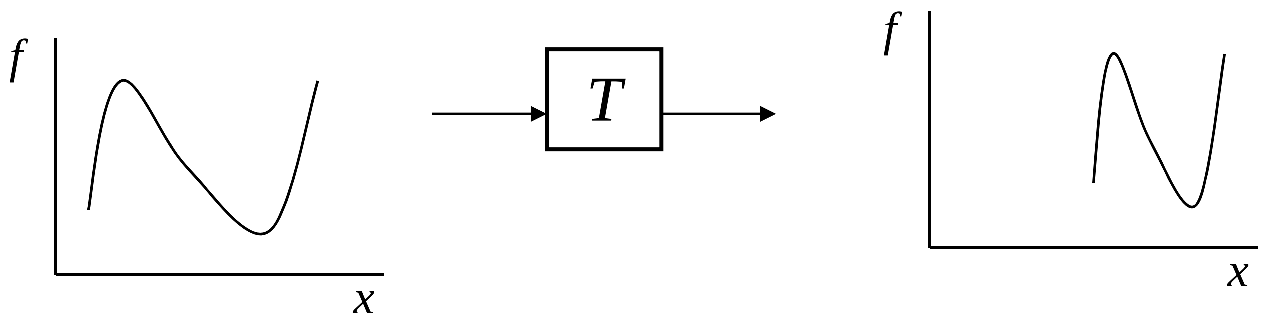
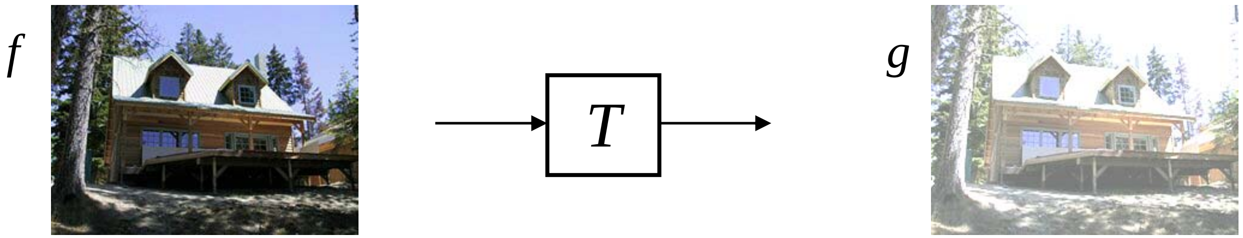


Image Warping

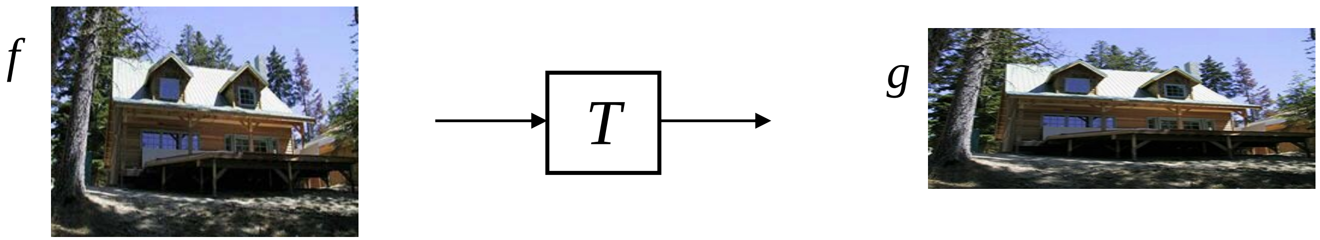
- image filtering: change ***range*** of image

$$g(x) = h(T(x))$$



- image warping: change ***domain*** of image

$$g(x) = f(T(x))$$



Parametric (global) warping

- Examples of parametric warps:

translation

rotation

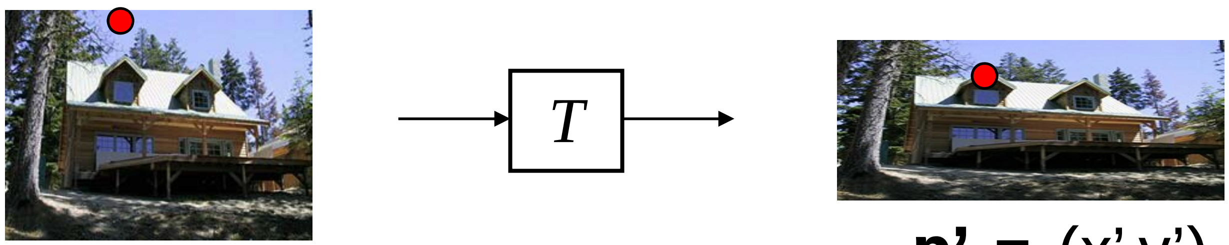
aspect

affine

perspective

cylindrical

Parametric (global) warping



p = (x,y)   **p'** = (x',y')

