Chapter2. Vector analysis

2-1 Introduction

1. vector algebra
   - addition, subtraction and multiplication

2. orthogonal coordinate systems

3. vector calculation
   - differentiation, integration (line, surface and volume integrals)
   - del(∇) operator, gradient, divergence and curl operation

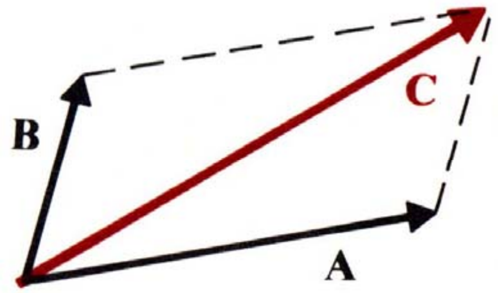
2-2 Vector Addition and Subtraction

1. Addition

- $\vec{A} = A\vec{a}$, $A = |\vec{A}|$, $\vec{a} = \dfrac{\vec{A}}{A}$
- $\vec{A} + \vec{B} = \vec{C}$

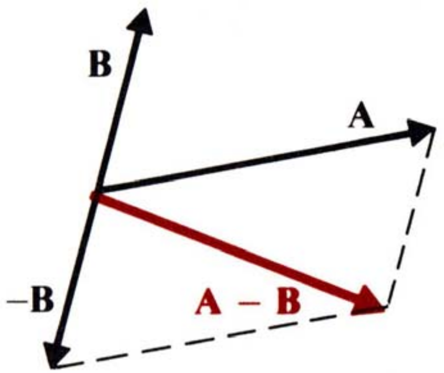
   ① Commutative law : $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

   ② Associative law : $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$



2. Subtraction

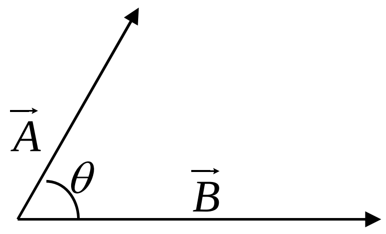
- $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$



2-3 Products of Vectors

1. Scalar or dot product

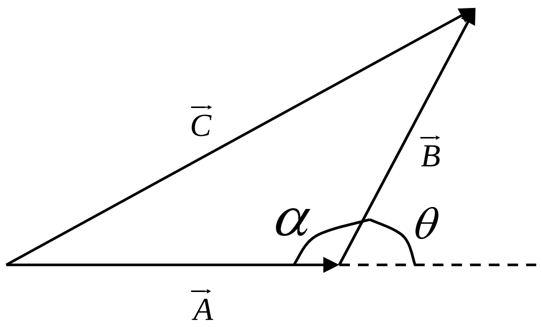
- $\vec{A} \bullet \vec{B} = AB\cos\theta$



① Commutative law : $\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$

② Distributive law : $\vec{A} \bullet (\vec{B} + \vec{C}) = \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C}$

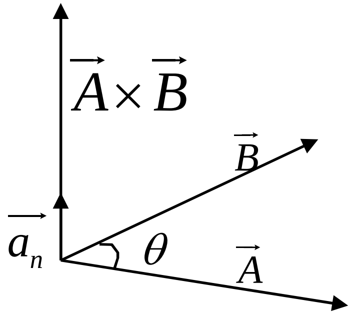
- $\vec{A} \bullet \vec{A} = A^2$ or $A = \sqrt{\vec{A} \bullet \vec{A}}$



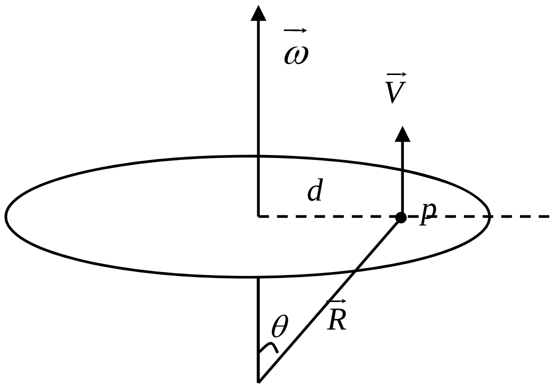
$$\left|\vec{C}\right| = \sqrt{\vec{C} \bullet \vec{C}} = \sqrt{(\vec{A} + \vec{B}) \bullet (\vec{A} + \vec{B})}$$
$$= \sqrt{A^2 + B^2 + 2AB\cos\theta}$$
$$= \sqrt{A^2 + B^2 - 2AB\cos\alpha}$$

2. Vector or cross product

- $\vec{A} \times \vec{B} = \widehat{a_n} AB \sin\theta$



☞ $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$



☞ $|\vec{V}| = \omega d = \omega R \sin\theta$

☞ $|\vec{V}| = \vec{\omega} \times \vec{R}$

3. Product of three vectors

- $\vec{A}\bullet(\vec{B}\times\vec{C}) = \vec{B}\bullet(\vec{C}\times\vec{A}) = \vec{C}\bullet(\vec{A}\times\vec{B})$

- $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\bullet\vec{C}) - \vec{C}(\vec{A}\bullet\vec{B})$

2.4 Orthogonal Coordinate Systems

- ✓ Base vector $(\widehat{u_1},\ \widehat{u_2},\ \widehat{u_3})$
  - $\widehat{u_1}\times\widehat{u_2}=\widehat{u_3},\ \widehat{u_2}\times\widehat{u_3}=\widehat{u_1},\ \widehat{u_3}\times\widehat{u_1}=\widehat{u_2}$
  - $\widehat{u_1}\bullet\widehat{u_2}=\widehat{u_2}\bullet\widehat{u_3}=\widehat{u_3}\bullet\widehat{u_1}=0$
  - $\widehat{u_1}\bullet\widehat{u_1}=\widehat{u_2}\bullet\widehat{u_2}=\widehat{u_3}\bullet\widehat{u_3}=1$

- ✓ $\vec{A}=A_{u_1}\widehat{u_1}+A_{u_2}\widehat{u_2}+A_{u_3}\widehat{u_3},\ \vec{B}=B_{u_1}\widehat{u_1}+B_{u_2}\widehat{u_2}+B_{u_3}\widehat{u_3}$

$u_i(i=1,2,3)$: const does not mean that the surface in plane. It can be curved surfaces

e.g) spherical coordinate system(r,θ,ϕ)

r : const means the surface of a sphere

(it is not plane, but $\hat{r},\ \hat{\theta},\ \hat{\phi}$ are perpendicular to each other)

