Dielectrics in Static Electric Field

- No free charge in dielectrics to make interior charge density and electric field vanish

- Dielectrics contain bound charge
  → effect on the electric field

- E-field → small displacement of positive and negative charges (bound charge) → polarize a dielectric material

  - a. Polar molecules : permanent dipole moments

    ex) H_2O (two or more dissimilar atoms) → P ~ 10^{-30} (C·m)

    ☞Individual dipoles are randomly oriented
    → macroscopically no net dipole

    ☞Some have a permanent dipole moment even in the absence of external field → electrets

  - b. nonpolar molecules : no permanent dipole moments

Equivalent Charge Distribution of Polarized Dielectrics

❖ Define polarization vector, $\vec{P}$

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \vec{p}_k}{\Delta v} \ [\mathrm{C/m^2}]$$: volume density of electric dipole moment

$d\vec{P}$ of an elemental volume $d\vec{p} = \vec{P}dv'$

$$dV = \frac{\vec{P} \bullet \hat{R}}{4\pi\varepsilon_0 R^2} dv'$$

(cf) potential due to a dipole $V = \frac{\vec{P} \bullet \hat{R}}{4\pi\varepsilon_0 R^2}$)

$$\therefore \ V = \frac{1}{4\pi\varepsilon_0} \int_{v'} \frac{\vec{P} \bullet \hat{R}}{R^2} dv',$$ where R is the distance from dv' to a fixed field point.

cf) $R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$ in Cartesian coordinate.

$$\nabla'\left(\frac{1}{R}\right) = \nabla'\left(\frac{1}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}\right) = \frac{\vec{R}}{R^3} = \frac{\hat{R}}{R^2}$$

$$\vec{R} = (x - x')\,\hat{x} + (y - y')\,\hat{y} + (z - z')\,\hat{z}$$

Equivalent Charge Distribution of Polarized Dielectrics

$$\therefore V = \frac{1}{4\pi\varepsilon_0}\int_{v'} \vec{P}\bullet\nabla'\left(\frac{1}{R}\right)dv'$$

cf) $\nabla'\bullet(f\vec{A}) = f\nabla'\bullet\vec{A} + \vec{A}\bullet\nabla' f \Rightarrow \therefore\ \vec{P}\bullet\nabla'\left(\frac{1}{R}\right) = \nabla'\bullet\left(\frac{\vec{P}}{R}\right) - \frac{1}{R}(\nabla'\bullet\vec{P})$

$$\therefore V = \frac{1}{4\pi\varepsilon_0}\left[\int_{v'}\nabla'\bullet\left(\frac{\vec{P}}{R}\right)dv' - \int_{v'}\frac{\nabla'\bullet\vec{P}}{R}dv'\right]$$

$$= \underbrace{\boxed{\frac{1}{4\pi\varepsilon_0}\oint_{S'}\frac{\vec{P}\bullet\hat{n}'}{R}ds'}}_{\text{surface}} + \underbrace{\boxed{\frac{1}{4\pi\varepsilon_0}\int_{v'}\frac{(-\nabla'\bullet\vec{P})}{R}dv'}}_{\text{volume}}$$

cf) potential due to surface and volume charge

$$V = \frac{1}{4\pi\varepsilon_0}\int_{v'}\frac{\rho_v}{R}dv' \quad (3\text{-}61), \qquad V = \frac{1}{4\pi\varepsilon_0}\int_{S'}\frac{\rho_s}{R}ds' \quad (3\text{-}62)$$

$$\therefore \boxed{\boxed{\rho_{ps} = \vec{P}\bullet\hat{n}} \qquad \boxed{\rho_p = -\nabla'\bullet\vec{P}}} \longrightarrow \text{Polarization charge density or bound charge density}$$

ρ_{ps}: Equivalent polarization surface charge density

ρ_p: Equivalent polarization volume charge density

Equivalent Charge Distribution of Polarized Dielectrics

cf) Imaginary elemental surface Δs of a nonpolar dielectric, net charge crossing the surface Δs is,

$\Delta Q = nq(\vec{d} \cdot \hat{n})\Delta s$, where n is the number of molecules per unit volume

$nq\vec{d}$: dipole moment per unit volume $\Rightarrow$ polarization vector $\vec{P}$

$$\Delta Q = \vec{P} \bullet \hat{n}(\Delta s) \quad \Rightarrow \quad \rho_{ps} = \frac{\Delta Q}{\Delta s} = \vec{P} \bullet \boxed{\hat{n}}$$

$\hat{n}$ ↘ outward normal

cf) For a surface S bounding a volume V,

the net total charge flowing out of V
= negative of the net charge remaining within the volume V

$$Q = -\oint_S \vec{P} \bullet \hat{n}\, ds = \int_v -(\nabla \bullet \vec{P})\, dv = \int_v \rho_p\, dv$$