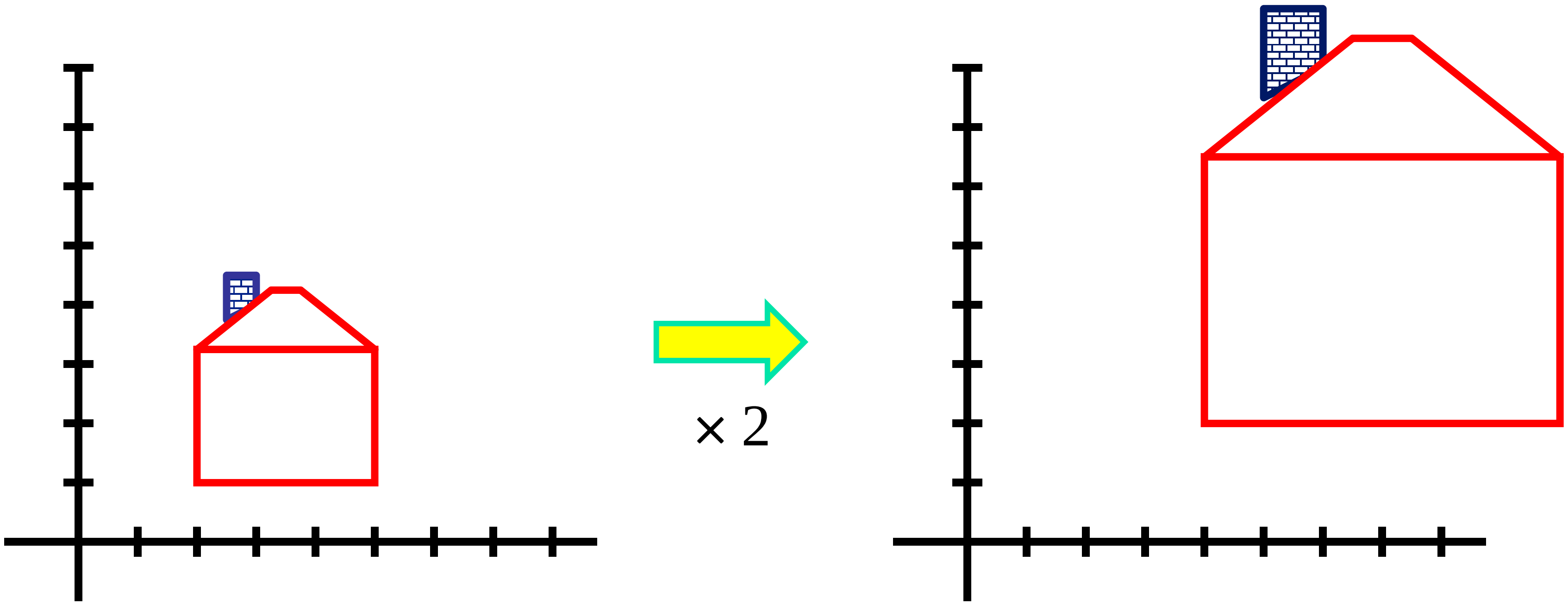
- Transformation T is a coordinate-changing machine: p' = *T*(p)
- What does it mean that *T* is global?
  - Is the same for any point p
  - can be described by just a few numbers (parameters)
- Let's represent *T* as a matrix:

p' = **M***p

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{M} \begin{bmatrix} x \\ y \end{bmatrix}$$

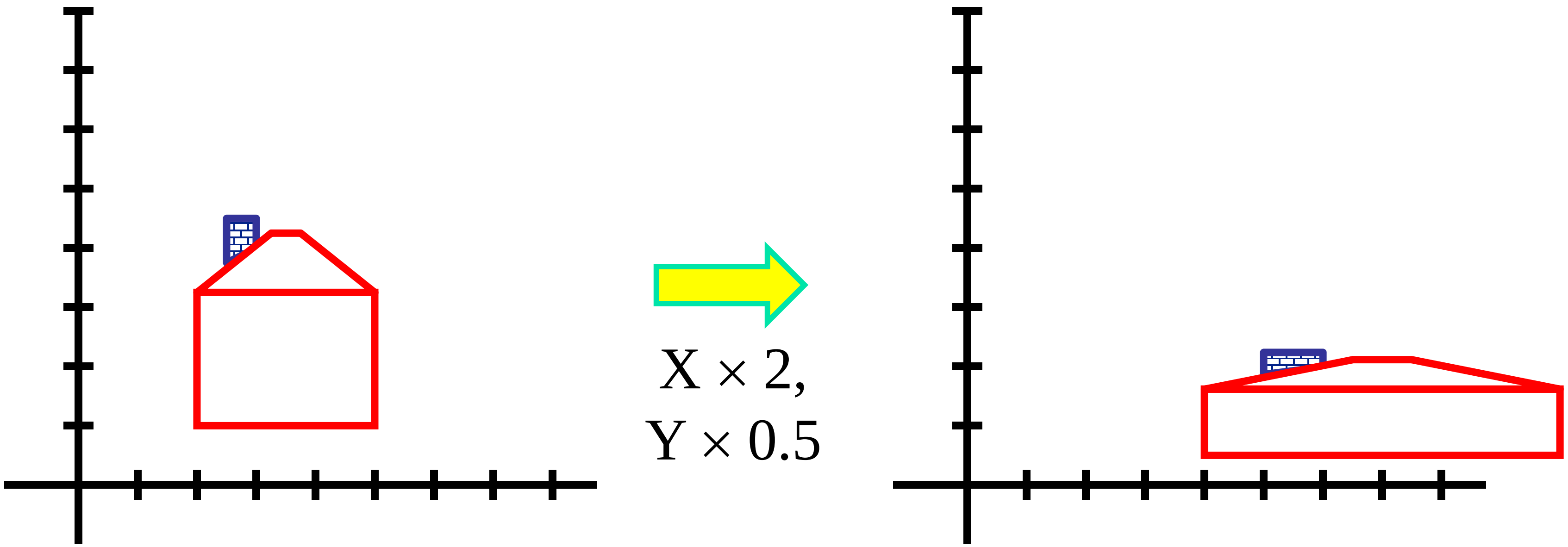
Scaling

- *Scaling* a coordinate means multiplying each of its components by a scalar
- *Uniform scaling* means this scalar is the same for all components:



Scaling

- *Non-uniform scaling*: different scalars per component:



