Curves and Surfaces

Intro. to Computer Graphics
Spring 2008, Y. G. Shin

Representation of Curves and Surfaces

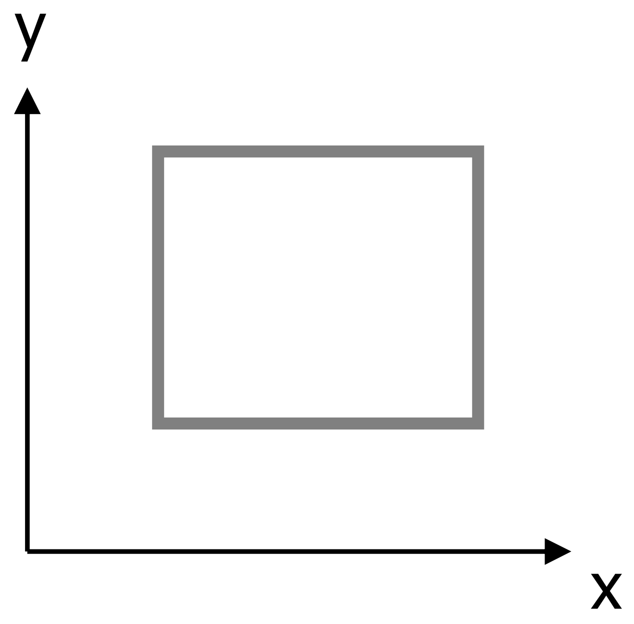
- Key words: surface modeling, parametric surface, continuity, control points, basis functions, Bezier curve, B-spline curve

Why we need surface models?

- All shapes can be described in terms of points. But, it is impractical to enumerate the points that comprise a shape
- We define shape indirectly through expressions that relate certain *properties* of points that comprise them.

Intrinsic and extrinsic properties

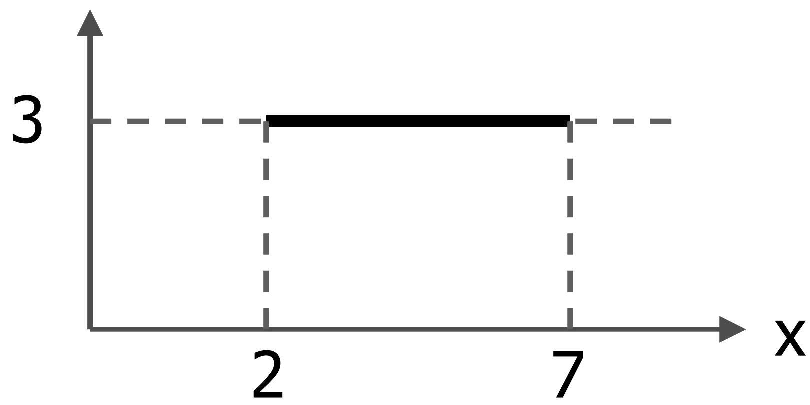
- Intrinsic properties
  - B has four sides
  - All four sides have equal length
  - All four angles are 90^{o}, ...
- Extrinsic properties
  - two horizontal sides
  - two vertical sides vertices of B are at P_0, P_1, P_2 and P_3



Intrinsic and extrinsic properties

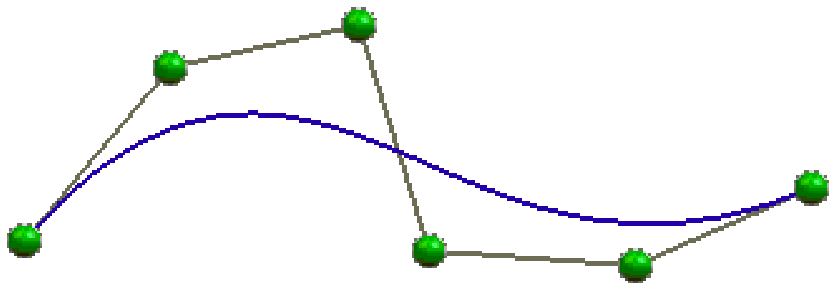
- Shape definitions that use extrinsic properties of the shape are dependent on the coordinate system used.
- a line: $y = 3, \ 2 \le x \le 7$

  🡐 *Axis dependency*



Axis Independence

A mathematical representation of a line/curve is axis independent if its shape depends on only the *relative position of the points* defining its characteristic vectors and is independent of the coordinate system used.



Axis-independent shape definition

- Shape definitions that use intrinsic properties of the shape are axis-independent.

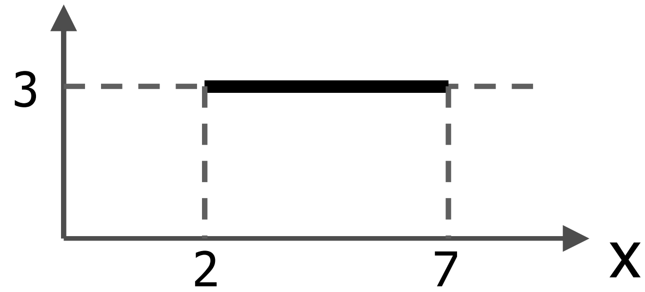
$$x = (1-t)\,p_{1x} + t p_{2x}$$

$$y = (1-t)\,p_{1y} + t p_{2y}$$

$$0 \le t \le 1$$

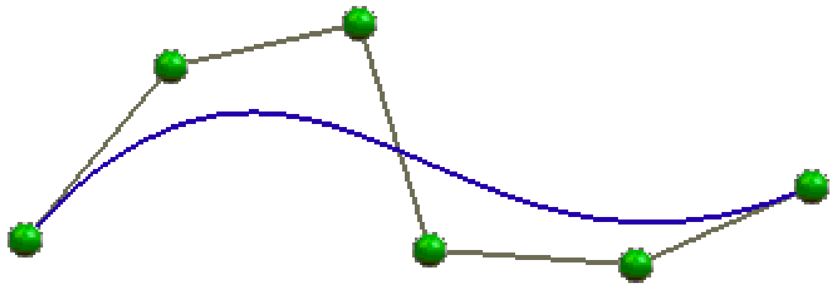
$$p_1 = (p_{1x}, p_{1y}) = (2,3)$$

$$p_2 = (p_{2x}, p_{2y}) = (7,3)$$



Curve & Surface Models

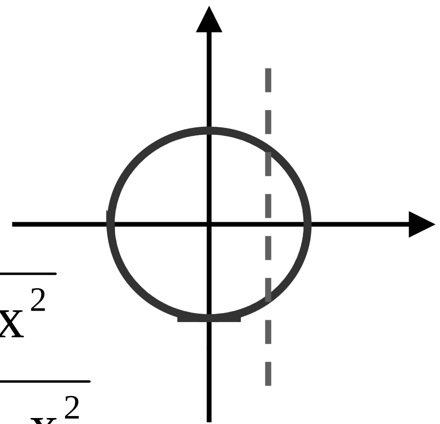
- Explicit/implicit
- Parametric/non-parametric
- Approximation
  - polygon mesh : a collection of edges, vertices, and polygons



Nonparametric explicit representation

- x = x

  y = f(x)
- successive values of y can be obtained by plugging in successive values of x.
- easy to generate polygons or line segments
- single-valued function



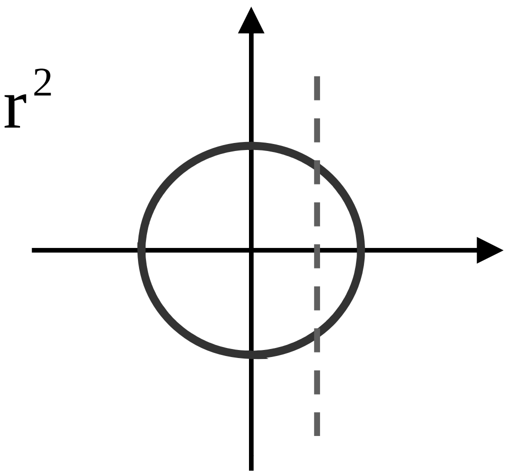
$$y = \sqrt{r^2 - x^2}$$

$$y = -\sqrt{r^2 - x^2}$$

Nonparametric implicit representation

- $f(x, y, z) = 0$
- Define curves as solution of equation system
- E.g., a circle:

$$x^2 + y^2 = r^2$$



Nonparametric implicit representation

- algebraic quadric surfaces
  : f is a polynomial of degree <= 2

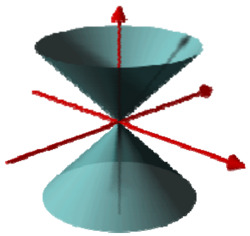
$$f(x,y,z) = ax^2 + by^2 + cz^2 + 2dxy + 2eyz + 2fxz + 2gx + 2hy + 2jz + k = 0$$

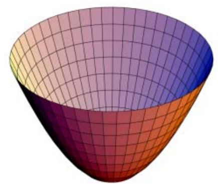
$$sphere: x^2 + y^2 + z^2 - 1 = 0$$

$$cylinder: x^2 + y^2 - 1 = 0$$

$$corn: x^2 + y^2 - z^2 = 0$$

$$paraboloid: x^2 + y^2 + z = 0$$

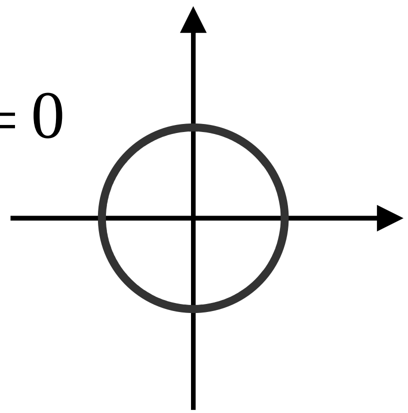




Nonparametric implicit representation

- Coefficients determine geometric properties
- Hard to render (have to solve non-linear equation system)
- Can represent closed or multi-valued curves
- Easy to classify point-membership

$$sphere: x^2 + y^2 + z^2 - 1 = 0$$

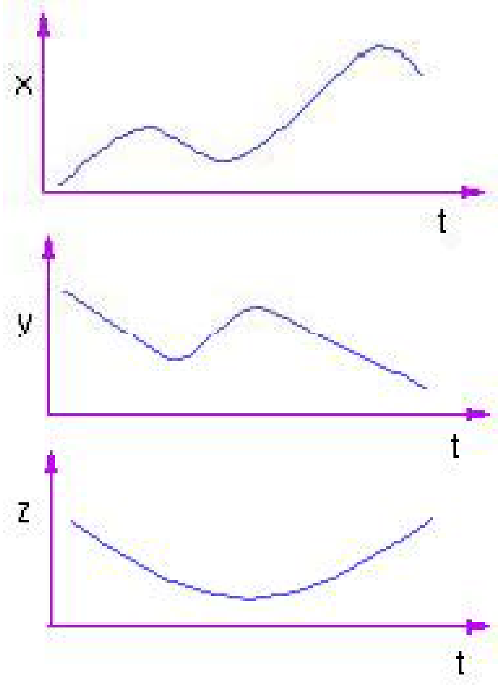


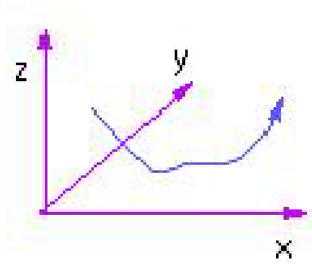
Parametric Curve

$$x(t) = a_{13}t^3 + a_{12}t^2 + a_{11}t + a_{10}$$

$$y(t) = a_{23}t^3 + a_{22}t^2 + a_{21}t + a_{20}$$

$$z(t) = a_{33}t^3 + a_{32}t^2 + a_{31}t + a_{30}$$





Parametric Curve (Example)

line from $P_1 = (x_1, y_1)$ to $P_2 = (x_2, y_2)$

$$x = (1-t)x_1 + tx_2$$

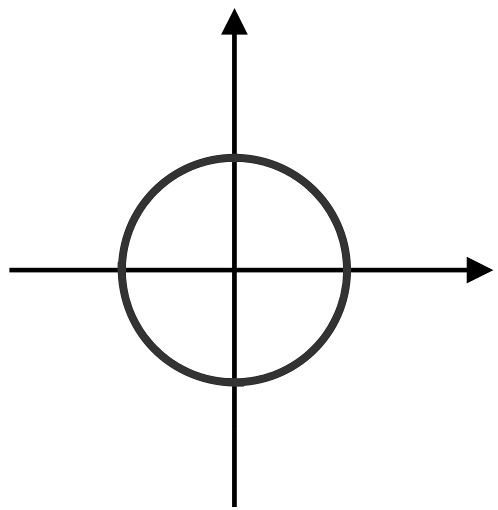
$$y = (1-t)y_1 + ty_2 \qquad 0 \le t \le 1$$