- ✓ $\vec{A}\bullet\vec{B} = A_{u_1}B_{u_1} + A_{u_2}B_{u_2} + A_{u_3}B_{u_3}$

- ✓ $\vec{A}\times\vec{B} = \begin{vmatrix} \widehat{u}_1 & \widehat{u}_2 & \widehat{u}_3 \\ A_{u_1} & A_{u_2} & A_{u_3} \\ B_{u_1} & B_{u_2} & B_{u_3} \end{vmatrix}$

$$= \widehat{u}_1(A_{u_2}B_{u_3} - A_{u_3}B_{u_2}) - \widehat{u}_2(A_{u_1}B_{u_3} - A_{u_3}B_{u_1}) + \widehat{u}_3(A_{u_1}B_{u_2} - A_{u_2}B_{u_1})$$

1. Differential change in one of the coordinates 와 Diffetential length change의 상관 관계

- $dl_i = h_i du_i$, (h_i : metric coefficient, may be a function of u_1, u_2, u_3)

  e.g) polar coordinate(the subset of cylindrical coordinate)

  $(u_1,u_2)=(r,\phi)$ $d\phi$ in ϕ $\rightarrow$ $dl_2 = r d\phi$ in $\vec{\phi}$ direction

2. 임의의 $\vec{l}$ 방향의 differential length change : vector

- $\vec{dl} = dl\hat{l} = \widehat{u_1} dl_1 + \widehat{u_2} dl_2 + \widehat{u_3} dl_3 = \widehat{u_1}(h_1 du_1) + \widehat{u_2}(h_2 du_2) + \widehat{u_3}(h_3 du_3)$

3. Differential volume change : scalar

- $dv = h_1 h_2 h_3 du_1 du_2 du_3$

4. Differential area change : vector

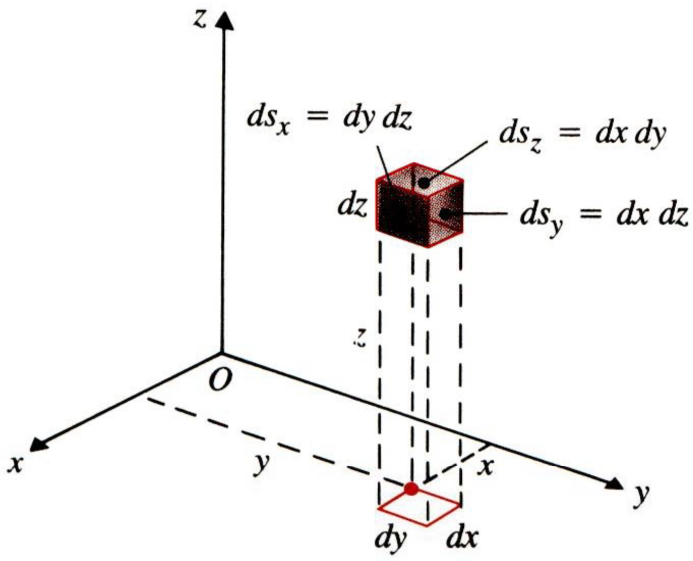
- $\vec{ds} = ds\hat{n}$ (right hand rule)

  cf) Differential area ds_1 normal to the unit vector $\vec{u_1}$

  $ds_1 = dl_2 dl_3 = h_2 h_3 du_2 du_3$

2-4.1 Cartesian Coordinates

✓ $(u_1, u_2, u_3) = (x, y, z)$ → distance element. so $h_1 = h_2 = h_3 = 1$



① $\hat{u_1} = \hat{x},\ \hat{u_2} = \hat{y},\ \hat{u_3} = \hat{z}$

$\hat{x} \times \hat{y} = \hat{z}$

② position vector to the point (x, y, z)

$\overrightarrow{OP} = x_1\hat{x} + y_1\hat{y} + z_1\hat{z}$

$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y + A_z B_z$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

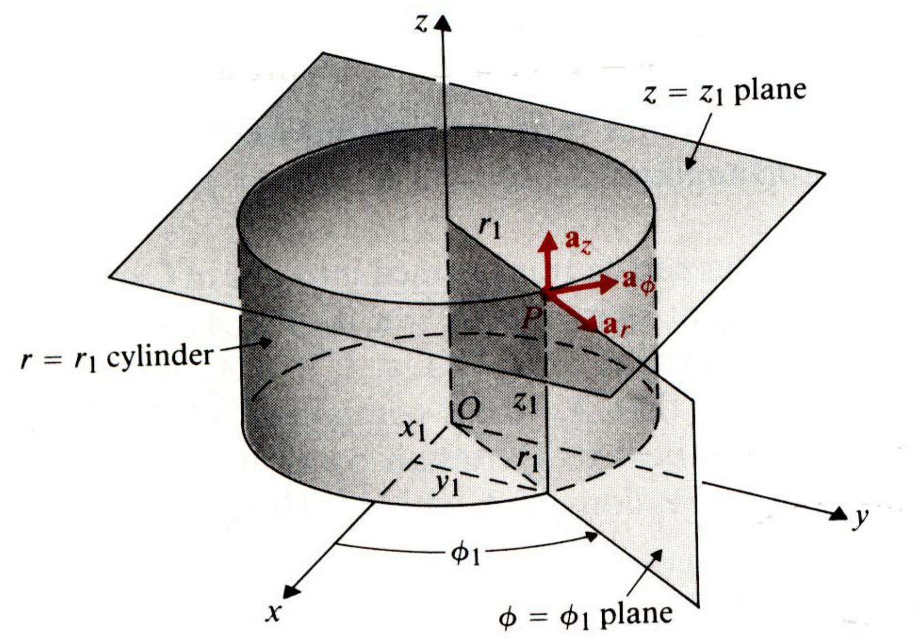
③ $\vec{dl} = \hat{x}dx + \hat{y}dy + \hat{z}dz$