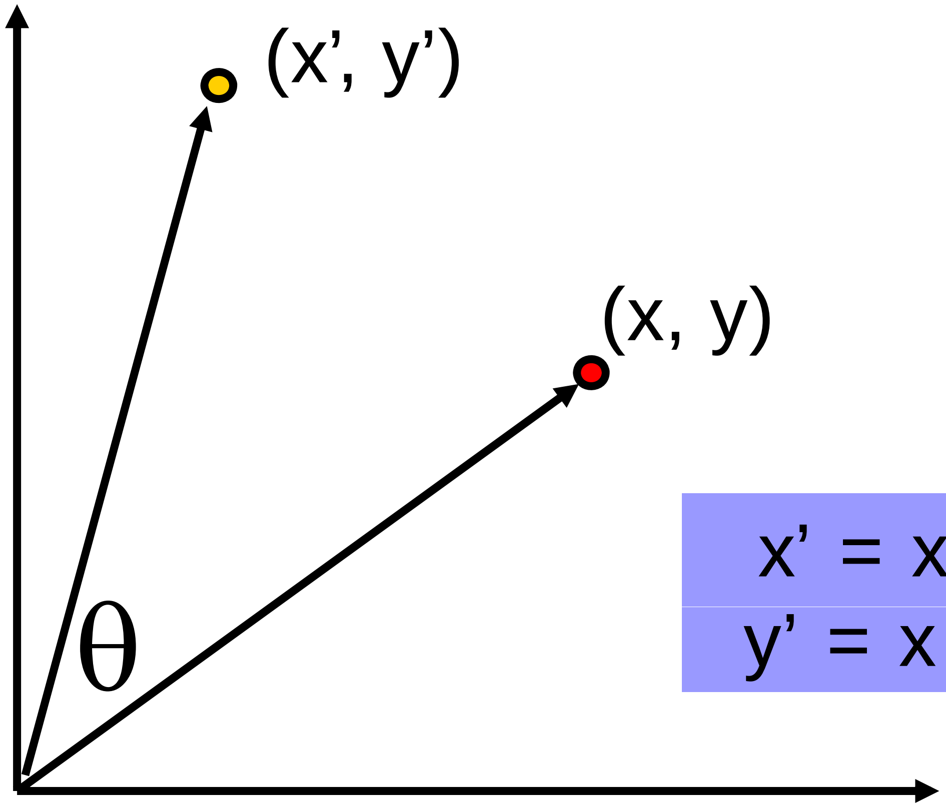
Scaling

- Scaling operation:
$$x' = ax$$
$$y' = by$$

- Or, in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}}_{\textit{scaling matrix } S} \begin{bmatrix} x \\ y \end{bmatrix}$$

2-D Rotation



$$x' = x \cos(\theta) - y \sin(\theta)$$
$$y' = x \sin(\theta) + y \cos(\theta)$$

2-D Rotation

- This is easy to capture in matrix form:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Even though sin(θ) and cos(θ) are nonlinear functions of θ,
  - *x' is a linear combination of x and y*
  - *y' is a linear combination of x and y*
- What is the inverse transformation?
  - Rotation by $-\theta$
  - For rotation matrices, det(R) = 1 so $\mathbf{R}^{-1} = \mathbf{R}^{T}$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Identity?

$$x' = x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Scale around (0,0)?

$$x' = s_x * x$$
$$y' = s_y * y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Rotate around (0,0)?

$$x' = \cos\theta * x - \sin\theta * y$$
$$y' = \sin\theta * x + \cos\theta * y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear?

$$x' = x + sh_x * y$$
$$y' = sh_y * x + y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Mirror about Y axis?

$$x' = -x$$
$$y' = y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Mirror over (0,0)?

$$x' = -x$$
$$y' = -y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2x2 Matrices

- What types of transformations can be represented with a 2x2 matrix?

2D Translation?

$$x' = x + t_x$$

$$y' = y + t_y$$

NO!

Only linear 2D transformations can be represented with a 2x2 matrix

2x2 Matrices

- Translation : Not linear transformation

$$\begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t_x + x \\ t_y + y \end{bmatrix}$$

$$T(P) + T(Q) = \begin{bmatrix} x_1 + t_x \\ y_1 + t_y \end{bmatrix} + \begin{bmatrix} x_2 + t_x \\ y_2 + t_y \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 2t_x \\ y_1 + y_2 + 2t_y \end{bmatrix}$$

$$T(P + Q) = \begin{bmatrix} x_1 + x_2 + t_x \\ y_1 + y_2 + t_y \end{bmatrix}$$

All 2D Linear Transformations

- **2D Linear transformations are combinations of**
  - Scale,
  - Rotation,
  - Shear, and
  - Mirror

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

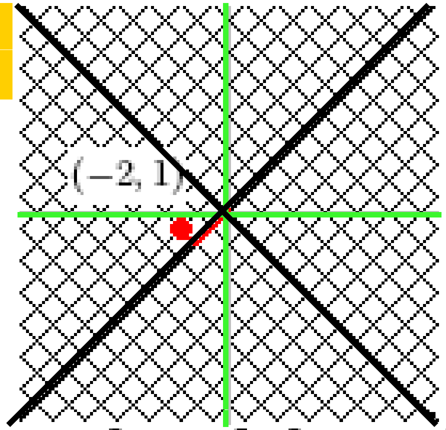
- **Properties of linear transformations:**
  - Origin maps to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Coordinate Systems

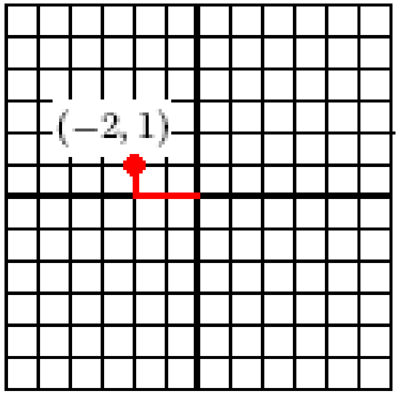
- **Every coordinate system is specified by a basis**
- **Linear transformations as a change of bases**

C_1



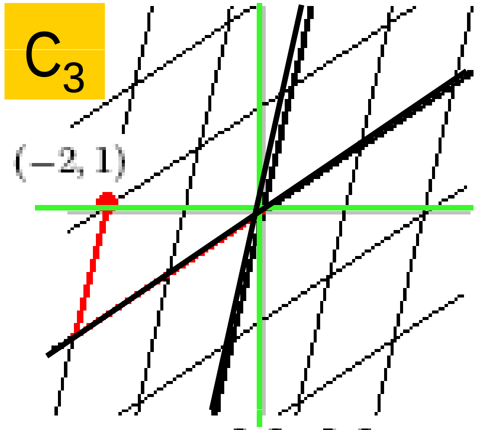
Basis: $\begin{bmatrix} \cos 45^\circ \\ \sin 45^\circ \end{bmatrix}, \begin{bmatrix} -\sin 45^\circ \\ \cos 45^\circ \end{bmatrix}$