P_1, P_2 : control points

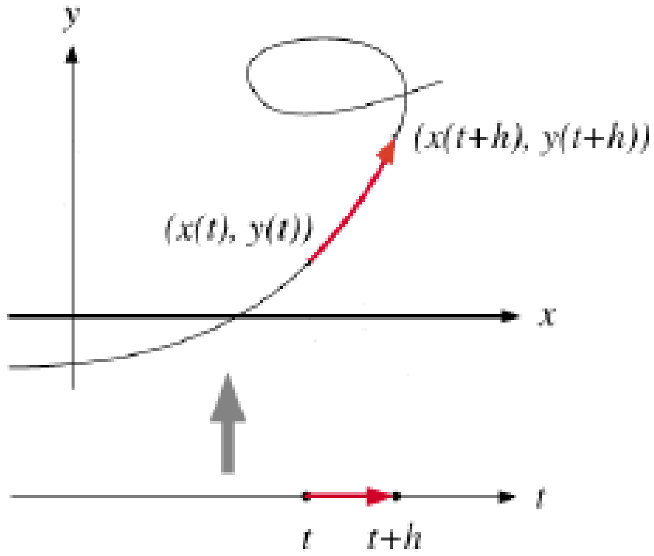
$t, 1-t$: blending functions

unit circle

$$Q(u) = (\cos(u/2\pi), \sin(u/2\pi))$$



Parametric Curve Characteristics



- Simple to render
  - evaluate parameter function
- Hard to check whether a point lies on curve
  - have to compute the inverse mapping from (x, y) to *t*
- Can represent closed or multi-valued curves
- Curve or surface can be easily translated or rotated
- Composite curves and surfaces can be formed by piecewise descriptions

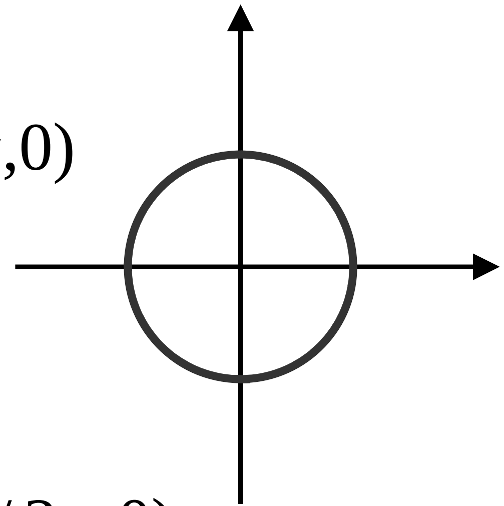
Parametric Curve Characteristics

- No infinite slope problem

parametric form:

$$Q'(u) = (-\sin(u/2\pi)/2\pi, \cos(u/2\pi)/2\pi, 0)$$

implicit form: $x^2 + y^2 - 1 = 0, \quad z = 0$



⇒ at $x = 1, y = 0$,

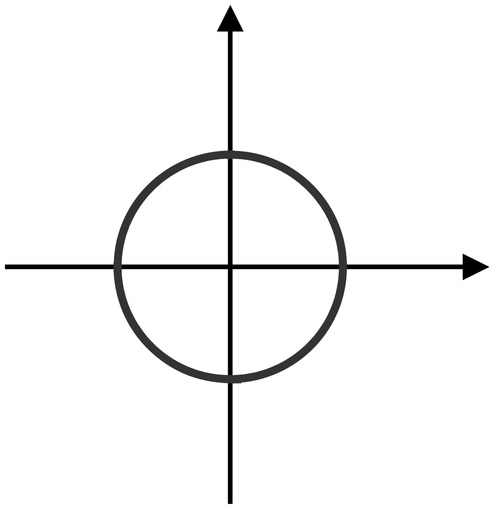
the parametric derivative is $(0, 1/2\pi, 0)$

implicit form $f'(x, y, z) = -x/y \Rightarrow \infty$

Parametric Curve Characteristics

- Not unit form

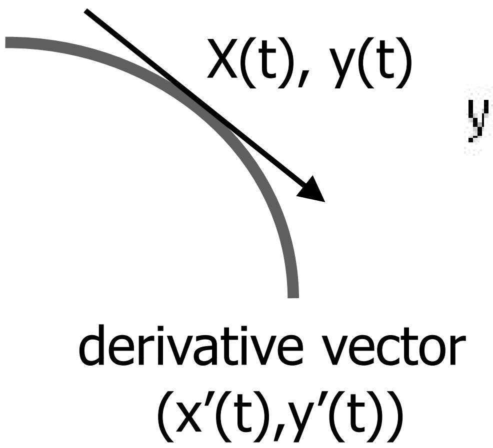
  (e.g.) a circle with radius 1 centered at the origin

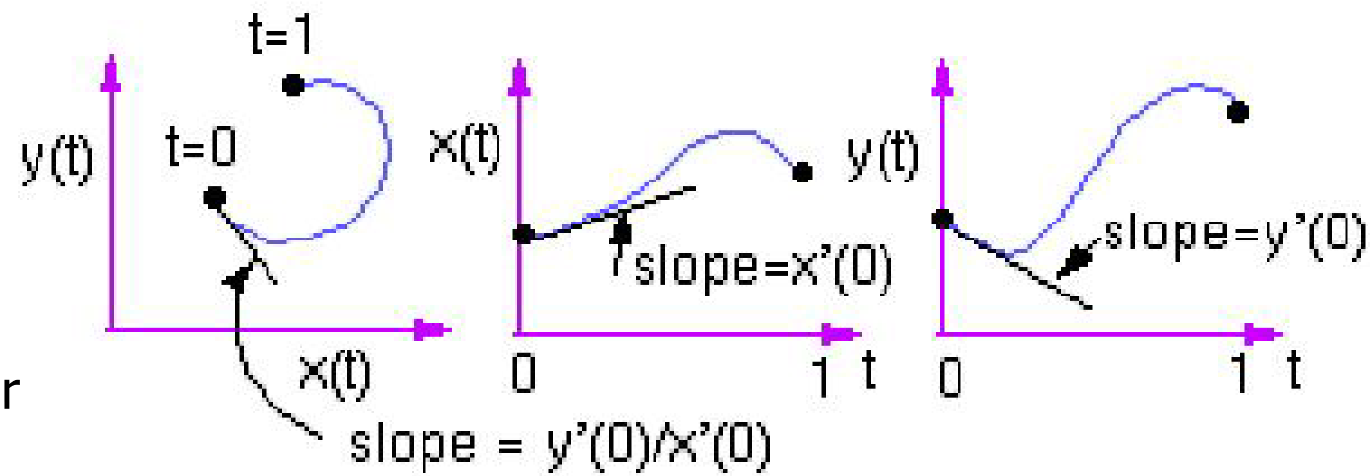


$$\begin{pmatrix} x = \cos\theta \\ y = \sin\theta \end{pmatrix} \equiv \begin{pmatrix} x = t \\ y = \sqrt{1 - t^2} \end{pmatrix}$$

Tangent line to a curve

- The straight line that gives the curve's slope at a point
- Deduced from the derivative of the curve at the point





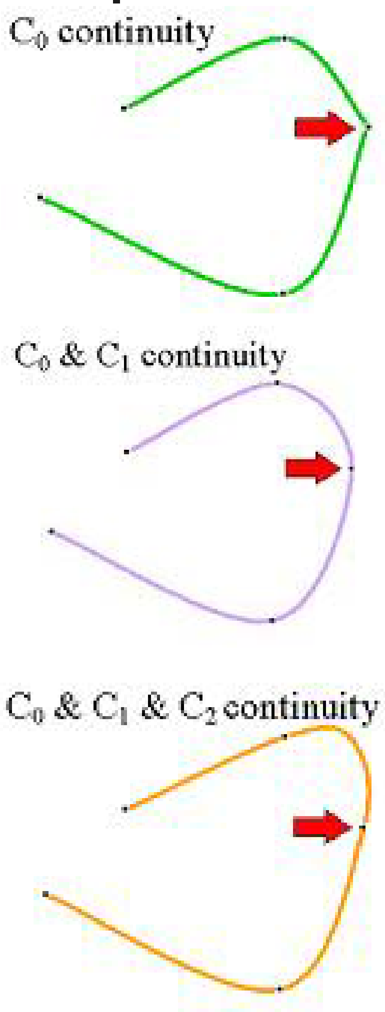
Piecewise Polynomial Curves

- Cut curve into segments and represent each segment as a polynomial curve
- But how do we guarantee smoothness at the joints? (continuity problem)

Continuity

- Implies a notion of smoothness at the connection points
- *Parametric continuity*
  - We view the curve or surface as a function rather than a shape.
  - Matching the parametric derivatives of adjoining curve sections at their common boundary
  - You need a parameterization

Parametric Continuity



C^0 : a curve is continuous if it can be drawn without lifting the pencil from the paper. (x, y, z) - values of the two curves agree.

C^1 : the derivative curve is also continuous, i.e., $(dx/dt, dy/dt, dz/dt)$ agree at their junction.

C^2 : the direction and magnitude of $d^2/dt^2[Q(t)]$ are equal at the join point

Geometric Continuity

- Geometric continuity is defined using only the shape of the curve
- Geometric smoothness independent of parametrization

G^0 : joining two segments at a common end point $(= C^0)$

G^1 : a curve's tangent direction changes continuously

(direction equal, but necessarily the magnitude)

The order of polynomial curves

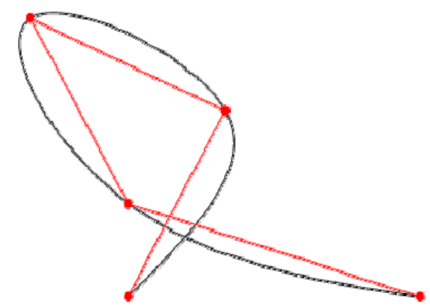
$$\text{a polynomial of order } k+1 \; (\equiv \text{degree } k)$$

$$P(u) = c_0 + c_1 u + c_2 u^2 + \cdots + c_k u^k$$

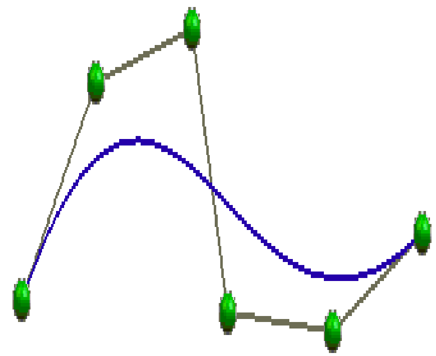
- In computer graphics, usually degree = 3
  - Sufficient flexibility w/o much cost
  - The cubic is the lowest degree polynomial that gives C^1 and C^2 continuity

Curve models

- Curve fitting techniques (interpolation techniques)
  - pass through each and every data point
  - linear approximation, natural cubic spline

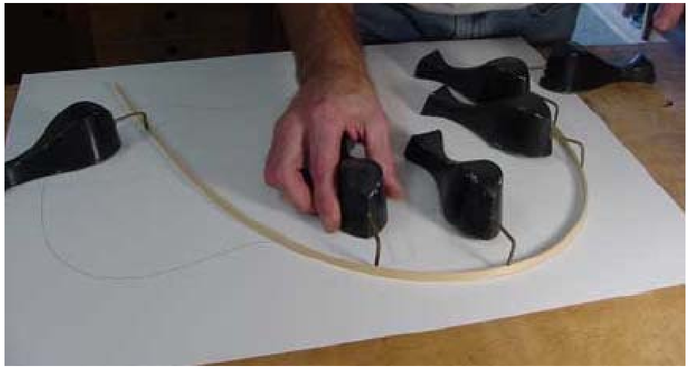


- Curve fairing techniques (approximation techniques)
  - few if any points on the curve pass through each and every data points
  - Hermite curve, Bezier curve, B-spline curve



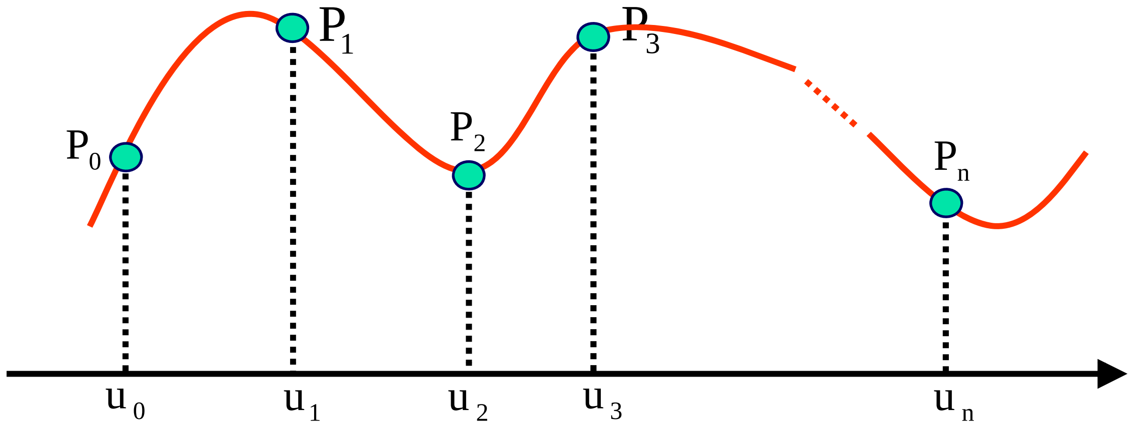
Natural Spline Curves

- Motivated by loftman's spline
  - Long narrow strip of wood or plastic
  - Shaped by lead weights (called ducks)
- a cubic spline curve, q(u), composed of cubic polynomials that interpolate the points $P_0, P_1, \ldots, P_n$
- C^{n-1} continuity can be achieved from splines of degree n



Natural Cubic Splines

- divide the interval [a,b] into *n* intervals $[u_i,u_{i+1}]$, for *i=0 to n-1* . The numbers u_i are called **knots.**
- The vector $[u_0,u_1, ..., u_{i-1}]$ is called a **knot vector** for the spline. If the knots are equidistantly distributed in the interval [*a*,*b*], we say the spline is **uniform**, otherwise we say it is **non-uniform**.