$ds_x = dydz,\ ds_y = dxdz,\ ds_z = dxdy$

$dv = dxdydz$

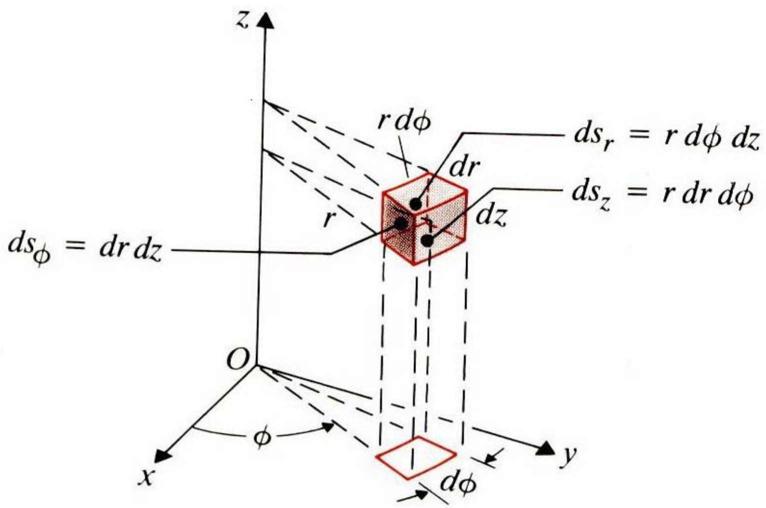
2-4.2 Cylindrical Coordinates

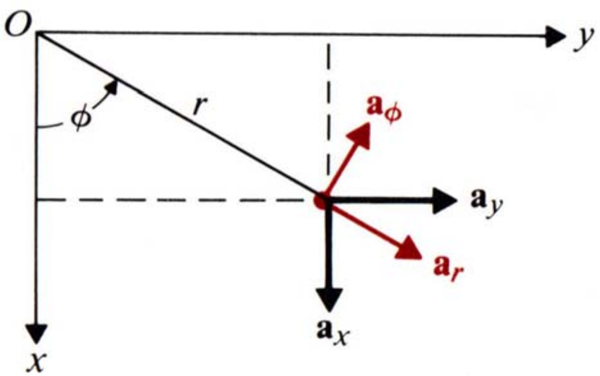
✓ $(u_1, u_2, u_3) = (r, \phi, z)$ → $h_1 = h_3 = 1,\ h_2 = r$



① $\hat{r} \times \hat{\phi} = \hat{z},\ \hat{\phi} \times \hat{z} = \hat{r},\ \hat{z} \times \hat{r} = \hat{\phi}$

② $\vec{A} = \hat{r} A_r + \hat{\phi} A_\phi + \hat{z} A_z$





③ $\vec{dl} = \hat{r}dr + \hat{\phi}rd\phi + \hat{z}dz$

$ds_r = rd\phi dz,\ ds_\phi = drdz,\ ds_z = rdrd\phi$

$dv = rdrd\phi dz$

$cf)\ \vec{A} = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z \Rightarrow \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$

$A_x = \vec{A}\bullet\hat{x} = A_r\hat{r}\bullet\hat{x} + A_\phi\hat{\phi}\bullet\hat{x} = A_r\cos\phi - A_\phi\sin\phi$

$A_y = \vec{A}\bullet\hat{y} = A_r\hat{r}\bullet\hat{y} + A_\phi\hat{\phi}\bullet\hat{y} = A_r\sin\phi + A_\phi\cos\phi$

$$\Rightarrow \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

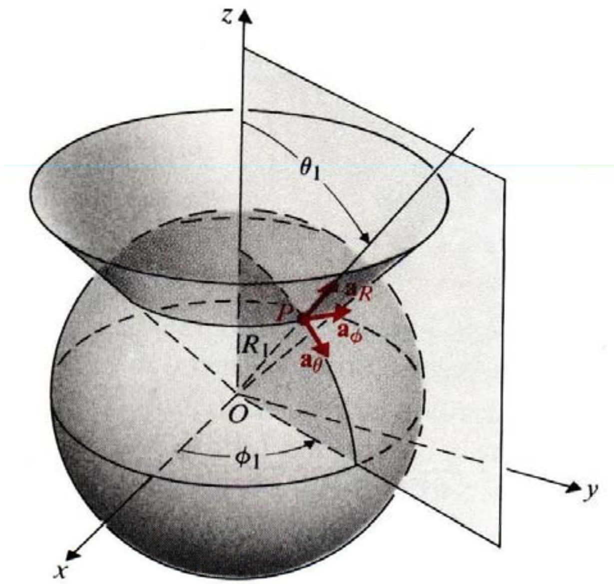
대입

$x = r\cos\phi,\ y = r\sin\phi,\ z = z$

$r = \sqrt{x^2 + y^2},\ \phi = \tan^{-1}(\frac{y}{x}),\ z = z$

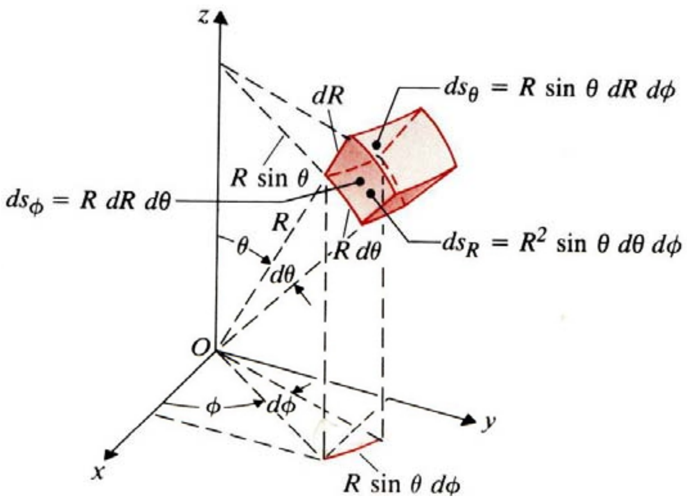
2-4.3 Spherical Coordinate

✓ $(u_1, u_2, u_3) = (R, \theta, \phi)$ → $h_1 = 1,\ h_2 = R,\ h_3 = R\sin\theta$



① $\hat{R} \times \hat{\theta} = \hat{\phi},\ \hat{\theta} \times \hat{\phi} = \hat{R},\ \hat{\phi} \times \hat{R} = \hat{\theta}$

② $\vec{A} = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi$



③ $\vec{dl} = \hat{R}dR + \hat{\theta}Rd\theta + \hat{\phi}R\sin\theta d\theta$

$ds_R = R^2\sin\theta d\theta d\phi,\ ds_\theta = R\sin\theta dRd\phi,\ ds_\phi = RdRd\theta$