C_2



Basis: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

C_3



Basis: $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

The point (-2,1) in C_3 is (-5,1) in C_2

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$$

The point (-2,1) in C_2 is (?,?) in C_3 ⇦

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Homogeneous Coordinates

- Add a 3rd coordinate to every 2D point
- Cartesian coordinate $\Rightarrow$ homogeneous coordinate

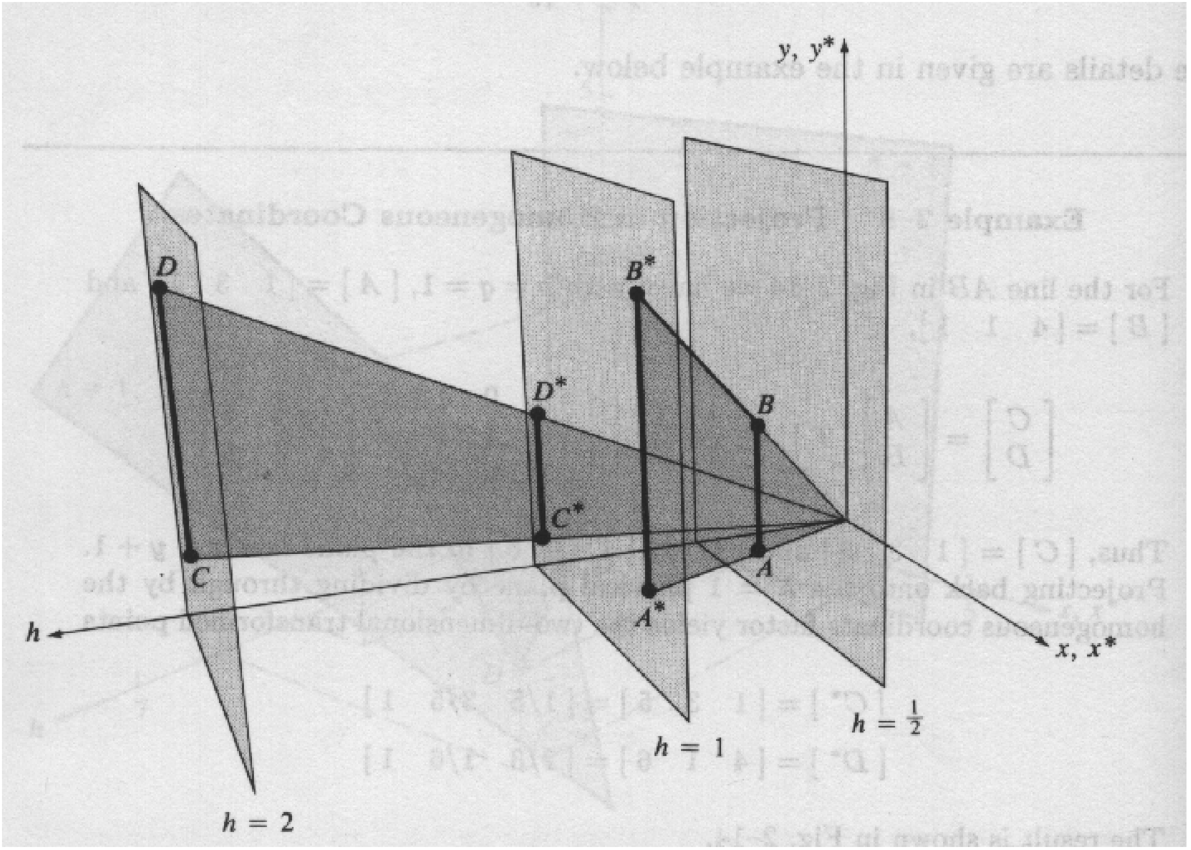
$$[x \;\; y] \quad \Rightarrow \quad [x' \;\; y' \;\; h]$$

h : real number

$$x = x'/h, \;\; y = y'/h$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{homogeneous coords}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

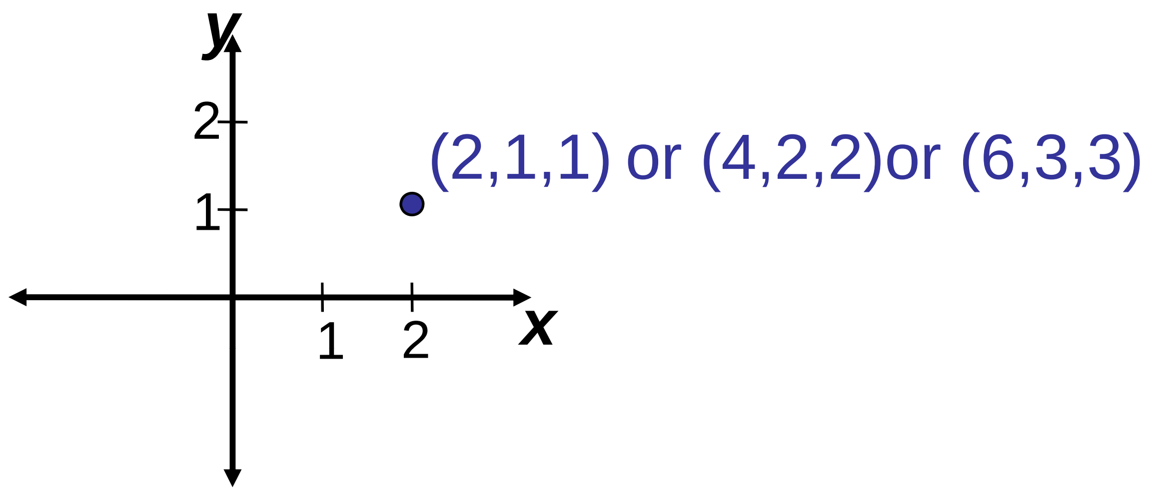
Homogeneous coordinates



- (x, y, 0) represents a point at infinity
- (0, 0, 0) is not allowed

Homogeneous Coordinates

- Convenient coordinate system to represent many useful transformations
- Possible to represent scaling, rotation, and translation in a matrix form
- *Any* sequence of translation, rotation, scale operations can be collapsed into a single homogeneous matrix.