Natural Cubic Splines

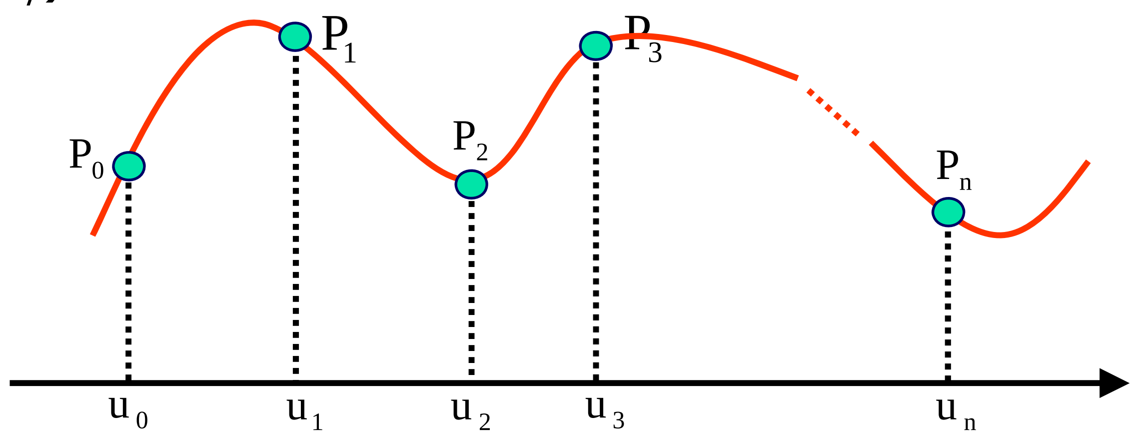
- each cubic spline curve is determined by the position vectors, tangent vectors and parameter values

$$q_i(u) = c_{i0} + c_{i1}u + c_{i2}u^2 + c_{i3}u^3$$

$$q_i(u_{i-1}) = p_{i-1} \text{ and } q_i(u_i) = p_i \qquad \text{for } i = 1 \text{ to } n$$

$$q_i'(u_i) = q'_{i+1}(u_i) \qquad \text{for } i = 1 \text{ to } n-1$$

$$q_i''(u_i) = q''_{i+1}(u_i) \qquad \text{for } i = 1 \text{ to } n-1$$



Cubic Splines

- the polynomial coefficients of a cubic spline are dependent on all *n* control points

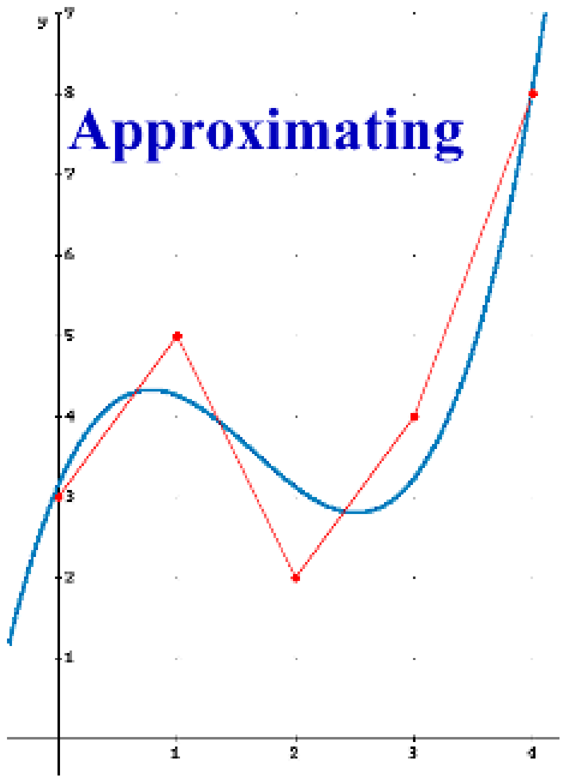
  → a change in any one segment affects the entire curve
- It is inconvenient to represent the curve directly using the coefficients c_i

  ← the relationship between the shape of the curve and the coefficients is not clear or intuitive

  ⇒ rearrange the polynomial form into *control points* and *basis functions* (GEOMETRIC FORM)

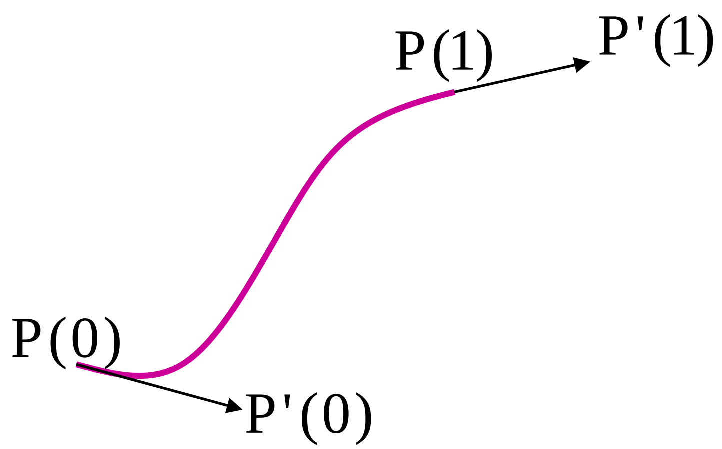
Specifying Curves

- Control Points
  - A set of points that influence the curve's shape
- Knots
  - Points that lie on the curve
  - Subinterval endpoints



Hermite Curves

- Parametric curves
- Defined by two end points with the derivative of the curve at these points



Hermite Curves(cubic polynomial)

$$x(t) = a_{13}t^3 + a_{12}t^2 + a_{11}t + a_{10}$$
$$y(t) = a_{23}t^3 + a_{22}t^2 + a_{21}t + a_{20}$$
$$z(t) = a_{33}t^3 + a_{32}t^2 + a_{31}t + a_{30}$$

$$P(t) = [x(t) \quad y(t) \quad z(t)] = \mathbf{a}_3 t^3 + \mathbf{a}_2 t^2 + \mathbf{a}_1 t + \mathbf{a}_0$$

$$\text{where } \mathbf{a}_i = (a_{1i}, a_{2i}, a_{3i})$$

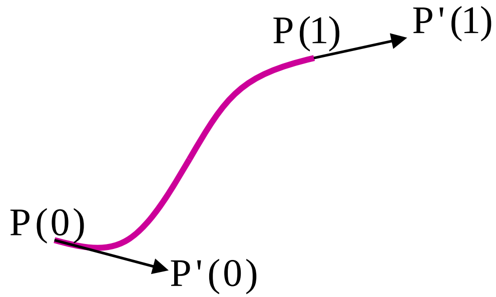
$$P(0) = \mathbf{a}_0 \qquad \mathbf{a}_0 = P(0)$$

$$P(1) = \mathbf{a_3} + \mathbf{a_2} + \mathbf{a_1} + \mathbf{a_0} \qquad \mathbf{a}_1 = P(0)$$

$$P'(0) = \mathbf{a_1} \qquad \mathbf{a}_2 = -3P(0)^2 + 3P(1) - 2P'(0) - P'(1)$$

$$P'(1) = 3\mathbf{a_3} + \mathbf{2a_2} + \mathbf{a_1} \qquad \mathbf{a}_3 = 2P(0) - 2P(1) + P'(0) + P(1)$$

Hermite Curves



$$P(t) = B_1(t)P(0) + B_2(t)P(1) + B_3P'(0) + B_4P'(1)$$

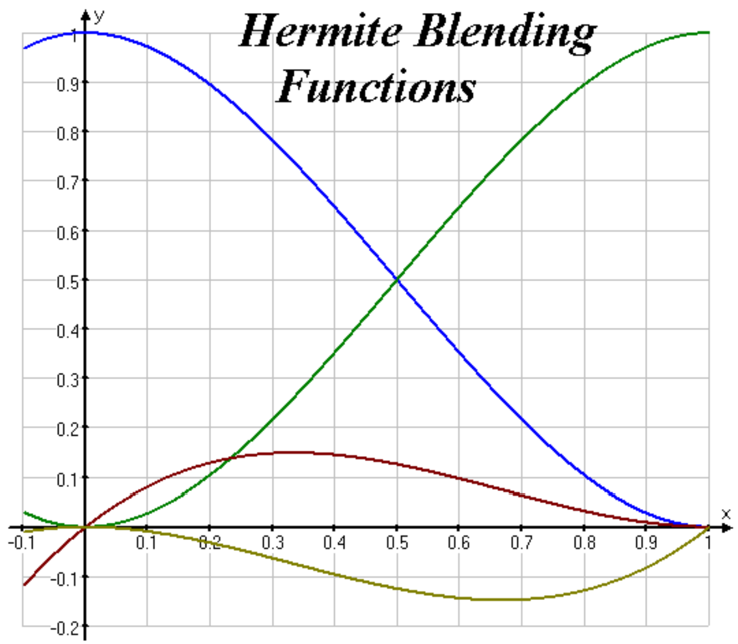
$$B_1(t) = 2t^3 - 3t^2 + 1 \qquad B_2(t) = -2t^3 + 3t^2$$

$$B_3(t) = t^3 - 2t^2 + t \qquad B_4(t) = t^3 - t^2$$

$B_i(t)$: blending functions

P(0), P(1), P'(0), P'(1)

: geometric coefficients



Hermite Curves

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P'_0 \\ P'_1 \end{bmatrix}$$

$$= T \cdot M_H \cdot G_H (= B \cdot G_H)$$

M_H :Hermite basis matrix

G_H :Hermite geometry vector

Represent Polynomials with basis functions

- Polynomials including degree k forms a vector space P^{k+1}
- Specify a curve *P(u)* as a position in the vector space P^{k+1} via the coordinate $(\mathrm{p}_0,\cdots,\mathrm{p}_\mathrm{k})$

and the basis $(1,t,t^2,\cdots,t^k)$

$$b_i(t)=t^k,\quad 0\le i\le k:\text{basis functions}$$

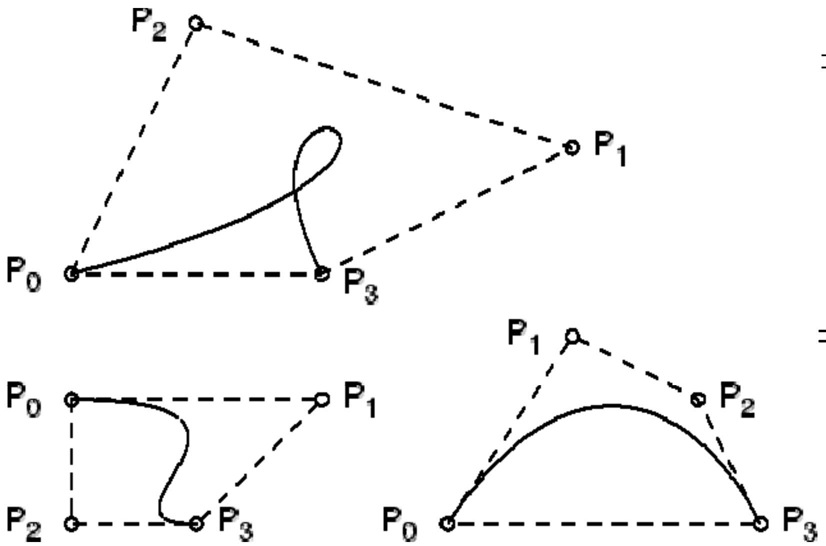
$$p_0,\cdots,p_k:\text{control points}$$

$$\Rightarrow Q(t)=\sum_{i=0}^{k} p_i b_i(t)$$

Properties shared by most useful bases

- **Convex hull property**

if $\sum_{i=0}^{k} b_i(t) = 1$ and basis functions are not negative over the interval they are defined then any point on the curve is a weighted average of its control points.

