Numerical Example: Critical Fluorescence Power of an Nd^{3+}: Glass Laser

The critical fluorescence power of an Nd^{3+}: glass laser is calculated using the following data:

$$l = 10 \text{ cm}$$

$$V = 10 \text{ cm}^3$$

$$\lambda = 1.06 \times 10^{-6} \text{ meter}$$

$$R = (\text{mirror reflectivity}) = 0.95$$

$$n \simeq 1.5$$

$$t_c \simeq \frac{nl}{(1-R)c} = 10^{-8} \text{ second}$$

$$\Delta\nu = 3 \times 10^{12} \text{ Hz}$$

The Nd^{3+}: glass is a four-level laser system (see Figure 7-11), since level 1 is about 2,000 cm^{-1} above the ground state so that at room temperature $E_1 \approx 10kT$. We can thus use (6.3-5), obtaining $N_t = 8.5 \times 10^{15}$ cm^{-3} and

$$P_s \simeq 150 \text{ watts}$$

6.4 POWER IN LASER OSCILLATORS

In Section 6.1 we derived an expression for the threshold population inversion N_t at which the laser gain becomes equal to the losses. We would expect that as the pumping intensity is increased beyond the point at which $N_2 - N_1 = N_t$ the laser will break into oscillation and emit power. In this section we obtain the expression relating the laser power output to the pumping intensity. We also treat the problem of optimum coupling—that is, of the mirror transmission that results in the maximum power output.

Rate Equations

Consider an ideal four-level laser such as the one shown in Figure 6-4. We take $E_1 \gg kT$ so that the thermal population of the lower laser level 1 can be neglected. We assume that the critical inversion density N_t is very small compared to the ground-state population, so during oscillation the latter is hardly affected. We can consequently characterize the pumping intensity by R_2 and R_1, the density of atoms pumped per second into levels 2 and 1, respectively. Process R_1, which populates the lower level 1, causes a reduction of the gain and is thus detrimental to the laser operation. In many laser systems, such as discharge gas lasers, considerable pumping into the lower laser level is unavoidable, and therefore a realistic analysis of such systems must take R_1 into consideration.

The rate equations that describe the populations of levels 1 and 2 become

$$\frac{dN_2}{dt} = -N_2\omega_{21} - W_i(N_2 - N_1) + R_2 \tag{6.4-1}$$

$$\frac{dN_1}{dt} = -N_1\omega_{10} + N_2\omega_{21} + W_i(N_2 - N_1) + R_1 \tag{6.4-2}$$

ω_{ij} is the decay rate per atom from level i to j; thus the density of atoms per second undergoing decay from i to j is $N_i\omega_{ij}$. If the decay rate is due entirely to spontaneous transitions, then ω_{ij} is equal to the Einstein A_{ij} coefficient introduced in Section 5.1. W_i is the probability per unit time that an atom in level 2 will undergo an *induced* (stimulated) transition to level 1 (or vice versa). W_i, given by (5.2-15), is proportional to the energy density of the radiation field inside the cavity.

Implied in the foregoing rate equations is the fact that we are dealing with a homogeneously broadened system. In an inhomogeneously broadened atomic transition, atoms with different transition frequencies $(E_2 - E_1)/h$ experience different induced transition rates and a single parameter W_i is not sufficient to characterize them.

In a steady-state situation we have $\dot{N}_1 = \dot{N}_2 = 0$. In this case we can solve (6.4-1) and (6.4-2) for N_1 and N_2, obtaining

$$N_2 - N_1 = \frac{R_2[1 - (\omega_{21}/\omega_{10})(1 + R_1/R_2)]}{W_i + \omega_{21}} \tag{6.4-3}$$