Transformations in Homogeneous Coordinates

- Translation

$$P' = M \cdot P$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix}$$

  - When we use a row vector P

$$P' = P \cdot M^{-1}$$

Transformations in Homogeneous Coordinates

- Scaling

$$M = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

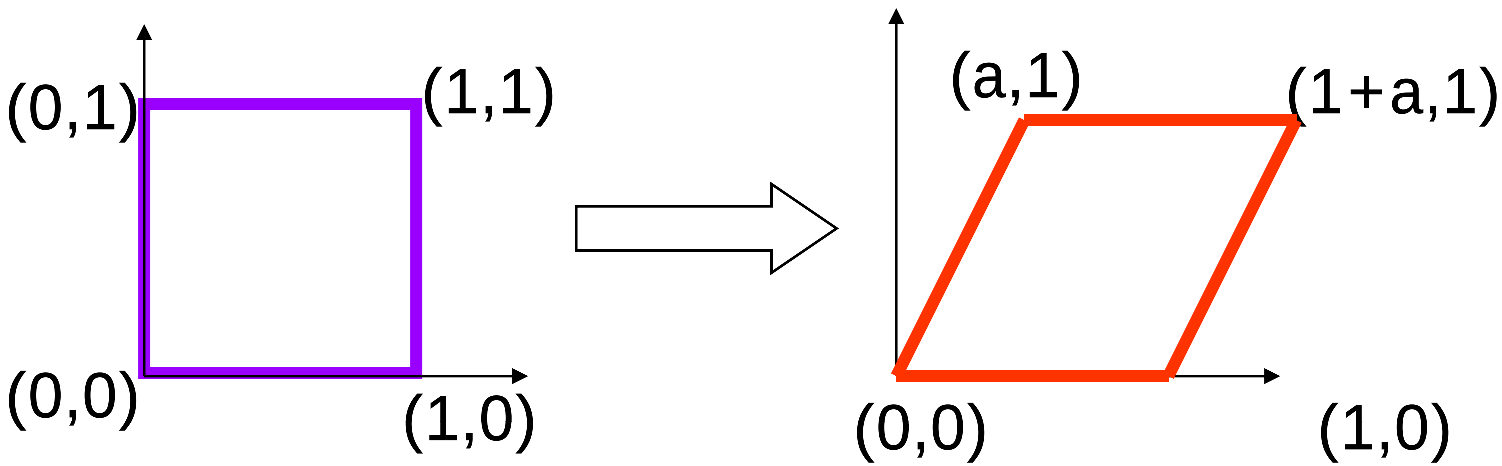
- Rotation (counter-clockwise)

$$M = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformations in Homogeneous Coordinates

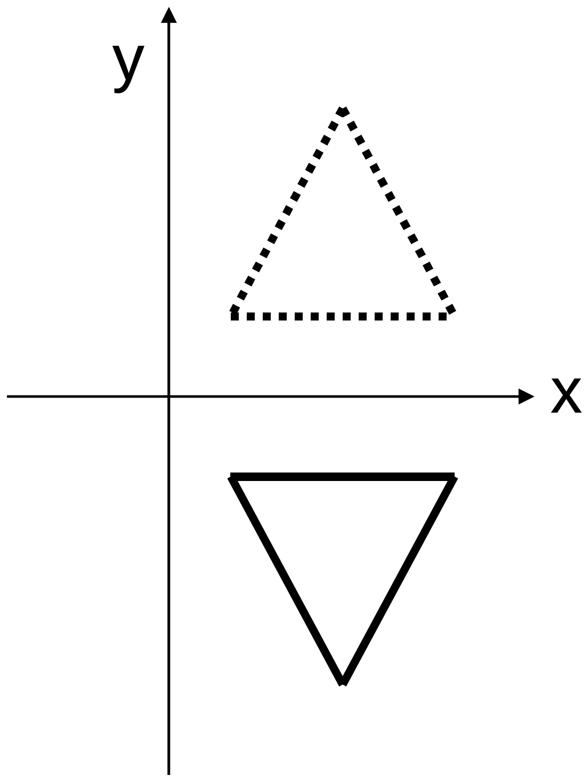
- Shearing
  - shear along x-axis
  - x' = x + ay, y' = y

$$M = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Transformations in Homogeneous Coordinates

- Reflection



$$\mathrm{Ref}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Affine Transformation

- Affine space

  Set of vectors and set of points $A = (V, P)$

  - Vectors in V form a vector space
  - Points in P can be added to vectors to generate new point $P + \vec{v} = P'$

Affine Transformation

Affine transformation $T : A_1 \rightarrow A_2$

Where A_1, A_2 are affine spaces.

- T maps vectors to vectors and points to points

$$T(P+\vec{u}) = T(P) + T(\vec{u})$$

- T is linear transformation + translation

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Affine Transformations

- **Properties of affine transformations:**
  - Origin does not necessarily map to origin
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios are preserved
  - Closed under composition
  - Models change of basis

Projective Transformations

- Perspective projection effect

$$\begin{bmatrix} x' \\ y' \\ h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ p & q & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$= [x \quad y \quad (px + qy + 1)^t]$$

We get

$$x^* = x'/h$$

$$y^* = y'/h$$

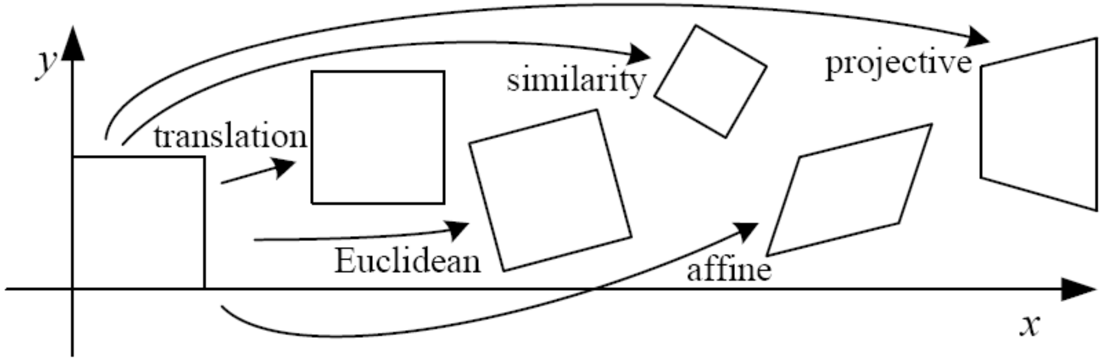
Projective Transformations

- **Projective transformations …**
  - Affine transformations, and
  - Projective warps