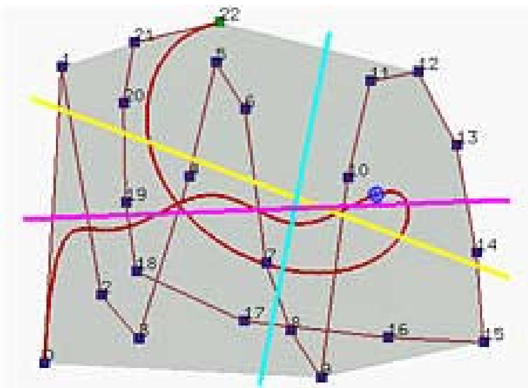


$\Rightarrow$ **no points on the curve lies outside the polygon formed by joining the control points together**

$\Rightarrow$ **inexpensive means for calculating the bound of a curve or surface in space**

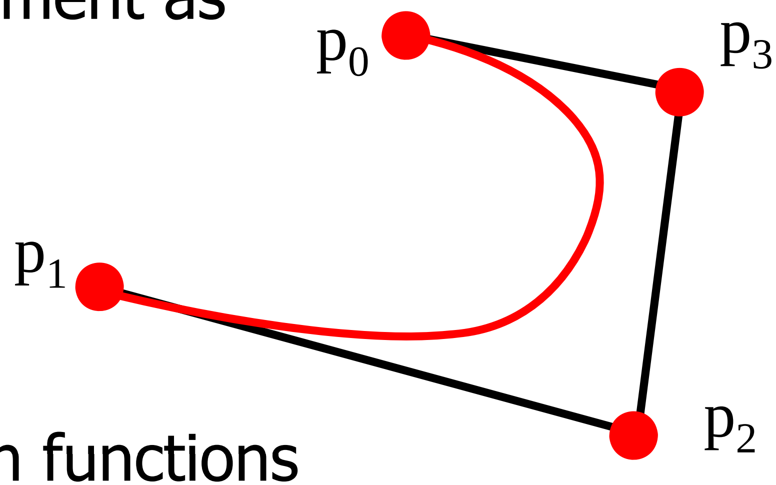
Properties shared by most useful bases

- **Affine invariance** - any linear transformation or translation of the *control points* defines a new curve that is the just the transformation or translation of the original curve. (Perspective transform is not affine.)
- **Variation diminishing** - no straight line intersects a curve more times than it intersects the curve's control polyline. It implies that the complexity (i.e., turning and twisting) of the curve is no more complex than the control polyline.



Bezier Curves

- Developed by Pierre Bézier in the 1970's for CAD/CAM operations. (PostScript drawing model)
- Represent a polynomial segment as



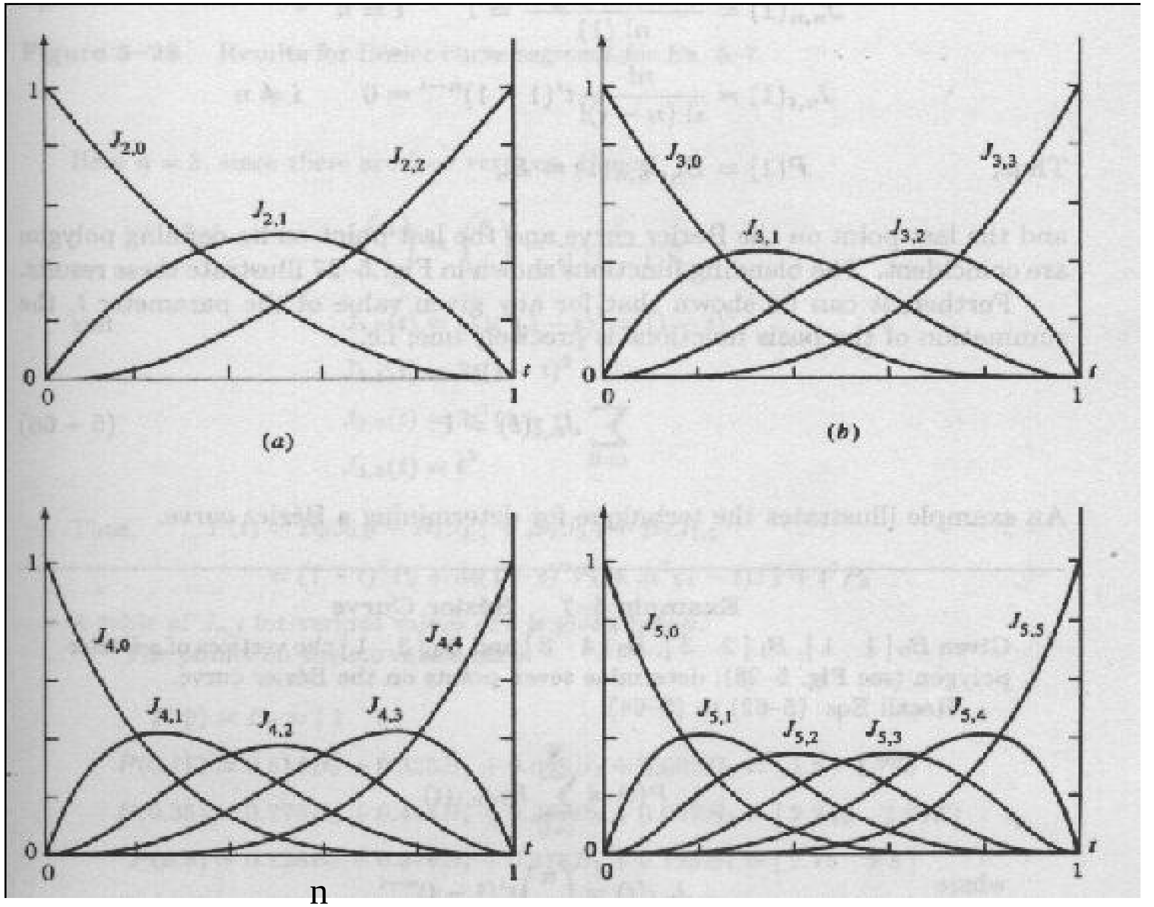
$$P(t) = \sum_{i=0}^{n} p_i J_{n,i}(t), \ 0 \le t \le 1$$

$$J_{n,i}(t) = {}_nC_i t^i (1-t)^{n-i}$$

$J_{n,i}(t)$ are the Bernstein functions

- basis or blending function of degree *n*
- used to scale or blend the control points

Bezier blending functions



Note that $\sum_{i=0}^{n} J_{n,i}(t) = 1 \rightarrow$ convex hull property

Bezier Curves (example)

Given $p_0(1,1)$, $p_1(2,3)$, $p_2(4,3)$ and $p_3(3,1)$,

find the Bezier curve.

$$P(t) = \sum_{i=0}^{n} J_{n,i}(t), \quad 0 \le t \le 1$$

$\rightarrow$ Since there are four vertices, n = 3.

$$J_{3,0}(t) = (1-t)^3 \qquad J_{3,1}(t) = 3t(1-t)^2$$

$$J_{3,2}(t) = 3t^2(1-t) \qquad J_{3,3}(t) = t^3$$

Thus, $P(t) = p_0 J_{3,0} + p_1 J_{3,1} + p_2 J_{3,2} + p_3 J_{3,3}$

$$= (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t) p_2 + t^3 p_3$$

Bezier Curves (Matrix Form)

$$P(t) = T \cdot M_B \cdot G = B \cdot G$$

$$\text{where} \quad G = \begin{bmatrix} p_1 & p_2 & \cdots & p_n \end{bmatrix}^T$$

$$B = \begin{bmatrix} J_{n,0} & J_{n,1} & \cdots & J_{n,n} \end{bmatrix}$$

$$P(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Bezier Curves

- The Bezier curve of order ***n***+1 (degree ***n***) has ***n***+1 control points.
- We can think a Bezier curve as a weighted average of all of its control points

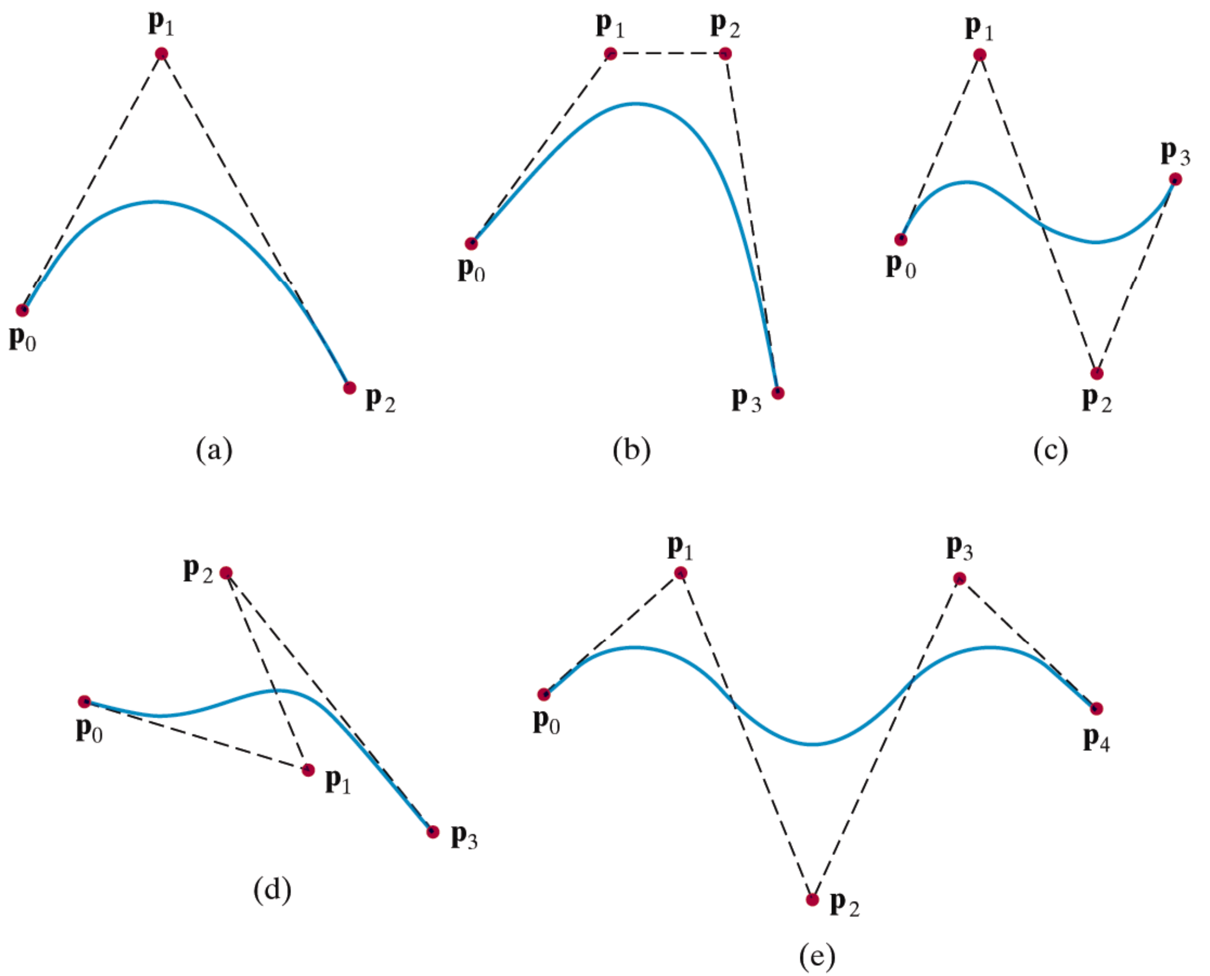
Linear (n=1) : $P(t) = (1-t)P_0 + tP_1$

Quadratic (n=2) : $P(t) = (1-t)[(1-t)P_0 + tP_1] + t[(1-t)P_1 + tP_2]$

$$\Longrightarrow P(t) = (1-t)^2 P_0 + 2(1-t)tP_1 + t^2 P_2$$

Cubic (n=3) : $P(t) = (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t) p_2 + t^3 p_3$