Electric Flux density and Dielectric Constant

$$\nabla \bullet \vec{E} = \frac{1}{\varepsilon_0}(\rho + \rho_p)$$

ρ_p → equivalent volume charge density of polarization

$$\rho_p = -\nabla \bullet \vec{P}$$

$$\therefore \nabla \bullet (\varepsilon_0 \vec{E} + \vec{P}) = \rho$$

❖ Define new fundamental quantity

$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ (C/m^2) : electric flux density or electric displacement

$\nabla \bullet \vec{D} = \rho$ (C/m^3) → valid everywhere

ρ → free charge density note no ε_0 appear

cf) $\nabla \times \vec{E} = 0$

→ Two of static maxwell equations

$\therefore$ $\oint_s \vec{D} \bullet d\vec{s} = Q$ $\Rightarrow$ Gauss's law $\left(\oint \vec{E} \bullet d\vec{s} = \frac{Q}{\varepsilon_0} \right)$

Electric Flux density and Dielectric Constant

- Permittivity of dielectric material
  - linear and isotropic dielectric media :
    - Polarization is directly proportional to the electric field intensity

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$

where χ_e : electric susceptibility (dimensionless quantity)

cf) medium is linear if χ_e is independent of $\vec{E}$
medium is homogeneous if χ_e is independent of space coordinate.

$$\begin{aligned} \vec{D} &= \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} \\ &= \varepsilon_0 (1 + \chi_e) \vec{E} \\ &= \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} \end{aligned}$$

$\varepsilon_r = 1 + \chi_e = \dfrac{\varepsilon}{\varepsilon_0}$: relative permittivity or dielectric constant of the medium

$\varepsilon = \varepsilon_0 \varepsilon_r$: absoulte permittivity (permittivity)

Electric Flux density and Dielectric Constant

- anisotropic medium
  - Dielectric constant is different for different directions of the electric field

    $\Rightarrow$ $\vec{D}$ and $\vec{E}$ vectors generally have different directions.

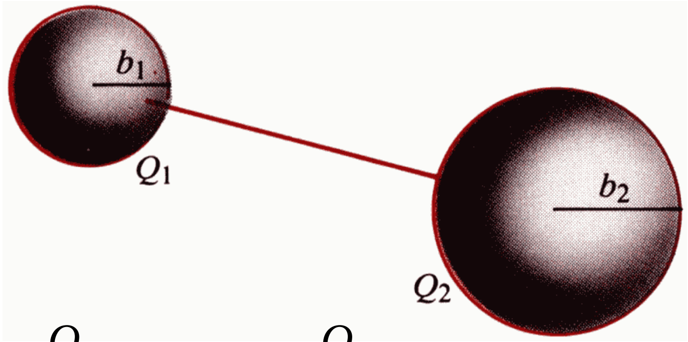
    $\Rightarrow$ permittivity is a tensor.

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

- Dielectric strength
  - Maximum electric field intensity that dielectric material can withstand without breakdown.

Ex) 3-13. Two connected conducting spheres

- ✓ Two spherical conductors with radius b_1 and b_2 $(b_2 > b_1)$
- ✓ Connected by a conducting wire
- ✓ Distance is large enough to ignore influence of each sphere on the other
- ✓ Total charge Q is deposited on the sphere



sol) Two conductors are at the same potential

a) $$\frac{Q_1}{4\pi\varepsilon_0 b_1} = \frac{Q_2}{4\pi\varepsilon_0 b_2}$$

$$\frac{Q_1}{Q_2} = \frac{b_1}{b_2}, \quad Q_1 + Q_2 = Q$$

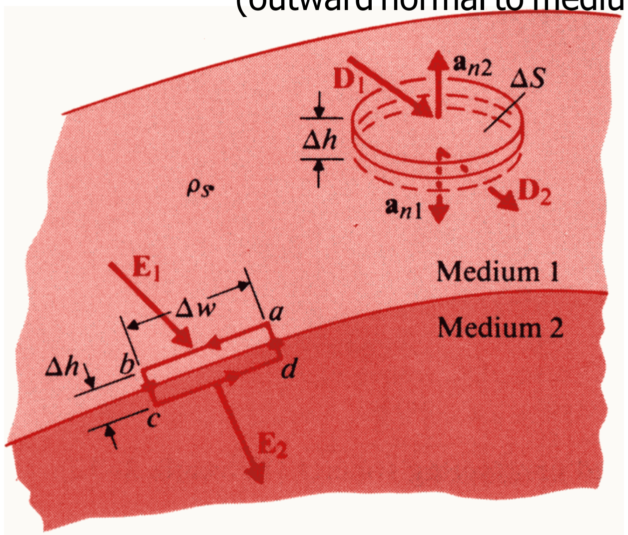
$$Q_1 = \frac{b_1}{b_1 + b_2}Q, \quad Q_2 = \frac{b_2}{b_1 + b_2}Q$$