$dv = R^2\sin\theta dRd\theta d\phi$

cf) $x = R\sin\theta\cos\phi,\quad y = R\sin\theta\sin\phi,\quad z = R\cos\theta$

$$R = \sqrt{x^2+y^2+z^2},\quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z},\quad \phi = \tan^{-1}\frac{y}{x}$$

H.W1 : Find the conversion matrix from spherical into cartesian

TABLE 2–1
Three Basic Orthogonal Coordinate Systems

Coordinate System Relations		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
Base vectors	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
Metric coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R\sin\theta$
Differential volume	dv	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

2-5 Integrals Containing Vector Functions

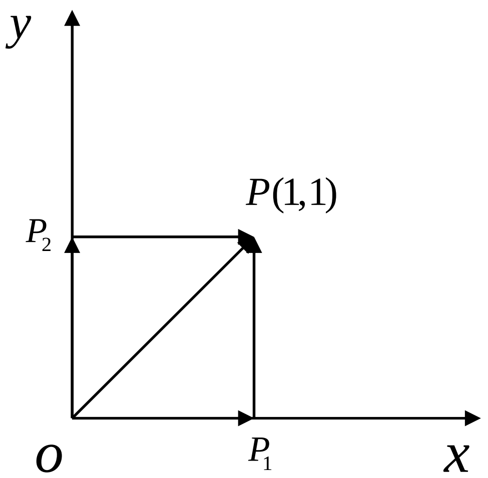
✓ $\int_v \vec{F}dv$, $\int_c v\vec{dl}$, $\int_c \vec{F}\bullet\vec{dl}$, $\int_s \vec{A}\bullet\vec{ds}$

1. $\int_v \vec{F}dv = \int_z\int_y\int_x \vec{F}dxdydz$ (for cartesian)

2. $\int_c v\vec{dl} = \int_c v(x,y,z)[\hat{x}dx + \hat{y}dy + \hat{z}dz] = \hat{x}\int_c v(x,y,z)dx + \hat{y}\int_c v(x,y,z)dy + \hat{z}\int_c v(x,y,z)dz$

   cf) from point P_1 to point P_2

   $\int_{p_1}^{p_2} v\vec{dl}$, around path $C \Rightarrow \oint_c v\vec{dl}$



ex) evaluate $\int_0^P r^2 \vec{dr}$

where $r^2 = x^2 + y^2$ from the original to the point $P(1,1)$

a) along the direct path OP

b) along the path OP_1P

c) along the path OP_2P

$$a)\ \int_0^P r^2 \overrightarrow{dr} = \hat{r}\int_0^{\sqrt{2}} r^2 dr = \hat{r}\,\frac{2\sqrt{2}}{3}$$

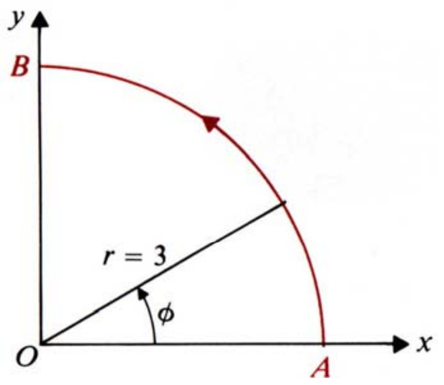
where $\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi$ and $\phi = \frac{\pi}{4}$. since $\tan\phi = \frac{y}{x} \Rightarrow \phi = \tan^{-1}\frac{y}{x} = \frac{\pi}{4}$

$$= \hat{x}\frac{2}{3} + \hat{y}\frac{2}{3}$$

$b)$ along OP_1P

$$\int_0^P (x^2+y^2)\overrightarrow{dr} = \int_0^{P_1}(0^2+y^2)dy\,\hat{y} + \int_{P_1}^P (x^2+1)dx\,\hat{x} = \frac{4}{3}\hat{x} + \frac{1}{3}\hat{y}$$

$c)$ same way, $\int_0^P (x^2+y^2)\overrightarrow{dr} = \frac{1}{3}\hat{x} + \frac{4}{3}\hat{y}$



3. $\int_C \vec{F} \bullet \vec{dl} = ?$

$ex)$ Given $\vec{F} = \hat{x}xy - \hat{y}2x$, evaluate $\int_A^B \vec{F} \bullet \vec{dl}$

$a)$ In cartesian coordinates

$b)$ In cylindrical coordinates

$a)$ In cartesian

$$\vec{F} \bullet \vec{dl} = (\hat{x}xy - \hat{y}2x) \bullet (\hat{x}dx + \hat{y}dy) = xydx - 2xdy$$

$$\int_{x=3,y=0}^{x=0,y=3} (xydx - 2xdy) = -9(1 + \frac{\pi}{2}) \text{ using } x^2 + y^2 = 9$$

$b)$ In cylindrical

$$\vec{F} = \hat{r}(xy\cos\phi - 2xy\sin\phi) - \hat{\phi}(xy\sin\phi + 2x\cos\phi)$$

where $x = r\cos\phi,\ y = r\sin\phi,\ \vec{dl} = \hat{\phi}rd\phi = \hat{\phi}3d\phi$

$$\vec{F} \bullet \vec{dl} = -3(xy\sin\phi + 2x\cos\phi)d\phi$$

$$\int_0^{\frac{\pi}{2}} -3(xy\sin\phi + 2x\cos\phi)d\phi = -9(1 + \frac{\pi}{2}), \text{ where } x = 3\cos\phi \text{ and } y = 3\sin\phi$$

4. $\oint_s \vec{A} \bullet \vec{ds} = \oint_s \vec{A} \bullet \hat{n} ds \Rightarrow$ the integral measures the flux of the vector field $\vec{A}$ flowing through the area S

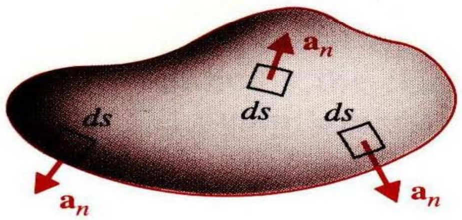
cf) the convention for the positive direction of $\vec{ds}$ or $\hat{n}$

① a closed surface

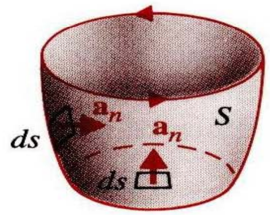
② an open surface

$\Rightarrow$ depend on the direction in which the perimeter of the open surface is traversed

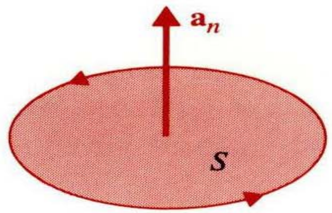
③ disk



(a) A closed surface.