Bezier Curves

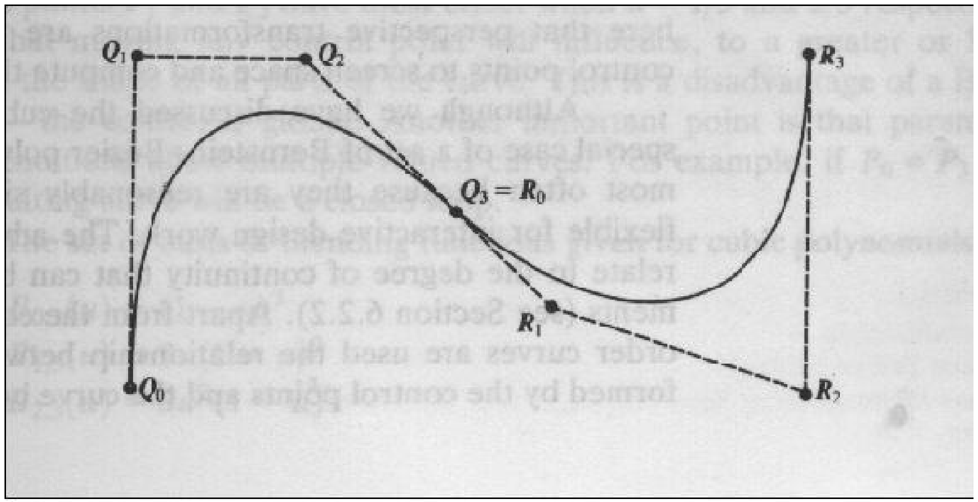


Bezier Curves

- A curve that is made of several Bézier curves is called a composite Bézier curve or a Bézier spline curve.
- Tangential continuity between Bezier segments :

$$(Q_3 - Q_2) = k(R_1 - R_0)$$

- Continuity conditions create restrictions on control points



C^1 continuity

$$Q'(1) = R'(0)$$

$$\Rightarrow (Q_3 - Q_2) = (R_1 - R_0)$$

$$\Rightarrow R_1 = Q_3 + R_0 - Q_2$$

$$= Q_3 + (Q_3 - Q_2)$$

Bezier Spline Curves

- *C^2* continuous two cubic Bezier segments *$V(t)$* and *$W(t)$* with the control points *(V_0, V_1, V_2, V_3)* and *(W_0, W_1, W_2, W_3)*.

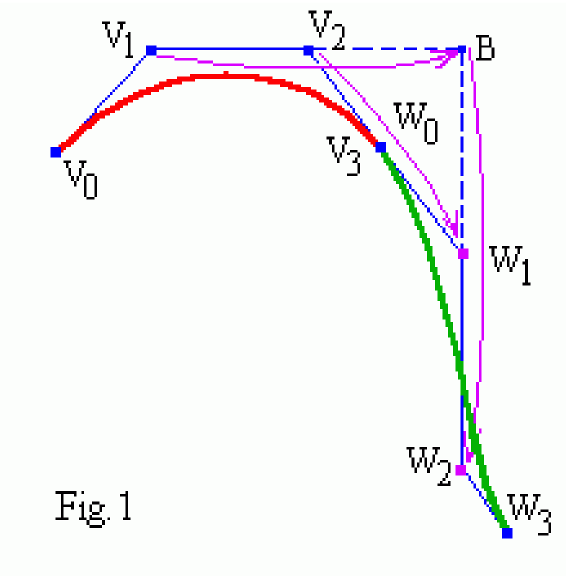


Fig. 1

  - For cubic Bezier spline:

    $V'(0) = 3(V_1 - V_0)$, $V'(1) = 3(V_3 - V_2)$,

    $V''(0) = 6(V_0 - 2V_1 + V_2)$, $V''(1) = 6(V_1 - 2V_2 + V_3)$

  - Continuity at the junction point ➔ $W_0 = V_3$.
  - Continuity of the first derivative $W'(0) = V'(1)$

    ➔ $W_1 - W_0 = V_3 - V_2$ => $W_1 = 2V_3 - V_2$

    i.e. W_1 depends on V_2 & V_3
  - Continuity of the second derivative $W''(0) = V''(1)$

    ➔ $W_0 - 2W_1 + W_2 = V_1 - 2V_2 + V_3$

    ➔ $W_2 = 2W_1 - (2V_2 - V_1)$

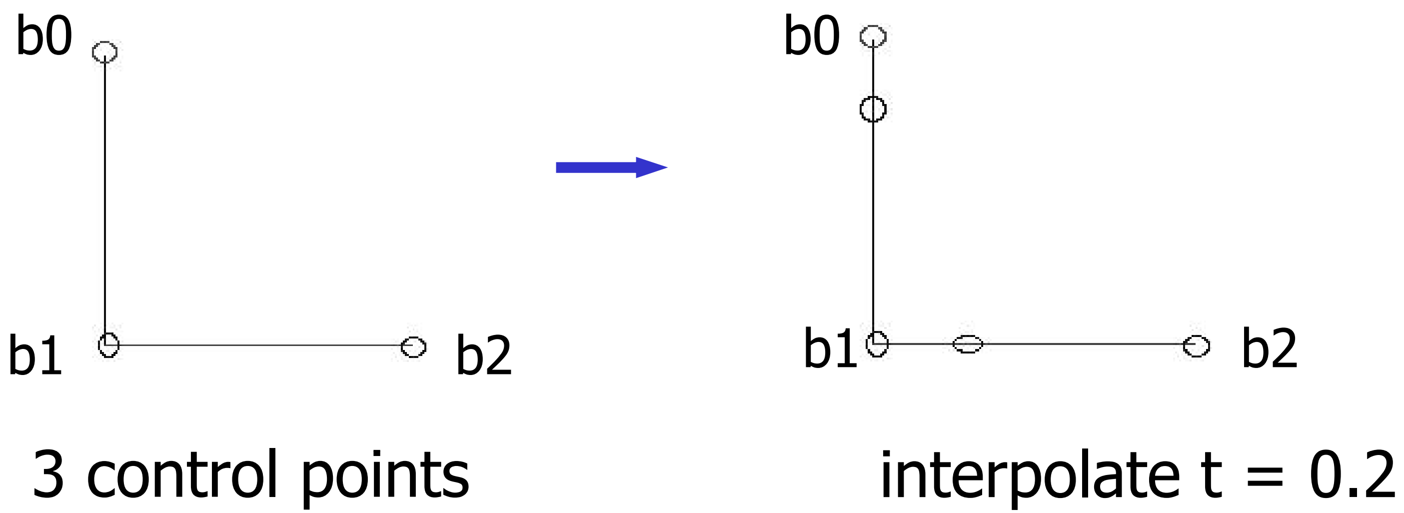
⟺ Only one control point *W_3* of the Bezier curve *$W(t)$* is really free.

Characteristics of Bezier Curves

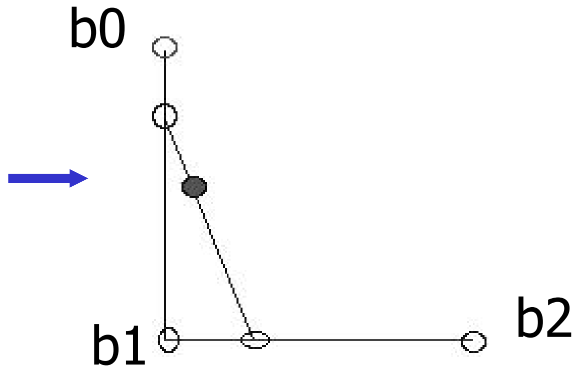
- Convex hull
- Affine invariance
- Variation diminishing
- The degree of the polynomial defining the curve segment is one less than the number of defining control points.
- In CAGD applications, a curve may have a so complicated shape that it cannot be represented by a single Bézier cubic curve
- Global control (disadv.) : change a control point affects the continuity of the curve.

The de Casteljau Algorithm

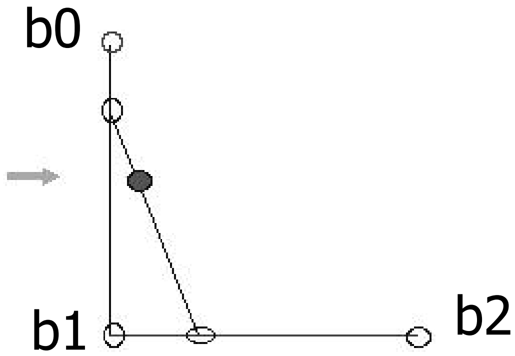
- Evaluation of the Bezier curve function
- Repeated linear interpolation
- Example of a quadratic (degree 2) Bezier curve



The de Casteljau Algorithm

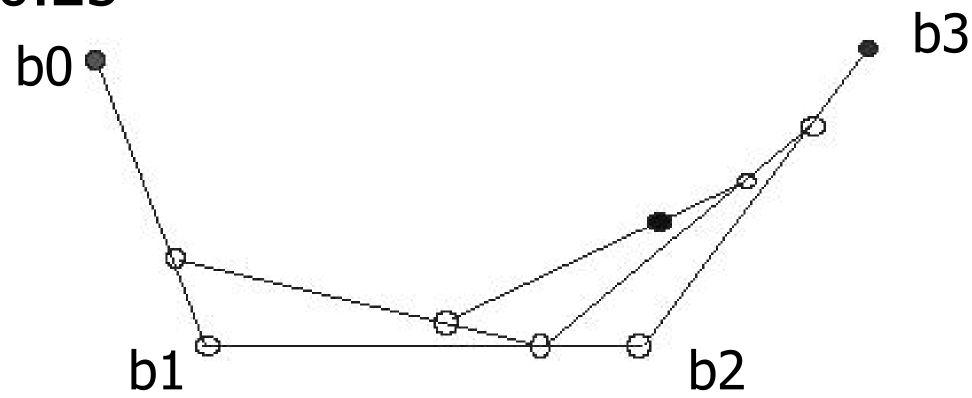


the point on the curve



repeating the procedure

Degree=3 and t=0.25

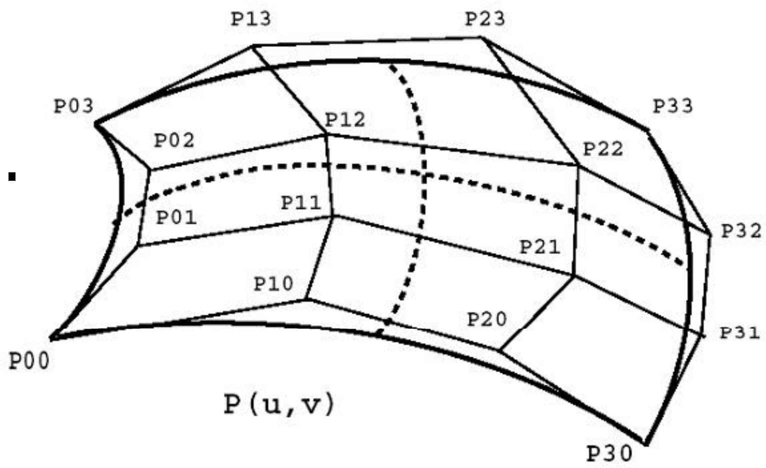


Parametric Surface

- Extend 2D parametric representation
  - increase the number of parameters from one to two, (s,t) in order to address each point in the 2D spaces.
  - express the 3D structure of the curved 2D surface by introducing a parameter z coordinate, z(s,t), *i.e.*, a patch

$$x = f_x(s,t), \quad y = f_y(s,t), \quad z = f_z(s,t).$$

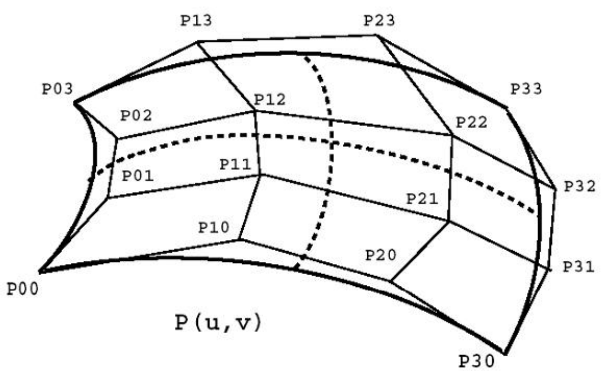
$$0 \le s,t \le 1$$



Bicubic Bezier Surface

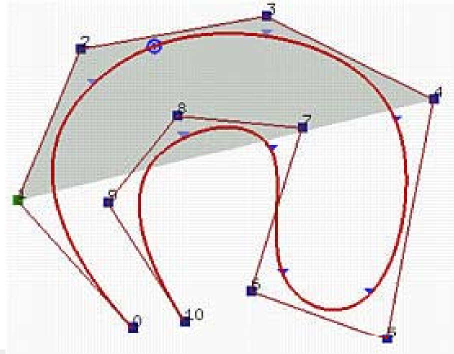
- Bezier patch:16 control points define one patch
- ease of interactivity & representation

$$\mathbf{P}(s,t) = \sum_{i=0}^{n} \binom{n}{i} (1-s)^{n-i} s^i \sum_{j=0}^{n} \binom{n}{j} (1-t)^{n-j} t^j \mathbf{P}_{i,j}$$



$$P(u,\ v) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} BPB^T \end{bmatrix} \begin{bmatrix} v^3 \\ v^2 \\ v \\ 1 \end{bmatrix}$$

where $B = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$ $\quad P = \begin{bmatrix} p_{00} & p_{01} & p_{02} & p_{03} \\ p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23} \\ p_{30} & p_{31} & p_{32} & p_{33} \end{bmatrix}$