b) $$E_{1n} = \frac{Q_1}{4\pi\varepsilon_0 b_1^{\ 2}}, \quad E_{2n} = \frac{Q_2}{4\pi\varepsilon_0 b_2^{\ 2}}$$

$$\therefore \frac{E_{1n}}{E_{2n}} = \left(\frac{b_2}{b_1}\right)^2 \frac{Q_1}{Q_2} = \frac{b_2}{b_1}$$

larger curvature $\Rightarrow$ smaller sphere : higher electric field intensity

Boundary Conditions for Electrostatic Fields



① $\Delta h \to 0$

$$\oint_{abcda} \vec{E} \bullet d\vec{l} = \vec{E}_1 \bullet \Delta\vec{w} + \vec{E}_2 \bullet (-\Delta\vec{w})$$

$$= E_{1t}\Delta w - E_{2t}\Delta w = 0$$

$\therefore$ $\boxed{E_{1t} = E_{2t}}$ or $\dfrac{D_{1t}}{\varepsilon_1} = \dfrac{D_{2t}}{\varepsilon_2}$

☞ the tangential component of an $\vec{E}$ field is continuous across an interface

② Cylinder $\Delta h \to 0$

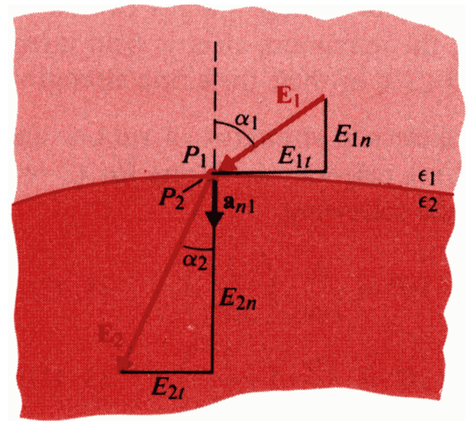
$$\oint_s \vec{D} \bullet d\vec{s} = (\vec{D}_1 \bullet \hat{n}_2 + \vec{D}_2 \bullet \hat{n}_1)\Delta S = \hat{n}_2 \bullet (\vec{D}_1 - \vec{D}_2)\Delta S = Q = \rho_s \Delta S$$

($\hat{n}_1 = -\hat{n}_2$)

$\therefore$ $\hat{n}_2 \bullet (\vec{D}_1 - \vec{D}_2) = \rho_s$ or $\hat{n}_1 \bullet (\vec{D}_2 - \vec{D}_1) = \rho_s$

i.e., $\boxed{D_{1n} - D_{2n} = \rho_s \ (\mathrm{C/m^2})}$ reference normal is $\hat{n}_2$

Ex) 3-15 Boundary conditions



- Tangential $\vec{E}$ should be continuous at boundary.

$$E_2 \sin\alpha_2 = E_1 \sin\alpha_1$$

- Normal $\vec{D}$ should be continuous at boudary.

$$\varepsilon_2 E_2 \cos\alpha_2 = \varepsilon_1 E_1 \cos\alpha_1$$

$$\therefore E_2 = \sqrt{E_{2t}{}^2 + E_{2n}{}^2} = \sqrt{(E_2 \sin\alpha_2)^2 + (E_2 \cos\alpha_2)^2}$$

$$= \sqrt{(E_1 \sin\alpha_1)^2 + \left(\frac{\varepsilon_1}{\varepsilon_2} E_1 \cos\alpha_1\right)^2} = E_1\left[\sin^2\alpha_1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\cos\alpha\right)^2\right]^{1/2}$$

note 1

The normal component of $\vec{D}$ field is discontinuous across an interface where a surface charge exists.

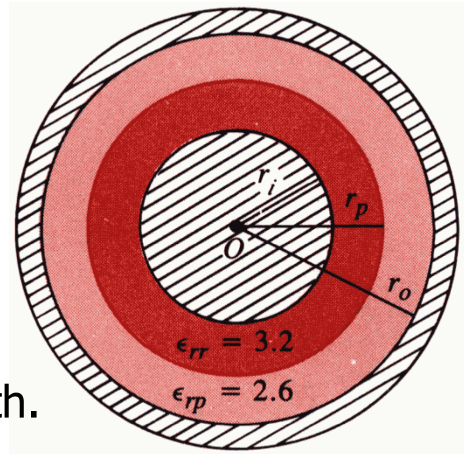
note 2

If, $\rho_s = 0$, then $D_{1n} = D_{2n}$

$$\text{Summary}: \boxed{\begin{aligned} E_{1t} &= E_{2t} \\ \hat{n}_1 \cdot (\vec{D}_2 - \vec{D}_1) &= \rho_s \end{aligned}}$$