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- **Properties of projective transformations:**
  - Origin does not necessarily map to origin
  - Lines map to lines
  - Parallel lines do not necessarily remain parallel
  - Ratios are not preserved
  - Closed under composition
  - Models change of basis

2D image transformations



Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation $+ \cdots$	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	3	lengths $+ \cdots$	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	4	angles $+ \cdots$	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$	6	parallelism $+ \cdots$	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

These transformations are a nested set of groups

⇔ Closed under composition and inverse is a member

Rigid Transformations

- A ***rigid transformation*** *T* is a mapping between affine spaces
  - *T* maps vectors to vectors, and points to points
  - *T* preserves distances between all points
  - *T* preserves cross product for all vectors (to avoid reflection)
- Preserves angles and lengths
- In 3-spaces, T can be represented as

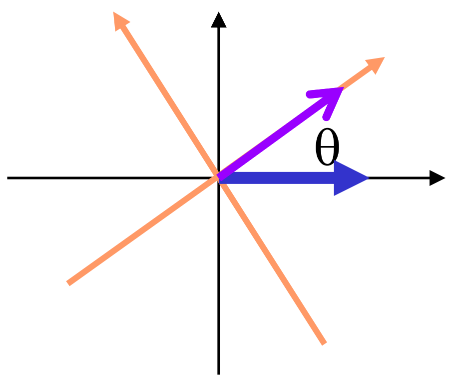
$$T(\mathbf{p}) = \mathbf{R}_{3\times3}\mathbf{p}_{3\times1} + \mathbf{T}_{3\times1}, \quad \text{where}$$

$$\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{I} \quad \text{and} \quad \det \mathbf{R} = 1$$

Orthogonal Matrix

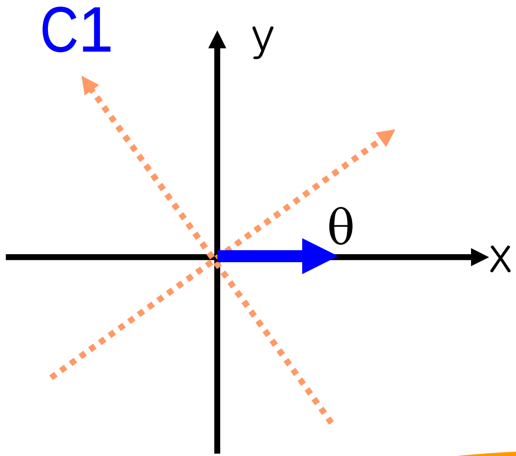
- Rotation matrix is a special orthogonal matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

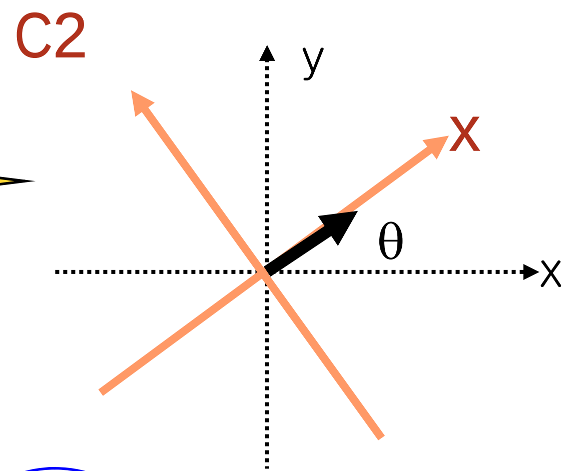


- R is normalized: the squares of the elements in any row or column sum to 1.
- R is orthogonal: the dot product of any pair of rows or any pair of columns is 0.
- The rows of R represent the coordinates in the original space of unit vectors along the coordinate axes of the rotated space.
- The columns of R represent the coordinates in the rotated space of unit vectors along the axes of the original space.

Orthogonal Matrix



$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



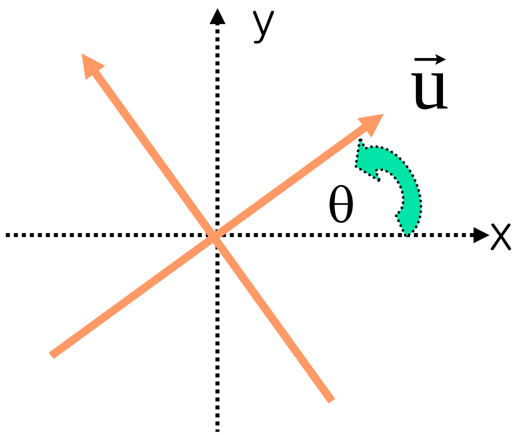
C2 C1

$$R \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$R \cdot \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The rows of R form a set of orthogonal unit vectors that are rotated by R onto x and y, respectively

Orthogonal Matrix



$$\vec{u}_x = \frac{\vec{u}}{|\vec{u}|} = \begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix}$$

$$\vec{u}_y = \begin{bmatrix} -\sin\theta & \cos\theta \end{bmatrix}$$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Composite Transformation