(b) An open surface.



(c) A disk.

ex) Given $\vec{F} = \widehat{a_r}\frac{k_1}{r} + \widehat{a_z}\frac{k_2}{r}$, $(z = \pm 3,\ r = 2)$

evaluate $\oint_s \vec{F} \bullet \vec{ds}$.

$$sol)\oint_S \vec{F}\bullet\vec{ds} = \int_{\substack{top\\face}} \vec{F}\bullet\hat{n}ds + \int_{\substack{bottom\\face}} \vec{F}\bullet\hat{n}ds + \int_{\substack{side\\wall}} \vec{F}\bullet\hat{n}ds$$

a)top face : $\hat{n} = \hat{z}, ds = rdrd\phi, z = 3 \quad (r = 0 \to 2, \phi = 0 \to 2\pi)$

$$\int_{\substack{top\\face}} \vec{F}\bullet\hat{n}ds = \int_0^{2\pi}\int_0^2 3k_2 rdrd\phi = 12\pi k_2$$

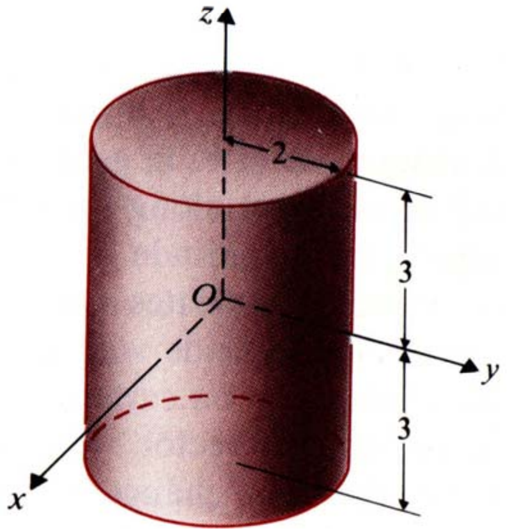
b)bottom face : $\hat{n} = -\hat{z}, ds = rdrd\phi, z = -3$

$$\int_{\substack{bottom\\face}} \vec{F}\bullet\hat{n}ds = \int_0^{2\pi}\int_0^2 3k_2 rdrd\phi = 12\pi k_2$$

c)side wall : $\hat{n} = \hat{r}, ds = rdrd\phi, r = 2$

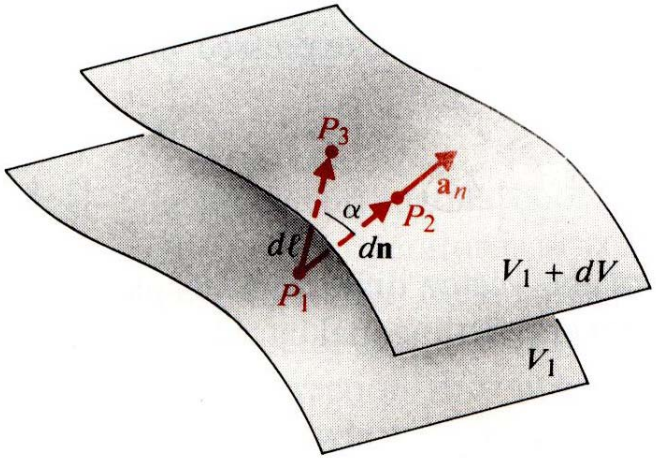
$$\int_{\substack{side\\wall}} \vec{F}\bullet\hat{n}ds = \int_0^{2\pi}\int_{-3}^3 k_1 rdrd\phi = 12\pi k_1$$

$$\therefore \oint_S \vec{F}\bullet\vec{ds} = 12\pi(k_1 + 2k_2)$$



2-6 Gradient of a Scalar Field

- scalar or vector fields : function of (t ; u_1, u_2, u_3)
- space rate of change of a scalar fields at a given time
- the gradient of the scalar : the vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar



- $\text{grad}V \triangleq \hat{n}\dfrac{dV}{dn}$ or $\nabla V \triangleq \hat{n}\dfrac{dV}{dn}$ (where, $\vec{dl} \neq \vec{dn}$)

$$cf)\ \frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha = \frac{dV}{dn}\hat{n}\bullet\hat{l} = (\nabla V)\bullet\hat{l}$$

$$\because dV = (\nabla V)\bullet\vec{dl}$$

- $dV = \frac{\partial V}{\partial l_1} dl_1 + \frac{\partial V}{\partial l_2} dl_2 + \frac{\partial V}{\partial l_3} dl_3$

  where, $\vec{dl} = \widehat{u_1} dl_1 + \widehat{u_2} dl_2 + \widehat{u_3} dl_3 = \widehat{u_1}(h_1 du_1) + \widehat{u_2}(h_2 du_2) + \widehat{u_3}(h_3 du_3)$

$$dV = \left(\widehat{u_1} \frac{\partial V}{\partial l_1} + \widehat{u_2} \frac{\partial V}{\partial l_2} + \widehat{u_3} \frac{\partial V}{\partial l_3} \right) \bullet \left(\widehat{u_1} dl_1 + \widehat{u_2} dl_2 + \widehat{u_3} dl_3 \right) = \left(\widehat{u_1} \frac{\partial V}{\partial l_1} + \widehat{u_2} \frac{\partial V}{\partial l_2} + \widehat{u_3} \frac{\partial V}{\partial l_3} \right) \bullet \vec{dl}$$

$$\therefore \ \nabla V = \widehat{u_1} \frac{\partial V}{\partial l_1} + \widehat{u_2} \frac{\partial V}{\partial l_2} + \widehat{u_3} \frac{\partial V}{\partial l_3} = \widehat{u_1} \frac{\partial V}{h_1 \partial u_1} + \widehat{u_2} \frac{\partial V}{h_2 \partial u_2} + \widehat{u_3} \frac{\partial V}{h_3 \partial u_3}$$

$$= \left(\widehat{u_1} \frac{\partial}{h_1 \partial u_1} + \widehat{u_2} \frac{\partial}{h_2 \partial u_2} + \widehat{u_3} \frac{\partial}{h_3 \partial u_3} \right) V$$

$$\Rightarrow \nabla \equiv \widehat{u_1} \frac{\partial}{h_1 \partial u_1} + \widehat{u_2} \frac{\partial}{h_2 \partial u_2} + \widehat{u_3} \frac{\partial}{h_3 \partial u_3}$$