B-Splines Curves

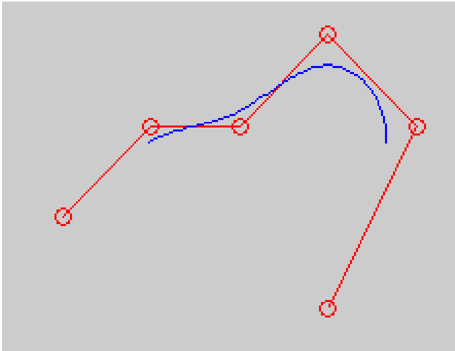
$$Q(u) = \sum_{k=0}^{n} P_k B_{k,d}(u)$$

P_k : an input set of *n+1* control points

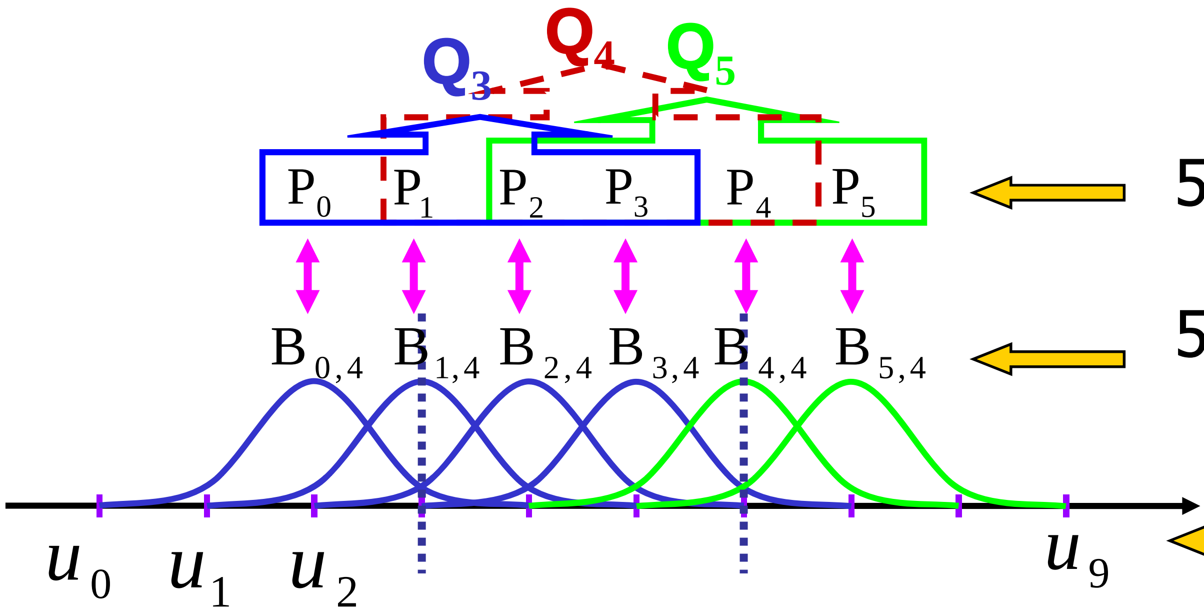
$B_{k,d}$: blending function of degree *d-1*

- The polynomial curve has degree *d-1* and C^{d-2} continuity over the range of u
- For $n+1$ control points, the curve is described with $n+1$ blending functions
- The range of u is divided into $n+d$ subintervals by the $n+d+1$ knot values

B-Splines Curves



A cubic b-spline which consists of three curve segments



n=5, d=4

5+1 control points

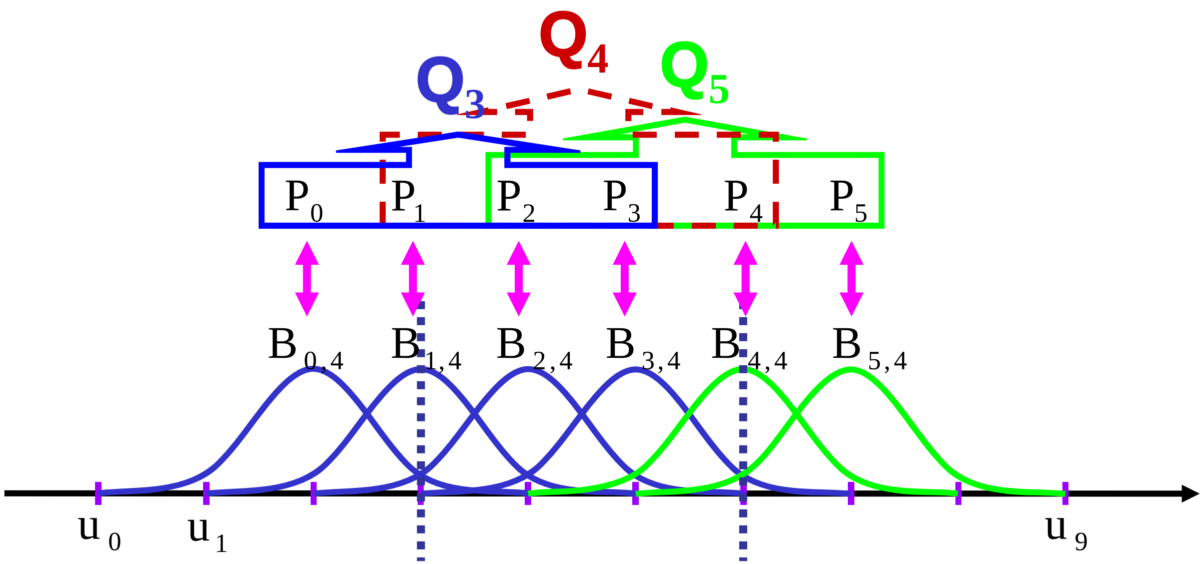
5+1 blending functions with degree 4-1

5+4+1 knots

Cubic B-Splines

- Each control point is associated with a unique blending function.

⇒ (*Local control*) Each control point affects the shape of a curve only over a range of a parameter values, *d* curve sections, where its associated basis function is nonzero.

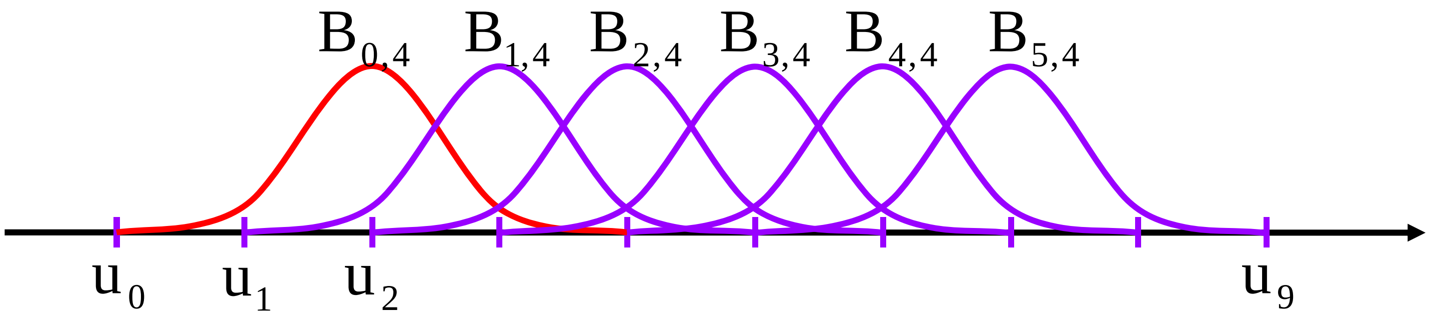


B-Splines Curves

- Knot vector : a set of subinterval endpoints in non-decreasing sequence

$$U = \{u_0, u_1, \ldots, u_{n+d}\}$$

☞ uniform, open uniform, nonuniform B-splines.



B-Splines Basis Functions

- <u>Cox-deBoor Algorithm</u>

:generate the basis functions recursively

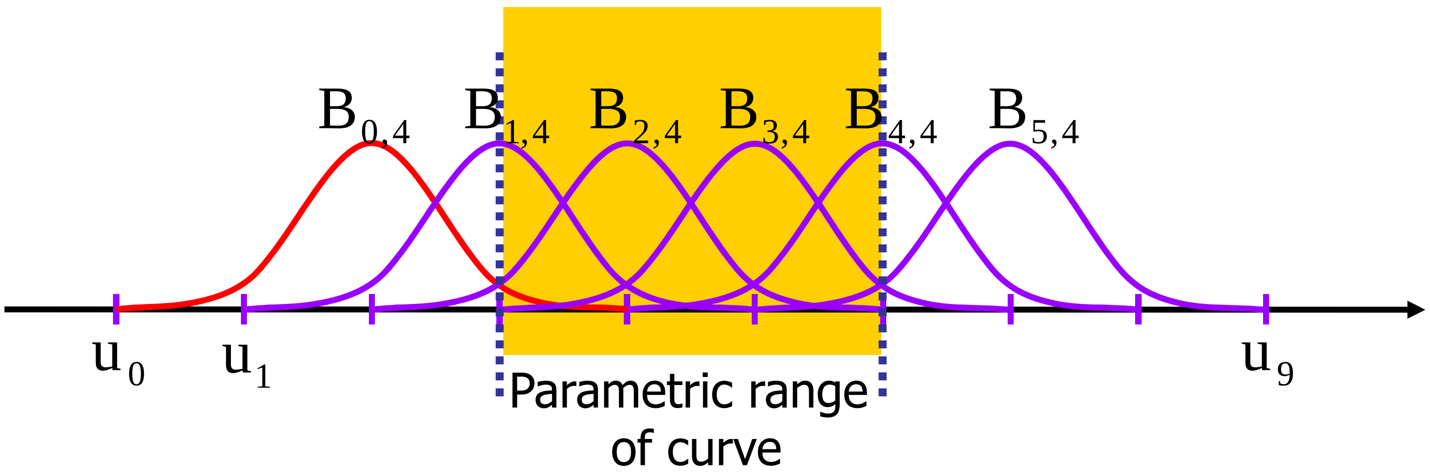
$$B_{k,1}(u) = 1, \quad \text{if} \quad u_k \le u \le u_{k+1}$$

$$0, \quad \text{otherwise}$$

$$B_{k,d}(u) = \frac{u - u_k}{u_{k+d+1} - u_k} B_{k,d-1}(u) + \frac{u_{k+d} - u}{u_{k+d} - u_{k+1}} B_{k+1,d-1}(u)$$

Uniform cubic B-spline basis functions

- Knots are spaced at equal intervals of parameter. e.g., {0,1,2,3,4,5,6,7,8,9}
- Bell-shaped basis function
- Each blending function $\mathbf{B_{k,4}}$ is defined over four subintervals starting at knot value u_k



Basis functions of Uniform Cubic B-splines

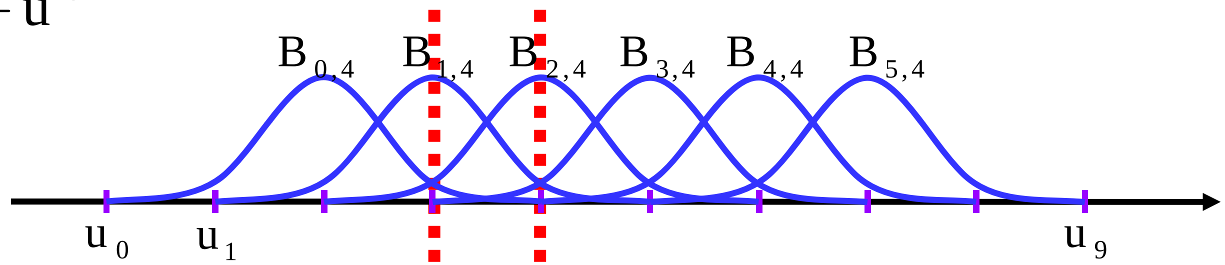
In $u_i \le u \le u_{i+1}$, we get basis functions by substituting $0 \le u \le 1$.