- Translations

$$\begin{bmatrix} 1 & 0 & t_{x2} \\ 0 & 1 & t_{y2} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & t_{x1} \\ 0 & 1 & t_{y1} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_{x1}+t_{x2} \\ 0 & 1 & t_{y1}+t_{y2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = T(t_{x2}, t_{y2}) \cdot T(t_{x1}, t_{y1})$$

$$= T(t_{x1}+t_{x2}, t_{y1}+t_{y2})$$

Composite Transformation

- Scaling

$$\begin{bmatrix} s_{x2} & 0 & 0 \\ 0 & s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_{x1} & 0 & 0 \\ 0 & s_{y1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_{x1} \cdot s_{x2} & 0 & 0 \\ 0 & s_{y1} \cdot s_{y2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = S(s_{x2}, s_{y2}) \cdot S(s_{x1}, s_{y1})$$

$$= S(s_{x1} \cdot s_{x2}, s_{y1} \cdot s_{y2})$$

Composite Transformation

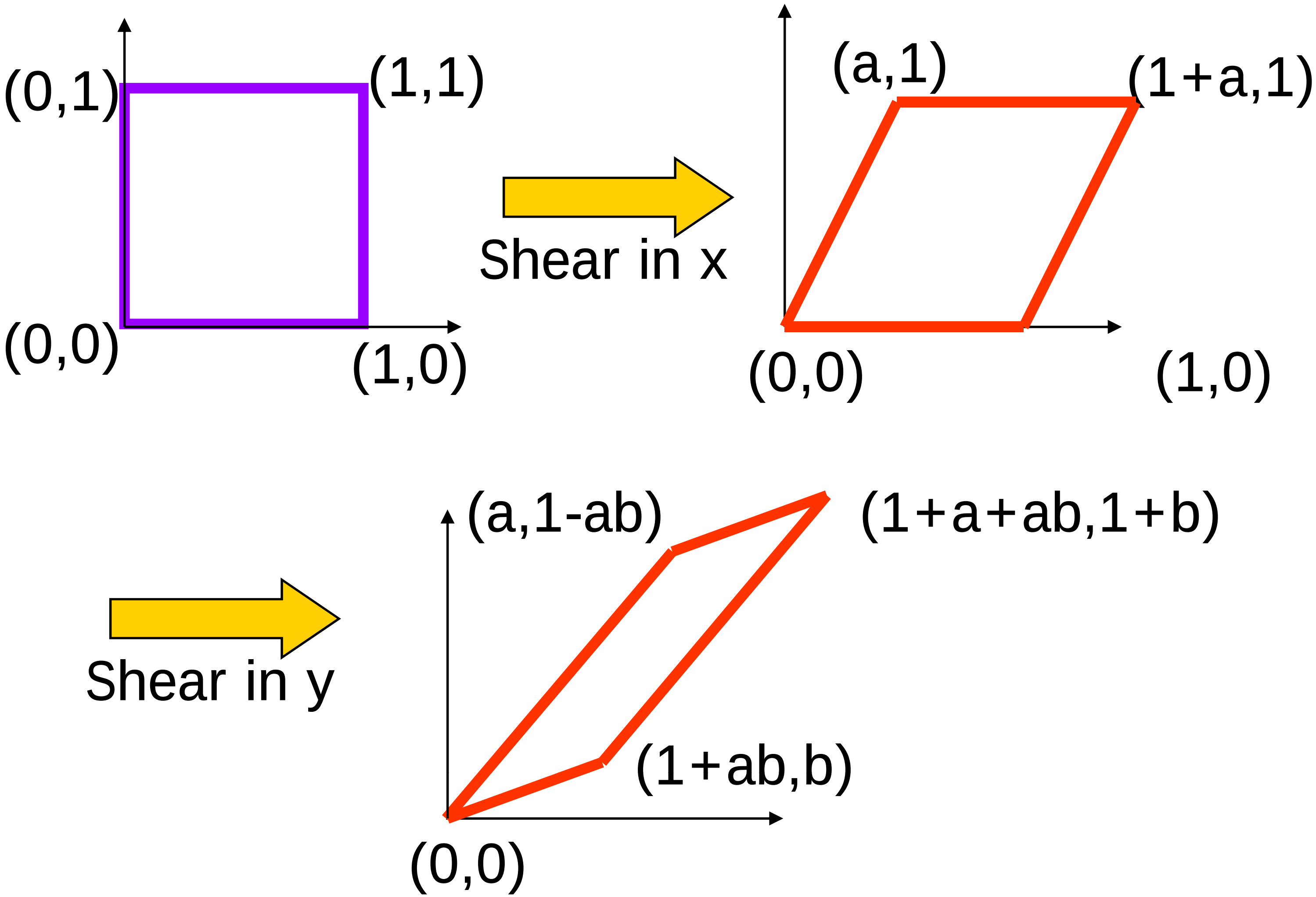
- Rotations

$$\begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

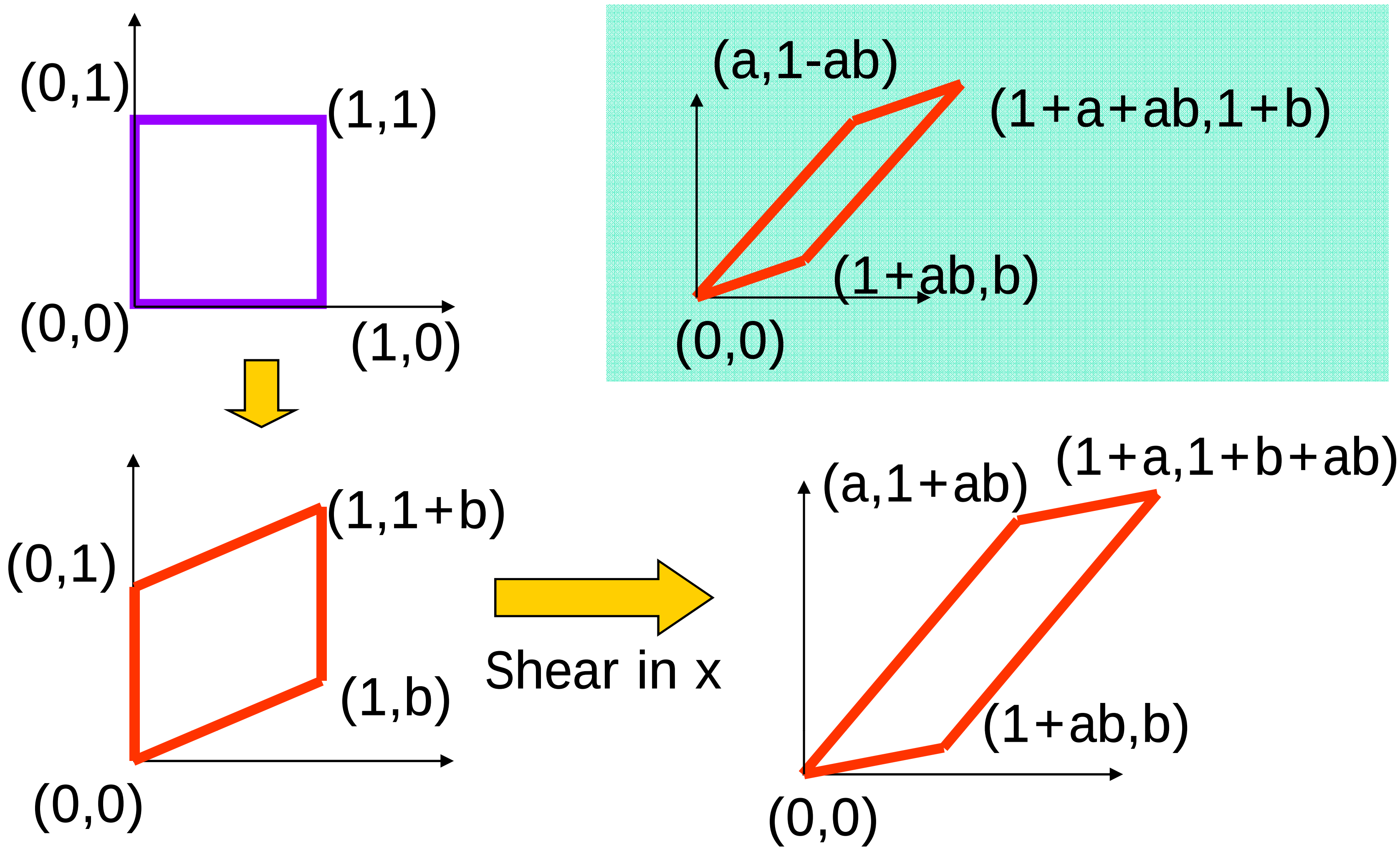
$$= \begin{bmatrix} \cos(\theta_2+\theta_1) & -\sin(\theta_2+\theta_1) & 0 \\ \sin(\theta_2+\theta_1) & \cos(\theta_2+\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = R(\theta_1) \cdot R(\theta_2) = R(\theta_1 + \theta_2)$$

Shear in x then in y



Shear in y then in x

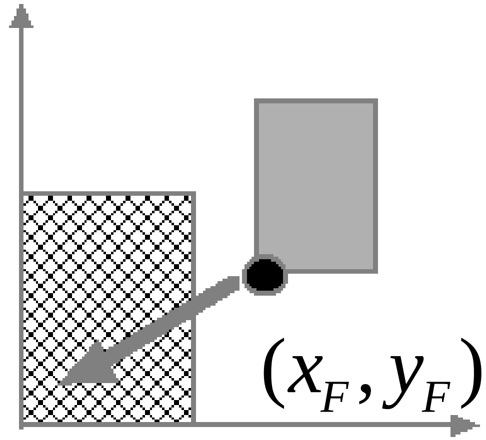


Scaling Relative to a Fixed Point

$$T(x_F, y_F) \cdot S(s_x, s_y) \cdot T(-x_F, -y_F)$$

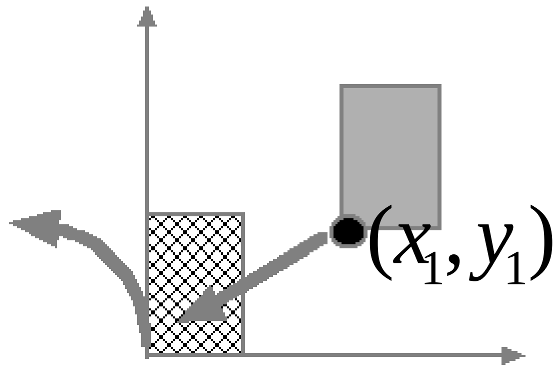