→ Vector differential operator

cf) $\nabla \bullet \ \Rightarrow$ divergence operation

$\nabla \times \ \Rightarrow$ curl opetation

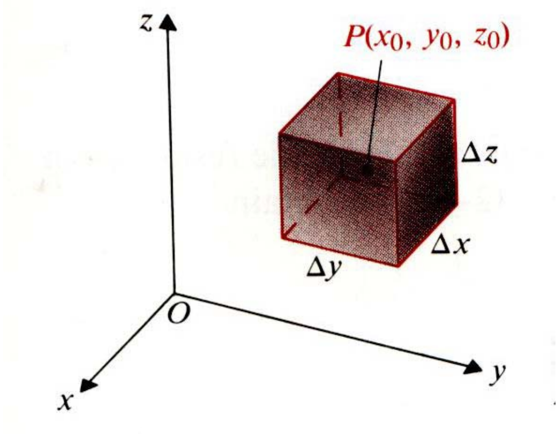
2-7 Divergence of a Vector Field

✓ Divergence of a vector field $\vec{A}$ at a point ($div\ \vec{A}$)

$\Rightarrow$ Net outward flux of $\vec{A}$ per unit volume as the volume about the point tends to zero

$$div\ \vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A} \bullet \vec{ds}}{\Delta v}$$

$\rightarrow$ (two dimensional surface/three dimensional volume) spatial derivatives as the volume approaches zero



$e.g)$ In cartesian coordinates

$$\oint_s \vec{A}\bullet\vec{ds}$$

$$= \left[\int_{\substack{front\\face}} + \int_{\substack{back\\face}} + \int_{\substack{right\\face}} + \int_{\substack{left\\face}} + \int_{\substack{top\\face}} + \int_{\substack{bottom\\face}} \right] \vec{A}\bullet\vec{ds}$$

① front face

$$\int_{\substack{front\\face}} \vec{A}\bullet\vec{ds} = \vec{A}_{front}\bullet\vec{\Delta s}_{front} = \vec{A}_{front}\bullet\hat{x}(\Delta y\Delta z) = A_x(x_0+\frac{\Delta x}{2}, y_0, z_0)\Delta y\Delta z$$

$$\therefore A_x(x_0+\frac{\Delta x}{2}, y_0, z_0) = A_x(x_0, y_0, z_0) + \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_{(x_0,y_0,z_0)} + high\ order\ term$$

② back face

$$\int_{\substack{back\\face}} \vec{A}\bullet\vec{ds} = \vec{A}_{back}\bullet\vec{\Delta s}_{back} = \vec{A}_{back}\bullet(-\hat{x}\Delta y\Delta z) = -A_x(x_0-\frac{\Delta x}{2}, y_0, z_0)\Delta y\Delta z$$

$$\therefore A_x(x_0-\frac{\Delta x}{2}, y_0, z_0) = A_x(x_0, y_0, z_0) - \frac{\Delta x}{2}\frac{\partial A_x}{\partial x}\bigg|_{(x_0,y_0,z_0)} + high\ order\ term$$

$$\therefore [\int_{\substack{front\\face}} + \int_{\substack{back\\face}}]\vec{A}\bullet\vec{ds} = \left(\frac{\partial A_x}{\partial x} + H.O.T.\right)\Bigg|_{(x_0,y_0,z_0)} \Delta x \Delta y \Delta z$$

$$\therefore \oint_s \vec{A}\bullet\vec{ds} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Bigg|_{(x_0,y_0,z_0)} \Delta x \Delta y \Delta z + H.O.T. \text{ in } \Delta x, \Delta y, \Delta z$$

$$\therefore div\ \vec{A} \equiv \nabla\bullet\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

$\Rightarrow$ depend on the position of the point on which it is evaluated

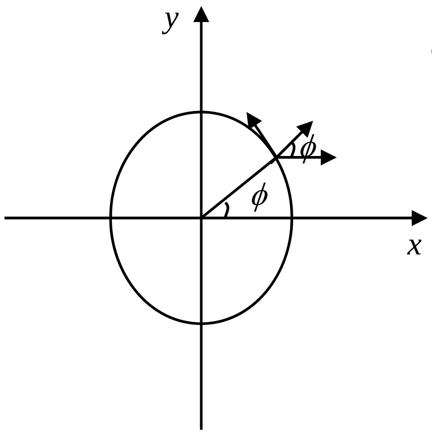
$$cf\,)\nabla \equiv \vec{u}_1 \frac{\partial}{h_1 \partial u_1} + \vec{u}_2 \frac{\partial}{h_2 \partial u_2} + \vec{u}_3 \frac{\partial}{h_3 \partial u_3}$$

✓ $\nabla \bullet \vec{A} = \left(\hat{u}_1 \frac{\partial}{h_1 \partial u_1} + \hat{u}_2 \frac{\partial}{h_2 \partial u_2} + \hat{u}_3 \frac{\partial}{h_3 \partial u_3} \right) \bullet \left(\hat{u}_1 A_1 + \hat{u}_2 A_2 + \hat{u}_3 A_3 \right)$

$$= \hat{u}_1 \frac{\partial}{h_1 \partial u_1} \bullet \left(\hat{u}_1 A_1 \right) + \hat{u}_2 \frac{\partial}{h_2 \partial u_2} \bullet \left(\hat{u}_2 A_2 \right) + \hat{u}_3 \frac{\partial}{h_3 \partial u_3} \bullet \left(\hat{u}_3 A_3 \right)$$