$$B_0(u) = \frac{1}{6}(1-u)^3$$

$$B_1(u) = \frac{1}{6}(3u^3 - 6u^2 + 4)$$

$$B_2(u) = \frac{1}{6}(-u^3 + 3u^2 + 3u + 1)$$

$$B_3(u) = \frac{1}{6}u^3$$



Uniform Cubic B-splines

- *ith* cubic segment

$$Q_i(u) = \sum_{k=0}^{3} p_{i-3+k} B_{i-3+k}(u)$$

k : local control point index

u : local control parameter, $0 \le u \le 1$

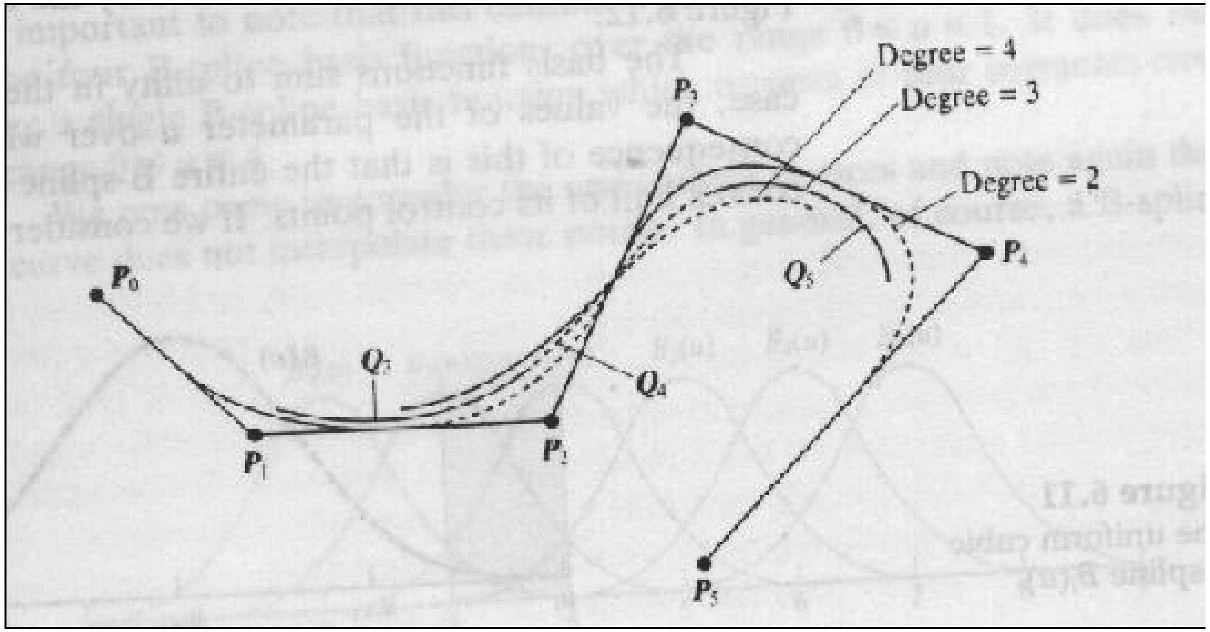
A cubic B-spline is a series of *m-2* curve segments, $Q_3, Q_4, \cdots, Q_m$, that approximate a series of *m+1* control points $P_0, P_1, \cdots, P_m, \quad m \ge 3$

Uniform Cubic B-splines

Q_3 is defined $P_0P_1P_2P_3$ which are scaled by $B_0B_1B_2B_3$

Q_4 is defined $P_1P_2P_3P_4$ which are scaled by $B_1B_2B_3B_4$

Q_5 is defined $P_2P_3P_4P_5$ which are scaled by $B_2B_3B_4B_5$



Uniform Quadratic B-splines

- Let *d=n=3*, we need *n+d+1=7* knot values: {0,1,2,3,4,5,6}.
- Get blending functions using Cox-deBoor Algorithm

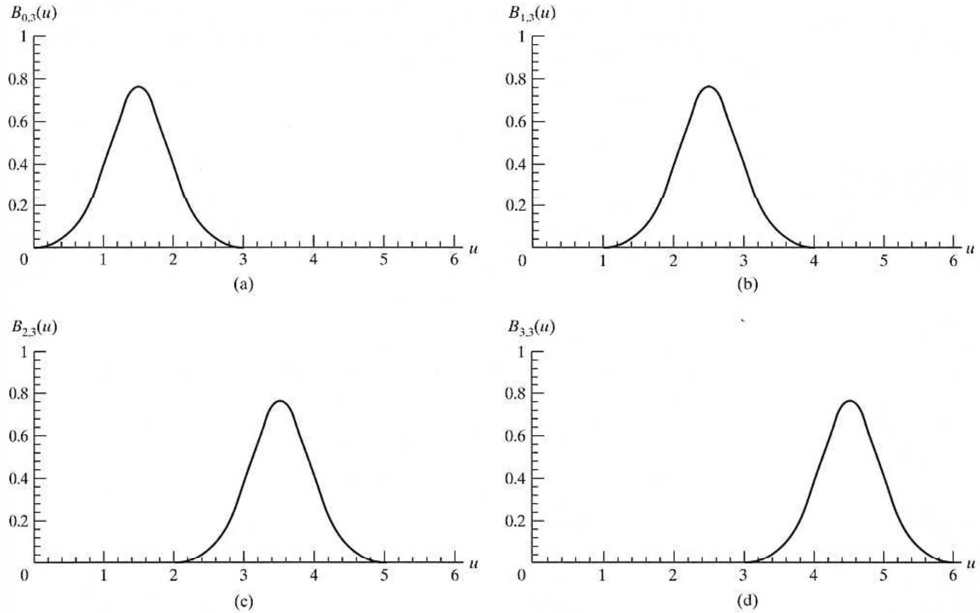


FIGURE 8-42 Periodic B-spline blending functions for $n = d = 3$ and a uniform, integer knot vector.

$$B_{0,3}(u) = \frac{u}{2} B_{0,2}(u) + \frac{3-u}{2} B_{1,2}(u)$$

Read text book!!

$$Q_i(u) = \sum_{k=0}^{2} p_{i-2+k} B_{i-2+k,3}(u)$$

Uniform B-splines(Example)

- The curve is defined from $u_{d-1}=2$ to $u_{n+1}=4$
- We can get starting and ending positions (boundary condition) of the curve:

$$Q_{begin} = \frac{1}{2}(p_0 + p_1), \quad Q_{end} = \frac{1}{2}(p_2 + p_3)$$

by applying $u=2$ and $u=4$ to the $Q(u)$.

In general, weighted average of $d-1$ control points.

- Derivatives at the starting and ending position

$$Q'_{begin} = p_1 - p_0, \quad Q'_{end} = p_3 - p_2$$

Uniform Cubic B-splines

- Using a general cubic polynomial expression and the following boundary conditions:

$$Q(0) = \frac{1}{6}(p_0 + 4p_1 + p_2) \qquad Q(1) = \frac{1}{6}(p_1 + 4p_2 + p_3)$$

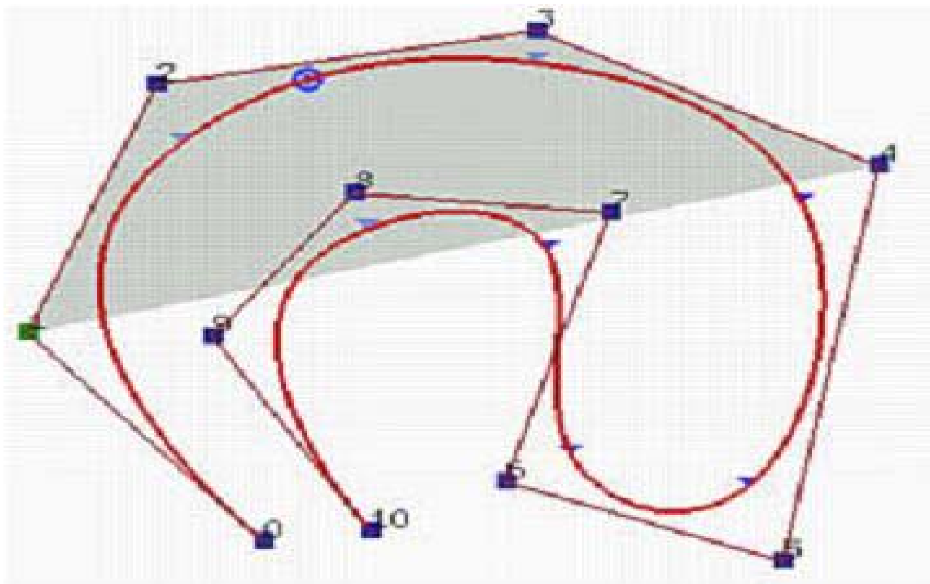
$$Q'(0) = \frac{1}{2}(p_2 - p_0) \qquad Q'(1) = \frac{1}{2}(p_3 - p_1)$$

→ We can get a matrix formulation:

$$Q_i(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Convex Hull Property of B-Splines Curves

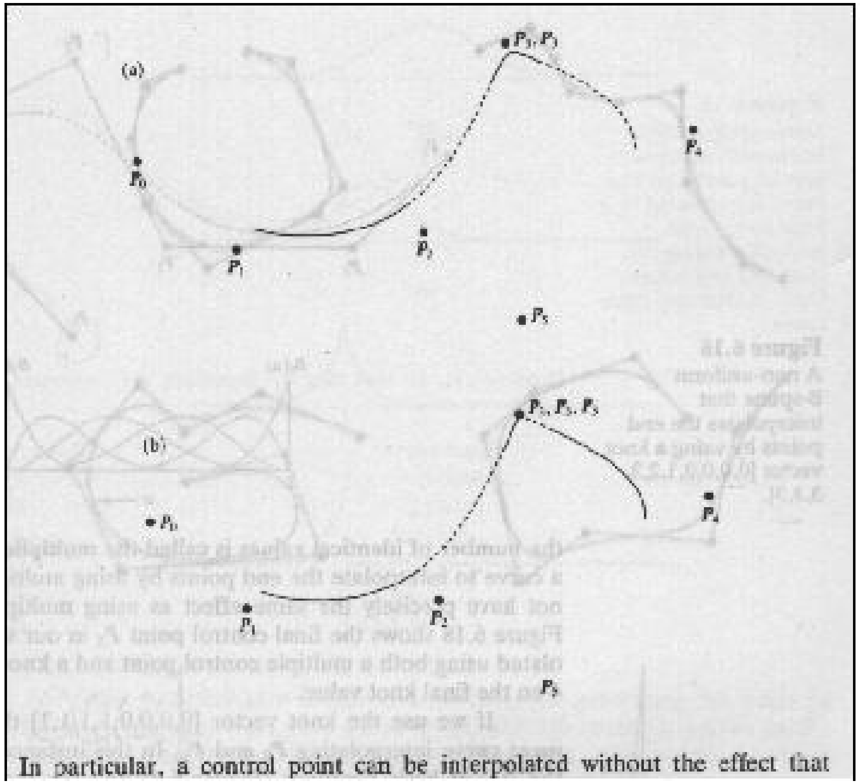
- B-spline curve of degree *d-1* must lie within the union of all such convex hulls formed by taking *d* successive defining polygon vertices.



Uniform Cubic B-splines

- The effect of multiple control points

⇒ interpolate control points but the loss of continuity.

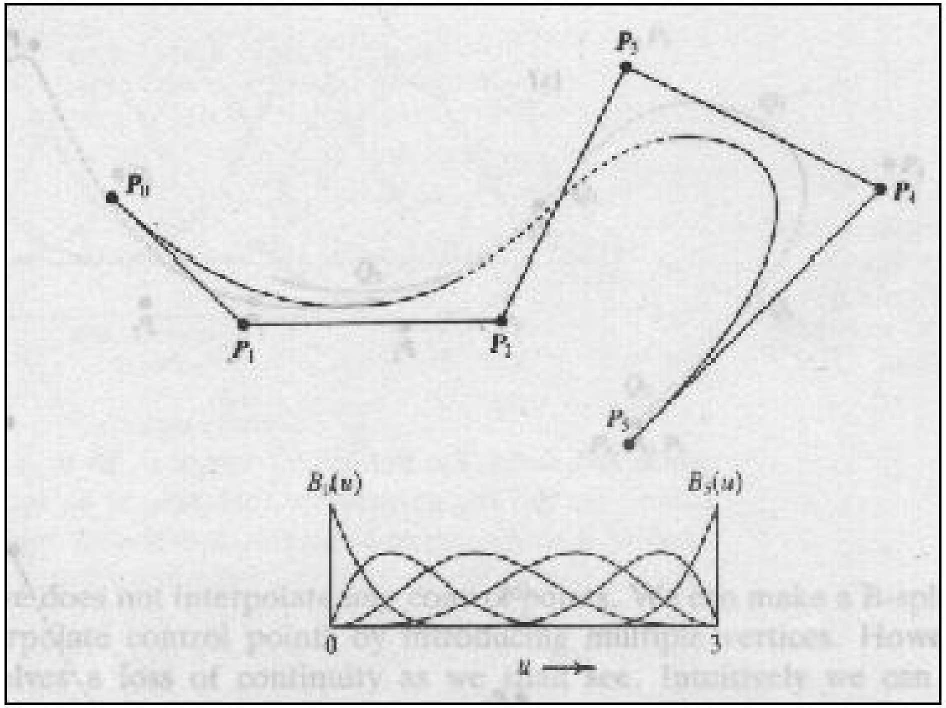


multiplicity

1	G_2	continuous
2	G_1	continuous
3	G_0	continuous

Non-uniform B-splines

- Non-uniform interval of knot values
- To permit the spline to interpolate control points by inserting multiple knots
- Knot vector is any non-decreasing sequence of knot values.