Ex) 3-16. coaxial cable

- ✓ The radius of the inner conductor : 0.4 cm
- ✓ Concentric layers of rubber $\varepsilon_{rr} = 3.2$
- ✓ Polystyrene , $\varepsilon_{rp} = 2.6$



➢ Design a cable that is to work at a voltage rating of 20kV.
$\rightarrow$ E field are not to exceed 25% of their dielectric strength.

sol) Dielctric strength of rubber : 25×10^6 V/m
Dielctric strength of polystyrene : 20×10^6 V/m
Cylindrical symmetry $\rightarrow$ Consider only E_r component

$$\max E_r = 0.25 \times 25 \times 10^6 = \frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{3.2\ r_i}\right)$$

$$\max E_p = 0.25 \times 20 \times 10^6 = \frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{2.6\, r_p}\right)$$

$$r_p = 1.54 r_i = 0.616\ [\text{cm}]$$

Ex) 3-16. coaxial cable

Potential difference

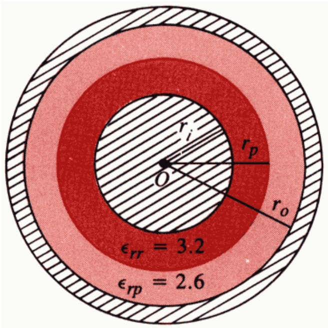
$$-\int_{r_{0.}}^{r_p} E_p\,dr - \int_{r_p}^{r_i} E_r\,dr = 20,000$$

$$\frac{\rho_l}{2\pi\varepsilon_0}\left[\frac{1}{\varepsilon_{rp}}\left(-\int_{r_o}^{r_p}\frac{1}{r}dr\right)+\frac{1}{\varepsilon_{rr}}\left(-\int_{r_p}^{r_i}\frac{1}{r}dr\right)\right]=\frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{2.6}\ln\frac{r_o}{r_p}+\frac{1}{3.2}\ln\frac{r_p}{r_i}\right)=20,000$$

$$r_i = 0.4,\ \ r_p = 0.616,$$

$$\therefore \frac{\rho_l}{2\pi\varepsilon_0} = 0.25\times 20\times 10^6 \times 2.6 r_p = 8\times 10^4, \quad \therefore\ r_o = 2.08 r_i = 0.832\ [\text{cm}]$$

cf) $\frac{5}{4} = \frac{2.6\,r_p}{3.2\,r_i} \Rightarrow r_p = \frac{5}{4}\times\frac{3.2}{2.6}r_i$



if order is reversed,

$$0.25\times 20\times 10^6 = \frac{\rho_l}{2\pi\varepsilon_0}\cdot\frac{1}{2.6}\cdot\frac{1}{r_i}, \qquad 0.25\times 25\times 10^6 = \frac{\rho_l}{2\pi\varepsilon_0}\cdot\frac{1}{3.2}\cdot\frac{1}{r_r}$$

$$\frac{4}{5} = \frac{3.2\,r_r}{2.6\,r_i}, \qquad \therefore\ r_r = \frac{4}{5}\times\frac{2.6}{3.2}r_i = \frac{10.4}{16}r_i,\ \ r_r < r_i$$ $\Rightarrow$ Non sense

Homework

✓ (Cheng 3 - 6) Two very small conducting spheres, each of a mass $1.0 \times 10^{-4} (kg)$, are suspended at a common point by very thin nonconducting threads of a length 0.2 (m). A charge Q is placed on a each sphere. The electric force of repulsion separates the spheres, and an equilibrium is reached when the suspending threads make an angle of 10^{o}. Assuming a gravitational force of 9.90 (N/kg) and a negligible mass of the threads, find Q

✓ (Cheng 3 - 12) Two infinitely long coaxial cylindrical surfaces, r = a and r = b (b > a), carry suirface charge densities ρ_{sa} and ρ_{sb}, respectively.

a) Determine $\bar{E}$ everywhere.

b) What must be the relation between a and b in order that E vanishes for r > b?

Homework

✓ (Cheng 3-27) What are the boundary conditions that must be satisfied by the electrics potential at an interface between two perfect dielectrics with dielectric constants ε_{r1} and ε_{r2}?

✓ A point charge q is enclosed in a linear, isotropic, and homogeneous dielectric medium of infinite extent. Calculate the $\bar{E}$ field, the $\bar{D}$ field, the polarization vector $\bar{P}$, the bound surface charge density ρ_{sb}, and the bound volume charge density ρ_{vb}.

✓ A very thin, finite and uniformly charged line of length 10 m carries a charge of 10 μC/m. Calculate the electric field intensity in a plane bisecting the line at ρ = 5m.

✓ Show that the magnitude of the electric field intensity of an electric dipole is

$$E = \frac{p}{4\pi\varepsilon_0 r^3}\left[1 + 3\cos^2\theta\right]^{1/2}$$