A necessary condition for population inversion in our model is thus $\omega_{21} < \omega_{10}$, which is equivalent to requiring that the lifetime of the upper laser level ω_{21}^{-1} exceed that of the lower one. The effectiveness of the pumping is, according to (6.4-3), reduced by the finite pumping rate R_1 and lifetime ω_{10}^{-1} of level 1 to an effective value

$$R = R_2\left[1 - \frac{\omega_{21}}{\omega_{10}}\left(1 + \frac{R_1}{R_2}\right)\right] \tag{6.4-4}$$

so (6.4-3) can be written as

$$N_2 - N_1 = \frac{R}{W_i + \omega_{21}} \tag{6.4-5}$$

Below the oscillation threshold the induced transition rate W_i is zero (since the oscillation energy density is zero) and $N_2 - N_1$ is, according to (6.4-5), proportional to the pumping rate R. This state of affairs continues until $R = N_t\omega_{21}$, at which point $N_2 - N_1$ reaches the threshold value [see (6.1-11)]

$$N_t = \frac{8\pi n^3\nu^2 t_{\text{spont}}}{c^3 t_c g(\nu_0)} = \frac{8\pi n^3 \nu^2 t_{\text{spont}}\Delta\nu}{c^3 t_c} \tag{6.4-6}$$

This is the point at which the gain at ν_0 due to the inversion is large enough to make up *exactly* for the cavity losses (the criterion that was used to derive N_t). Further increase of $N_2 - N_1$ with pumping is impossible in a *steady-state situation*, since it would result in a rate of induced (energy) emission that exceeds the losses so that the field energy stored in the resonator will increase with time in violation of the steady-state assumption.

This argument suggests that, under steady-state conditions, $N_2 - N_1$ must remain equal to N_t regardless of the amount by which the threshold pumping rate is exceeded. An examination of (6.4-5) shows that this is possible, provided W_i is allowed to increase once R exceeds its threshold value $\omega_{21}N_t$, so that the equality

$$N_t = \frac{R}{W_i + \omega_{21}} \tag{6.4-7}$$

is satisfied. Since, according to (5.2-15), W_i is proportional to the energy density in the resonator, (6.4-7) relates the electromagnetic energy stored in the resonator to the pumping rate R. To derive this relationship we first solve (6.4-7) for W_i, obtaining

$$W_i = \frac{R}{N_t} - \omega_{21} \qquad R \geqslant N_t\omega_{21} \tag{6.4-8}$$

The total power generated by stimulated emission is

$$P_e = (N_t V)W_i h\nu \tag{6.4-9}$$

where V is the volume of the oscillating mode. Using (6.4-8) in (6.4-9) gives

$$\frac{P_e}{Vh\nu} = N_t\omega_{21}\left(\frac{R}{N_t\omega_{21}} - 1\right) \qquad R \geqslant N_t\omega_{21} \tag{6.4-10}$$

This expression may be recast in a slightly different form, which we will find useful later on. We use expression (6.4-6) for N_t and, recalling that in our idealized model $\omega_{21}^{-1} = t_{\text{spont}}$, obtain

$$\frac{P_e}{Vh\nu} = N_t\omega_{21}\left(\frac{R}{p/t_c} - 1\right) \qquad R \geqslant \frac{p}{t_c} \tag{6.4-11}$$

where

$$p = \frac{8\pi n^3\nu^2}{c^3 g(\nu_0)} = \frac{8\pi n^3\nu^2\Delta\nu}{c^3} \tag{6.4-12}$$

According to (4.0-12), p corresponds to the density (meters^{-3}) of radiation modes whose resonance frequencies fall within the atomic transition linewidth $\Delta\nu$—that is, the density of radiation modes that are capable of interacting with the transition.

Returning to the expression for the power output of a laser oscillator (6.4-11), we find that the term $R/(p/t_c)$ is the factor by which the pumping rate R exceeds its threshold value p/t_c. In addition, in an ideal laser system, $\omega_{21} = t_{\text{spont}}^{-1}$, so we can identify $N_t\omega_{21}h\nu V$ with the power P_s going into spontaneous emission at threshold, which is defined by (6.3-5). We can consequently rewrite (6.4-11) as