$$= \begin{bmatrix} 1 & 0 & x_F \\ 0 & 1 & y_F \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -x_F \\ 0 & 1 & -y_F \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s_x & 0 & (1 - s_x) \cdot x_F \\ 0 & s_y & (1 - S_y) \cdot y_F \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation about a Pivot Point

$$T(x_1, y_1) \cdot R(\theta) \cdot T(-x_1, -y_1)$$



Note that

$$T(x_1, y_1) R(\theta) T(-x_1, -y_1) \neq T(x_1, y_1) T(-x_1, -y_1) R(\theta)$$

Concatenation Properties

- Associative

  $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

- Commutative
  - $T(t_{x1}, t_{y1}) \cdot T(t_{x2}, t_{y2}) = T(t_{x2}, t_{y2}) \cdot T(t_{x1}, t_{y1})$
  - $S(s_{x1}, s_{y1}) \cdot S(s_{x2}, s_{y2}) = S(s_{x2}, s_{y2}) \cdot S(s_{x1}, s_{y1})$
  - $R(\theta_1) \cdot R(\theta_2) = R(\theta_2) \cdot R(\theta_1)$
  - $S(s_x, s_y) \cdot R(\theta) = R(\theta) \cdot S(s_x, s_y)$ with $s_x = s_y$

General Transformation Eq.

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Efficiency Considerations

- x' = ax + cy + e, y' = bx + dy + f

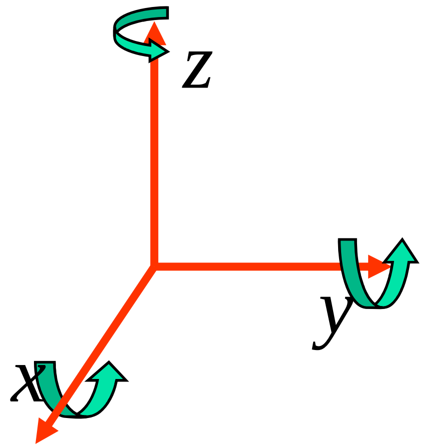
  Need x : 4   + : 4

- Without concatenation: increased number of calculations

  ⇒ Apply composite transformation after concatenation of each transformation matrix.

3D Transformation

- We can extend to 2D to 3D by considering one more dimension with very similar equations.



- Remember
  - Rigid body transformation
  - Affine transformation
  - Projective (Perspective) transformation