- $\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi, \quad \hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$

$$\frac{\partial \hat{r}}{\partial \phi} = \hat{\phi}, \quad \frac{\partial \hat{\phi}}{\partial \phi} = \hat{r}$$



eg) cylindrical coordinate

$u_1 = r,\ u_2 = \phi,\ u_3 = z$ and $h_1 = 1,\ h_2 = r,\ h_3 = 1$

$$\hat{r} \frac{\partial}{\partial r} \bullet \left(\hat{r} A_r \right) + \hat{r} \frac{\partial}{\partial r} \bullet \left(\hat{\phi} A_\phi \right) + \hat{r} \frac{\partial}{\partial r} \bullet \left(\hat{z} A_z \right)$$

$$= \hat{r} \left(\bullet \frac{\partial}{\partial r} \hat{r} \right) A_r + \hat{r} \bullet \hat{r} \frac{\partial}{\partial r} A_r + \left(\hat{r} \bullet \frac{\partial \hat{\phi}}{\partial r} \right) A_\phi + \hat{r} \bullet \hat{\phi} \frac{\partial}{\partial r} A_\phi + \hat{r} \left(\bullet \frac{\partial \hat{z}}{\partial r} \right) A_z + \hat{r} \bullet \hat{z} \frac{\partial}{\partial r} A_z = \frac{\partial}{\partial r} A_r$$

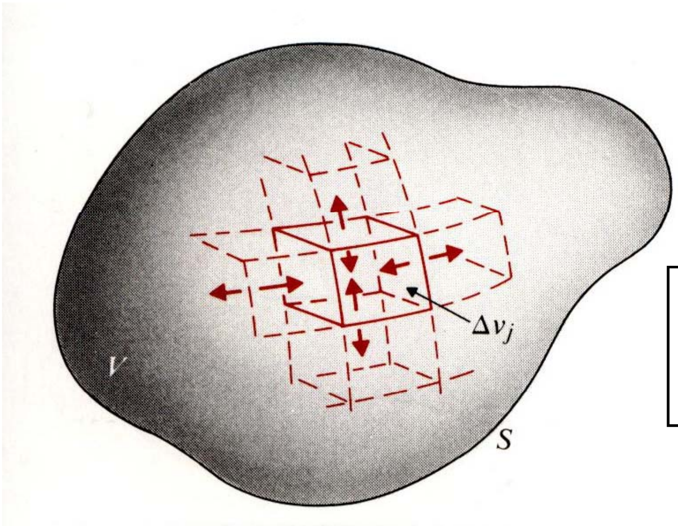
$$cf) \frac{1}{r} \frac{\partial}{\partial r} \left(r A_r \right) = \frac{\partial}{\partial r} A_r + \frac{A_r}{r}$$

2-8 Divergence Theorem

✓ $\int_V \nabla \bullet \vec{A} dv = \oint_s \vec{A} \bullet \vec{ds}$

from definition of divergence, $(\nabla \bullet \vec{A})_j \Delta v_j = \oint_{s_j} \vec{A} \bullet \vec{ds}$

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \bullet \vec{A})_j \Delta v_j \right] = \lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{s_j} \vec{A} \bullet \vec{ds} \right]$$



Volume integral

$$\int_V (\nabla \bullet \vec{A}) dv$$

$$\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \int_{s_j} \vec{A} \bullet \vec{ds} \right] = \oint_s \vec{A} \bullet \vec{ds}$$

$$\therefore \int_V (\nabla \bullet \vec{A}) dv = \oint_s \vec{A} \bullet \vec{ds}$$

2-9 Curl of a Vector Field

- ✓ Flow source
  - ➢ outward flux
  - ➢ divergence
- ✓ Vortex source
  - ➢ cause a circulation of a vector around it
  - ➢ circulation of $\vec{A}$ around contour C $\triangleq \oint_C \vec{A} \bullet \vec{dl}$

e.g) ▪ a force acting on an object

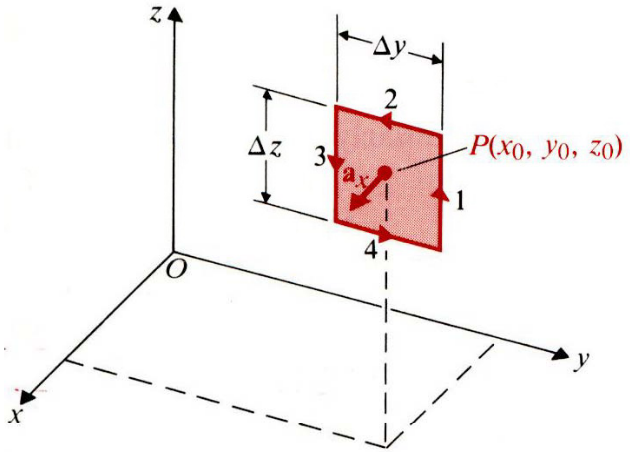
→ circulation will be the work done by the force in moving the object once around the contour

▪ electric field → e.m.f

▪ water whirling down a sink drain → vortex sink

✓ $curl\ \vec{A} \equiv \nabla \times \vec{A} \triangleq \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\widehat{a}_n \oint_C \vec{A} \bullet \vec{dl} \right]_{\max}$

$cf)\ \left(\nabla \times \vec{A}\right)_u = \widehat{a}_u \bullet \left(\nabla \times \vec{A}\right) = \lim_{\Delta s_u \to 0} \frac{1}{\Delta s_u} \left(\oint_{C_u} \vec{A} \bullet \vec{dl} \right)$



✓ $\hat{u} = \hat{x},\ \Delta s_u = \Delta x \Delta y,\ C_u$:①,②,③,④

$$\left(\nabla \times \vec{A}\right)_x = \lim_{\Delta y \Delta z \to 0} \frac{1}{\Delta y \Delta z}\left(\oint_{\substack{\text{sides}\\1,2,3,4}} \vec{A} \bullet \vec{dl}\right)$$

① side1

$$\vec{dl} = \hat{z}\Delta z,\ \vec{A} \bullet \vec{dl} = A_z(x_0, y_0 + \frac{\Delta y}{2}, z_0)\Delta z$$

$$\text{where, } A_z(x_0, y_0 + \frac{\Delta y}{2}, z_0) = A_z(x_0, y_0, z_0) + \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + H.O.T.$$

$$\therefore \int_{\text{side1}} \vec{A} \bullet \vec{dl} = \left\{A_z(x_0, y_0, z_0) + \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + H.O.T.\right\}\Delta z$$

② side3

$$\vec{dl} = -\hat{z}\Delta z,\ \vec{A} \bullet \vec{dl} = A_z(x_0, y_0 - \frac{\Delta y}{2}, z_0)\Delta z \qquad \therefore \int_{\text{side3}} \vec{A} \bullet \vec{dl} = \left\{A_z(x_0, y_0, z_0) - \frac{\Delta y}{2}\frac{\partial A_z}{\partial y}\bigg|_{(x_0, y_0, z_0)} + H.O.T.\right\}(-\Delta z)$$