$$P_e = P_s\left(\frac{R}{R_t} - 1\right) \tag{6.4-13}$$

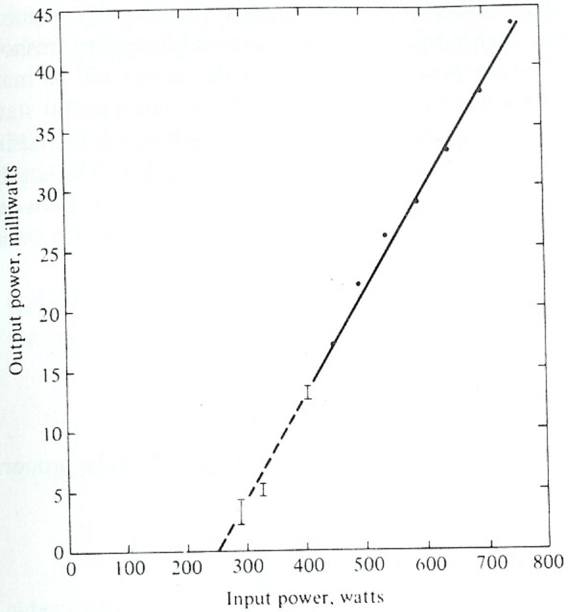

Figure 6-6 Plot of output power versus electric power input to a xenon lamp in a CW $CaF_2:U^{3+}$ laser. Mirror transmittance at 2.61 μm is 0.2 percent, $T = 77$ K. (After Reference [2].)

The main attraction of (6.4-13) is in the fact that, in addition to providing an extremely simple expression for the power emitted by the laser atoms, it shows that for each increment of pumping, measured relative to the threshold value, the power increases by P_s. An experimental plot showing the linear relation predicted by (6.4-13) is shown in Figure 6-6.

In the numerical example of Section 6.3, which was based on an Nd^{3+}:glass laser, we obtained $P_s = 150$ watts. We may expect on this basis that the power from this laser for, say $(R/R_t) \simeq 2$ (that is, twice above threshold) will be of the order of 150 watts.

6.5 OPTIMUM OUTPUT COUPLING IN LASER OSCILLATORS

The total loss encountered by the oscillating laser mode can conveniently be attributed to two different sources: (a) the inevitable residual loss due to absorption and scattering in the laser material and in the mirrors, as well as diffraction losses in the finite diameter reflectors; (b) the (useful) loss due to coupling of output power through the partially transmissive reflector. It is obvious that loss (a) should be made as small as possible since it raises the oscillation threshold without contributing to the output power. The problem of the coupling loss (b), however, is more subtle. At zero coupling (that is, both mirrors have zero transmission) the threshold will be at its minimum value and the power P_e emitted by the atoms will be maximum. But since none of this power is available as output, this is not a useful state of affairs. If, on the other hand, we keep increasing the coupling loss, the increasing threshold pumping will at some point exceed the actual pumping level. When this happens, oscillation will cease and the power output will again be zero. Between these two extremes there exists an optimum value of coupling (that is, mirror transmission) at which the power output is a maximum.

The expression for the population inversion was shown in (6.4-5) to have the form

$$N_2 - N_1 = \frac{R/\omega_{21}}{1 + W_i/\omega_{21}} \tag{6.5-1}$$

Since the exponential gain constant $\gamma(\nu)$ is, according to (5.3-3), proportional to $N_2 - N_1$, we can use (6.5-1) to write it as

$$\gamma = \frac{\gamma_0}{1 + W_i/\omega_{21}} \tag{6.5-2}$$

where γ_0 is the unsaturated ($W_i = 0$) gain constant (that is, the gain exercised by a very weak field, so that $W_i \ll \omega_{21}$). We can use (6.4-9) to express W_i in (6.5-2) in terms of the total emitted power P_e and then, in the resulting expression, replace $N_t V h\nu\omega_{21}$ by P_s. The result is

$$\gamma = \frac{\gamma_0}{1 + P_e/P_s} \tag{6.5-3}$$

where P_s, the saturation power, is given by (6.3-4). The oscillation condition (6.1-6) can be written as

$$e^{\gamma_t l}(1 - L) \doteq 1 \tag{6.5-4}$$

where $L = 1 - r_1 r_2 \exp(-\alpha l)$ is the fraction of the intensity lost per pass. In the case of small losses ($L \ll 1$), (6.5-4) can be written as