Non-uniform B-splines

- knot vector: [0,0,0,0,1,2,3,3,3,3]

  nine segment: $Q_0, Q_1, \cdots, Q_8$

  $Q_0, Q_1, Q_2, Q_6, Q_7,$ and Q_8 are reduced to a single point

  $Q_3, Q_4,$ and Q_5 are defined over the range $0 \le u \le 3$

- knot vector [0,0,0,0,1,1,1,1] ≡ Bezier curve

  $P_0, \cdots P_3$ control points

B-Spline Surfaces

- Given the following information:
  - a set of *m*+1 rows and *n*+1 column control points $p_{i,j}$, where 1<= i<= m, 1<=j<=n;
  - a knot vector of *h* + 1 knots in the *u*-direction,

    $$U = (u_0, u_1, u_2, \ldots, u_h)$$

  - a knot vector of *k* + 1 knots in the *v*-direction,

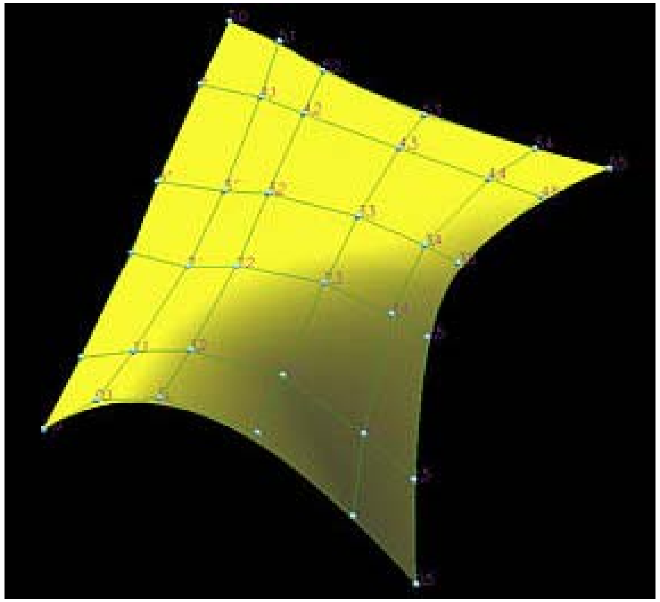
    $$V = (v_0, v_1, v_2, \ldots, v_k)$$

  - the degree *p* in the *u*-direction; and the degree *q* in the *v*-direction;

B-Spline Surfaces

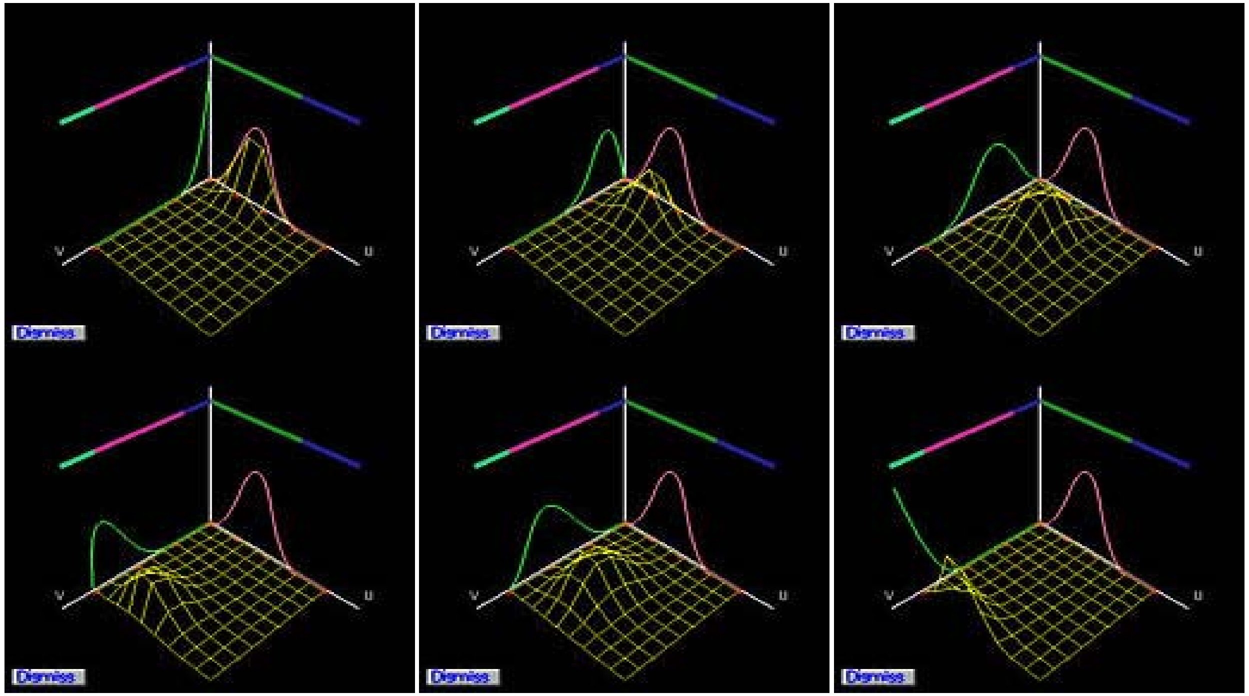
The B-spline surface defined by these information is the following:

$$Q(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{i,p}(u)\, B_{j,q}(v)\, p_{ij}$$



B-Spline Surfaces

The coefficient of control point $\mathbf{p}_{i,j}$ is the product of two one-dimensional B-spline basis functions, one in the *u*-direction, $B_{i,p}(u)$, and the other in the *v*-direction, $B_{j,q}(v)$. All of these products are two-dimensional B-spline functions. The following figures show the basis functions of control points $\mathbf{p}_{2,0}$, $\mathbf{p}_{2,1}$, $\mathbf{p}_{2,2}$, $\mathbf{p}_{2,3}$, $\mathbf{p}_{2,4}$ and $\mathbf{p}_{2,5}$



NURBS

- NURBS(non-uniform rational B-spline)
  - Adding some relative weight to the control point for extra control facility
  - Can represent more various curves such as circles and cylinders
  - More useful for interpolation
  - Invariant w.r.t a ***projective transformation***

NURBS

$$P_i^w = (w_i x_i, w_i y_i, w_i z_i, w_i)$$

$$P(u) = \frac{\sum_{i=o}^{n} p_i w_i B_{i,k}(u)}{\sum_{i=0}^{n} w_i B_{i,k}(u)}$$

$$= \sum_{i=0}^{n} p_i R_{i,k}(u)$$

$$\mathrm{R}_{\mathrm{i,k}}(u) = \frac{B_{i,k}(u) w_i}{\sum_{j=0}^{n} B_{j,k}(u) w_i}$$

w_i = weight

- $w_i = 1$ for all $i \Rightarrow R_{i,k}(u) = B_{i,k}(u)$
- extra shape parameter
- w_i increase $\Rightarrow$ curve is pulled toward control point P_i

Drawing Curves

- **Forward-differencing method** : to plot a curve or a surface, a polynomial must be evaluated at successive t values with fixed increments.

$$\text{For } P(t) = at^3 + bt^2 + ct + d, 0 \le t \le 1$$

$$P_i = P(i/n) = a(i/n)^3 + b(i/n)^2 + c(i/n) + d$$

$$P_{i+1} - P_i = a\{((i+1)/n)^3 - (i/n)^3\}$$

$$+ b\{((i+1)/n)^2 - (i/n)^2\} + c\{((i+1)/n) - (i/n)\}$$

$$\Delta_{1,i} = \frac{a}{n^3}(3i^2 + 3i + 1) + \frac{b}{n^2}(2i + 1) + \frac{c}{n}$$

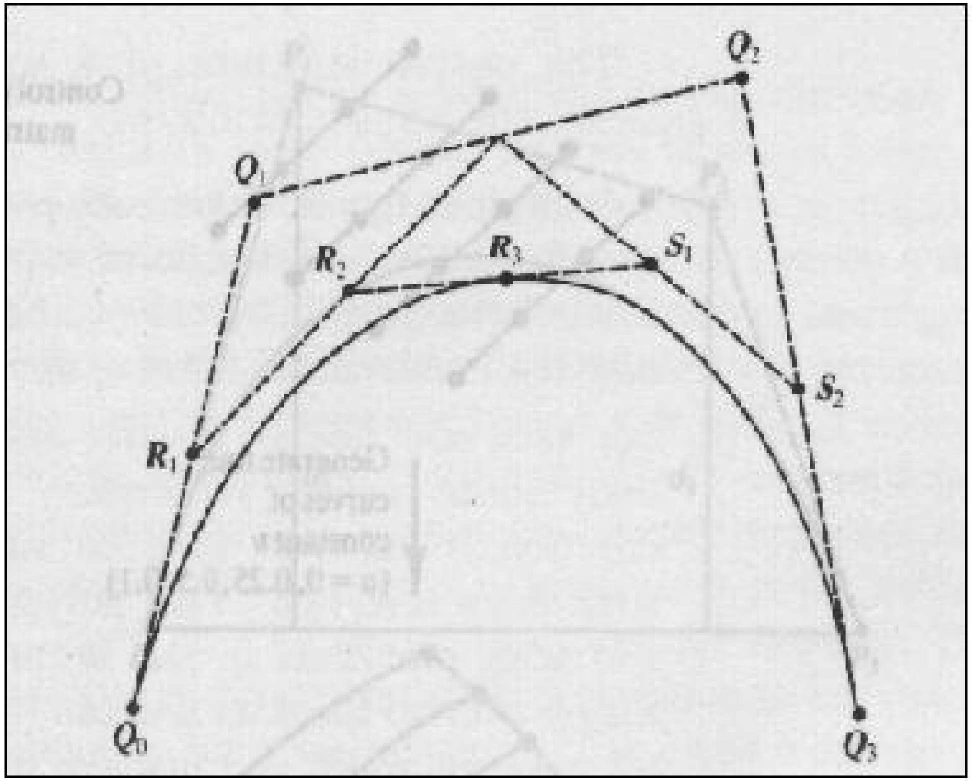
$$\Delta_{2,i} = \Delta_{1,i+1} - \Delta_{1,i} = 6(i+1)\frac{c}{n^3} + \frac{2b}{n^2}$$

$$\Delta_{3,i} = \Delta_{2,i+1} - \Delta_{2,i} = \frac{6a}{n^3}$$

Drawing Curves

- Recursive subdivision
  - stops when the control points get sufficiently close to the curve
  - need flatness test
  - Bezier curve - divide the control points

Drawing Bezier Curves



$$R_0 = Q_0$$
$$R_1 = (Q_0 + Q_1) / 2$$
$$R_2 = R_1 / 2 + (Q_1 + Q_2) / 4$$
$$R_3 = (R_2 + S_1) / 2$$

$$S_0 = R_3$$
$$S_1 = (Q_1 + Q_2) / 4 + S_2 / 2$$
$$S_2 = (Q_2 + Q_3) / 2$$
$$S_3 = Q_3$$

Comparison of Surface

Comparison of Four Different Forms of Parametric Cubic Curves

	Hermite	Bézier	Uniform B-Spline	Nonuniform B-spline
Convex hull defined by control points	N/A	Yes	Yes	Yes
Interpolates some control points	Yes	Yes	No	No
Interpolates all control points	Yes	No	No	No
Ease of subdivision	Good	Best	Average	High
Continuities inherent in representation	C^0 G^0	C^0 G^0	C^2 G^2	C^2 G^2
Continuities achieved easily	C^1 G^1	C^1 G^1	C^2 G^{2*}	C^2 G^{2*}
Number of parameters controlling a curve segment	4	4	4	5

*Except for special case discussed in Section 9.2.