- side1 + side3

$$= \left(\frac{\partial A_z}{\partial y} + H.O.T \right)\Bigg|_{(x_0, y_0, z_0)} \Delta y \Delta z$$

- side2 + side4

$$= \left(-\frac{\partial A_y}{\partial y} + H.O.T \right)\Bigg|_{(x_0, y_0, z_0)} \Delta y \Delta z$$

$$H.O.T \to 0 \text{ , when } \Delta y \to 0 \text{ and } \Delta z \to 0$$

$$\therefore (\nabla \times \vec{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

✓ General expression in cartesian

- $$\nabla \times \vec{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

✓ General expression for orthogonal curvilinear systems

- $$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{u}_1 h_1 & \hat{u}_2 h_2 & \hat{u}_3 h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

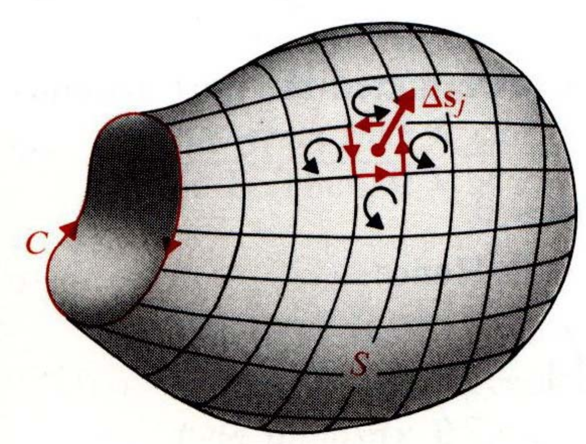
2-10 Stokes's Theorem

✓ $$\int_S (\nabla \times \vec{A}) \bullet \vec{ds} = \oint_C \vec{A} \bullet \vec{dl}$$

from definition of curl, $(\nabla \times \vec{A})_j \bullet (\Delta s_j) = \oint_{C_j} \vec{A} \bullet \vec{dl}$

$$\Rightarrow \lim_{\Delta s_j \to 0} \sum_{j=1}^{N} (\nabla \times \vec{A})_j \bullet (\Delta s_j) = \int_S (\nabla \times \vec{A}) \bullet \vec{ds}$$

$$\lim_{\Delta s_j \to 0} \sum_{j=1}^{N} \oint_{C_j} \vec{A} \bullet \vec{dl} = \oint_C \vec{A} \bullet \vec{dl}$$



$cf)\ \oint_S (\nabla \times \vec{A}) \bullet \vec{ds} = 0,$

closed surface $\Rightarrow$ no boundary contour

2-11 Two Null Identities

1. $\nabla \times (\nabla V) \equiv 0$

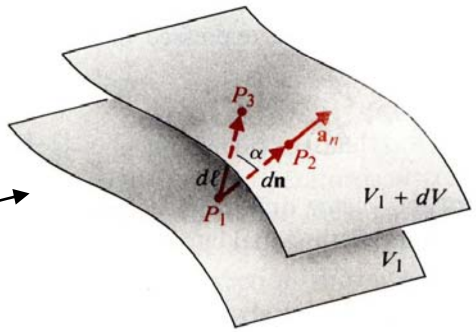
→ the curl of the gradient of any scalar field is identically zero

- $\int_S [\nabla \times \nabla V] \bullet \vec{ds} = \oint_C (\nabla V) \bullet \vec{dl} = \oint_C dV = 0$

$$\because \frac{dV}{dl} = \frac{dV}{dn}\frac{dn}{dl} = \frac{dV}{dn}\cos\alpha = \frac{dV}{dn}\hat{n} \bullet \hat{l} = (\nabla V) \bullet \hat{l}$$

$$\because dV = (\nabla V) \bullet \vec{dl}$$

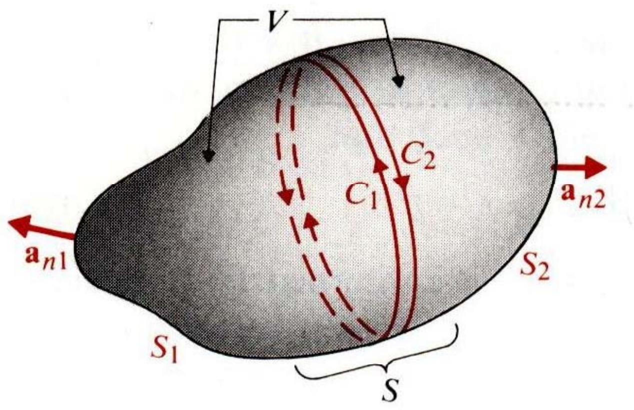
참 고



2. $\nabla \bullet (\nabla \times \vec{A}) = 0$

→ the divergence of the curl of any vector field is identically zero

- $\int_V \nabla \bullet (\nabla \times \vec{A}) dv = \oint_S (\nabla \times \vec{A}) \bullet \vec{ds} = 0$



cf) $\hat{a}_n$ of $\overrightarrow{ds}$ should be outward,

so $\hat{a}_{n1}$, $\hat{a}_{n2}$ also outward

$$i.e)\ \oint_S (\nabla \times \vec{A}) \bullet \overrightarrow{ds} = \int_{S_1} (\nabla \times \vec{A}) \bullet \hat{a}_{n1} ds + \int_{S_2} (\nabla \times \vec{A}) \bullet \hat{a}_{n2} ds$$

$$= \oint_{C_1} \vec{A} \bullet \overrightarrow{dl} + \oint_{C_2} \vec{A} \bullet \overrightarrow{dl} = 0$$

(C_1 and C_2 are same, but opposite in direction)

2-12 Helmholtz's Theorem

1. solenoidal and irrotational if $\nabla \bullet \vec{F} = 0$ and $\nabla \times \vec{F} = 0$
2. solenoidal but not irrotational if $\nabla \bullet \vec{F} = 0$ and $\nabla \times \vec{F} \neq 0$
3. irrotational but not solenoidal if $\nabla \times \vec{F} = 0$ and $\nabla \bullet \vec{F} \neq 0$
4. neither solenoidal nor irrotational if $\nabla \times \vec{F} \neq 0$ and $\nabla \bullet \vec{F} \neq 0$

✓ Helmholtz's theorem

: A vector field is determined to within an additive constant if both its divergence and its curl are specified everywhere