$$\gamma_t l = L \tag{6.5-5}$$

According to the discussion in the introduction to this chapter, once the oscillation threshold is exceeded, the actual gain γ exercised by the laser oscillation is clamped at the threshold value γ_t regardless of the pumping. We can thus replace γ by γ_t in (6.5-3) and, solving for P_e, obtain

$$P_e = P_s \left(\frac{g_0}{L} - 1\right) \tag{6.5-6}$$

where $g_0 = \gamma_0 l$ (that is, the unsaturated gain per pass in nepers). P_e, we recall, is the *total* power given off by the atoms due to stimulated emission. The total loss per pass L can be expressed as the sum of the residual (unavoidable) loss L_i and the useful mirror transmission[6] T, so

$$L = L_i + T \tag{6.5-7}$$

The fraction of the total power P_e that is coupled out of the laser as useful output is thus $T/(T + L_i)$. Therefore, using (6.5-6) we can write the (useful) power output as

$$P_o = P_s \left(\frac{g_0}{L_i + T} - 1\right) \frac{T}{T + L_i} \tag{6.5-8}$$

Replacing P_s in (6.5-8) by the right side of (6.3-5), and recalling from (4.7-2) that for small losses

$$t_c = \frac{nl}{(L_i + T)c} = \frac{nl}{Lc} \tag{6.5-9}$$

Equation (6.5-8) becomes

$$P_o = \frac{8\pi n^2 h\nu\Delta\nu A}{\lambda^2(t_2/t_{\text{spont}})} T \left(\frac{g_0}{L_i + T} - 1\right) = I_s A T \left(\frac{g_0}{L_i + T} - 1\right) \tag{6.5-10}$$

where $A = V/l$ is the cross-sectional area of the mode (assumed constant) and I_s is the saturation intensity as given in (5.6-9). Maximizing P_0 with respect to T by setting $\partial P_0/\partial T = 0$ yields

$$T_{\text{opt}} = -L_i + \sqrt{g_0 L_i} \tag{6.5-11}$$

as the condition for the mirror transmission that yields the maximum power output.

The expression for the power output at optimum coupling is obtained by substituting (6.5-11) for T in (6.5-10). The result, using (5.6-9), is

$$\begin{aligned}(P_o)_{\text{opt}} &= \frac{8\pi n^2 h\nu\Delta\nu A}{(t_2/t_{\text{spont}})\lambda^2}(\sqrt{g_0} - \sqrt{L_i})^2 = I_s A(\sqrt{g_0} - \sqrt{L_i})^2 \\ &\equiv S(\sqrt{g_0} - \sqrt{L_i})^2\end{aligned} \tag{6.5-12}$$

where the parameter $S = I_s A$ is defined by (6.5-12) and is independent of the excitation level (pumping) or losses.

Theoretical plots of (6.5-10) with L_i as a parameter are shown in Figure 6-7. Also shown are experimental data points obtained in a He–Ne 6328-Å laser. Note that the value of g_0 is given by the intercept of the $L_i = 0$ curve and is equal to 12 percent. The existence of an optimum coupling resulting in a maximum power output for each L_i is evident.

It is instructive to consider what happens to the energy $\mathscr{E}$ stored in the laser resonator as the coupling T is varied. A little thinking will convince us that $\mathscr{E}$ is

[6] For the sake of simplicity we can imagine one mirror as being perfectly reflecting, whereas the second (output) mirror has a transmittance T.

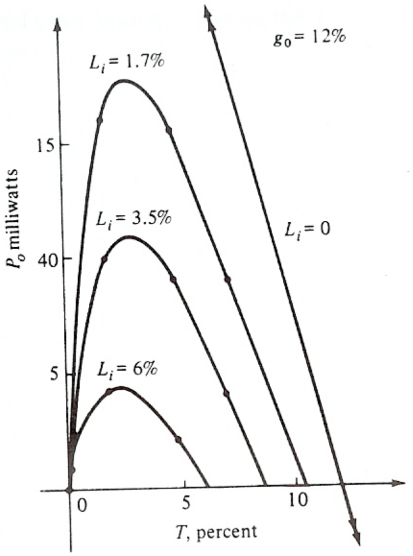

Figure 6-7 Useful power output (P_o) versus mirror transmission T for various values of internal loss L_i in an He–Ne 6328 Å laser. (After Laures, *Phys. Lett.* 10:61, 1964.)

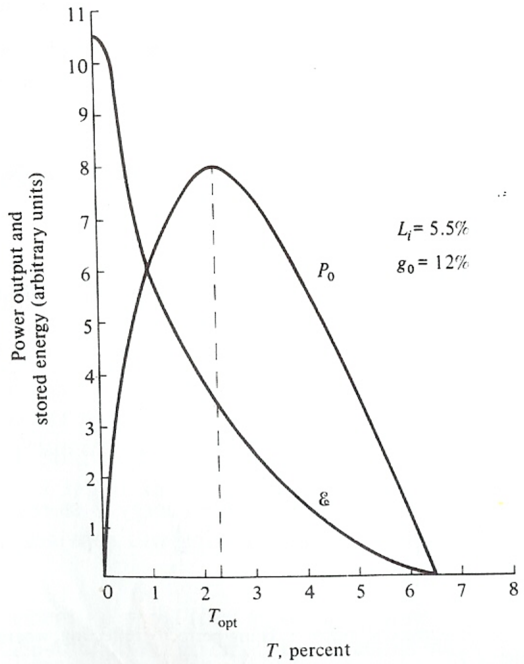

Figure 6-8 Power output P_o and stored energy $\mathscr{E}$ plotted against mirror transmission T.

proportional to P_o/T.[7] A plot of P_o (taken from Figure 6-7) and $\mathscr{E} \propto P_o/T$ as a function of the coupling T is shown in Figure 6-8. As we may expect, $\mathscr{E}$ is a monotonically decreasing function of T.

6.6 MULTIMODE LASER OSCILLATION AND MODE LOCKING

In this section we contemplate the effect of homogeneous or inhomogeneous broadening (in the sense described in Section 5.1) on the laser oscillation.

We start by reminding ourselves of some basic results pertinent to this discussion:

1. The actual gain constant prevailing inside a laser oscillator *at the oscillation frequency* ν is clamped, at steady state, at a value that is equal to the losses

$$\gamma_t(\nu) = \alpha - \frac{1}{l} \ln r_1 r_2 \qquad (6.1\text{-}8)$$

   where l is the length of the gain medium as well as the distance between the mirrors which are taken here to be the same.

2. The gain constant of a distributed laser medium is given, according to (5.3-3), by

$$\gamma(\nu) = (N_2 - N_1) \frac{c^2}{8\pi n^2 \nu^2 t_{spont}} g(\nu)$$

3. The optical resonator can support oscillations, provided sufficient gain is present to overcome losses, at frequencies[8] ν_q separated according to (4.6-3) by

$$\nu_{q+1} - \nu_q = \frac{c}{2nl}$$

Now consider what happens to the gain constant $\gamma(\nu)$ inside a laser oscillator as the pumping is increased from some value below threshold. Operationally, we can imagine an extremely weak wave of frequency ν launched into the laser medium and then measuring the gain constant $\gamma(\nu)$ as "seen" by this signal as ν is varied.

We treat first the case of a homogeneous laser. Below threshold the inversion $N_2 - N_1$ is proportional to the pumping rate and $\gamma(\nu)$, which is given by (5.3-3), is proportional to $g(\nu)$. This situation is illustrated by curve *A* in Figure 6-9(a). The spectrum (4.6-3) of the passive resonances is shown in Figure 6-9(b). As the pumping rate is increased, the point is reached at which the gain per pass at the center resonance frequency ν_0 is equal to the average loss per pass. This is shown in curve *B*. At this point, oscillation at ν_0 starts. An increase in the pumping cannot increase the inversion since this will cause $\gamma(\nu_0)$ to increase beyond its clamped value as given by Equation (6.1-8). Since the spectral lineshape function $g(\nu)$ describes the response

[7] The internal one-way power P_i incident on the mirrors is related, by definition, to P_o by $P_o = P_i T$. The total energy $\mathscr{E}$ is proportional to P_i.

[8] The high-order transverse modes discussed in Section 4.5 are ignored here.