# Field and Wave Electromagnetic

## Chapter.4

## Solution of electrostatic Problems

# Poisson's and Laplace's Equations

✓ $\begin{cases} \nabla \bullet \vec{D} = \rho \\ \nabla \times \vec{E} = 0 \end{cases}$ : Two fundamental equations for electrostatic problem

✓ $\begin{cases} \vec{E} = -\nabla V \\ \vec{D} = \varepsilon \vec{E} \end{cases}$ Where, V is scalar electric potential V

$$\therefore \nabla \cdot (\varepsilon \nabla V) = -\rho$$

# Poisson's and Laplace's Equations

✓ Assuming $\varepsilon$ constant for simple and homogeneous media

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$ : Poisson's equation.

$\nabla^2$ : Laplacian operator

✓ $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2},$$ for cartesian coordinate

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2},$$ for cylindrical coordinate

$$= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial^2 V}{\partial \phi^2}$$ , for spherical coordinate

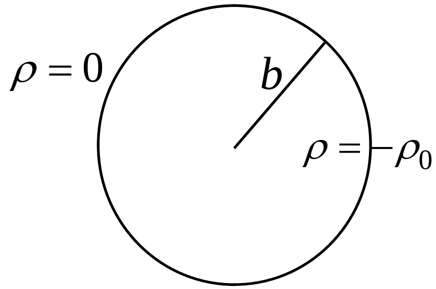
$$= -\frac{\rho}{\varepsilon}$$

note : No free charge

$$\boxed{\nabla^2 V = 0}$$ : Laplace's equation

# Poisson's and Laplace's Equations

ex) Spherical cloud problem




(a) $0 \le R \le b, \quad \rho = -\rho_0$

$$\because \nabla^2 V_i = -\frac{\rho}{\varepsilon_0} = \frac{\rho_0}{\varepsilon_0}$$

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial^2 V}{\partial \phi^2} = \frac{\rho_0}{\varepsilon_0}$$

By symmetry, $\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \phi} = 0$

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2 \frac{dV_i}{dR}\right) = \frac{\rho_0}{\varepsilon_0}$$

$$\frac{d}{dR}\left(R^2 \frac{dV_i}{dR}\right) = \frac{\rho_0}{\varepsilon_0} R^2$$

$$\therefore \frac{dV_i}{dR} = \frac{\rho_0}{3\varepsilon_0} R + \frac{C_1}{R^2}$$

$\rightarrow R = 0$ is singular point unless $C_1 = 0$

# Poisson's and Laplace's Equations

$$\vec{E_i} = -\nabla V_i = -\hat{R}(\frac{\partial V_i}{\partial R})$$

$C_1$=0 , since $E$ can not be infinite at $R = 0$

$$\therefore \vec{E_i} = -\hat{R}\frac{\rho_0}{3\varepsilon_0}R \quad ,0 \le R \le b$$

$$V_i = \frac{\rho_0}{6\varepsilon_0}R^2 + C_1$$

$(b)$ $R \ge b$ , $\rho$=0 $\quad \therefore \nabla^2 V_0 = 0$

$$\frac{1}{R^2}\frac{\partial}{\partial R}(R^2\frac{dV_o}{dR}) = 0 \qquad \therefore \frac{dV_o}{dR} = \frac{C_2}{R^2}$$

$E_o = -\nabla V_o = -\hat{R}\frac{dV_o}{dR} = -\hat{R}\frac{C_2}{R^2}$ at $R = b,\ E_i = E_0$ (homogeneous medium)

$$\frac{C_2}{b^2} = \frac{\rho_0}{3\varepsilon_0}b \to C_2 = \frac{\rho_0}{3\varepsilon_0}b^3$$

$$\therefore \vec{E_0} = -\hat{R}\frac{\rho_0}{3\varepsilon_0 R^2}b^3 \ ,\ R \ge b$$

# Poisson's and Laplace's Equations

Total charge in the cloud

$$Q = -\rho_0 \frac{4\pi}{3} b^3 \qquad \therefore E_o = \hat{R} \frac{Q}{4\pi\varepsilon_0 R^2}$$

$$\frac{dV_o}{dR} = \frac{\rho_0}{3\varepsilon_0} \frac{b^3}{R^2} \qquad \therefore V_o = -\frac{\rho_0}{3\varepsilon_0} \frac{b^3}{R^2} + C_2', \quad \text{As } R \to \infty \quad V_o = 0 \quad \therefore C_2' = 0$$

Boundary condition at $R = b$

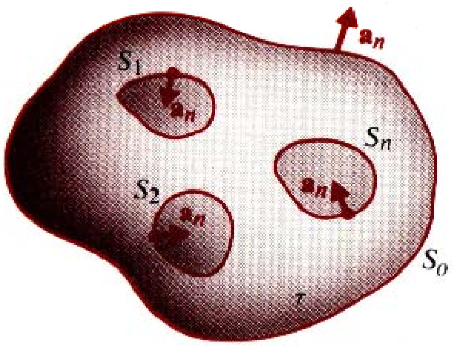
$$-\frac{\rho_0}{3\varepsilon_0} b^2 = \frac{\rho_0}{6\varepsilon_0} b^2 + C_1'$$

$$\therefore C_1' = -\frac{\rho_0}{3\varepsilon_0} b^2 - \frac{\rho_0}{6\varepsilon_0} b^2 = \frac{\rho_0}{\varepsilon_0} b^2 \left(-\frac{1}{3} - \frac{1}{6}\right) = -\frac{\rho_0}{2\varepsilon_0} b^2$$

$$\therefore V_i = -\frac{\rho_0}{3\varepsilon_0} \left(\frac{3b^2}{2} - \frac{R^2}{2}\right)$$

# Uniqueness of Electrostatic Solution

✓ Uniqueness theorem
A solution of Poisson's equation that satisfies the given boundary conditions is a unique solution




✓ proof of uniqueness theorem
charged conducting bodies with surface $S_1, S_2, \ldots, S_n$ at specified potential

① Assume two solutions $V_1$ and $V_2$ of Poisson's equation in $\tau$

$$\nabla^2 V_1 = -\frac{\rho}{\varepsilon}, \qquad \nabla^2 V_2 = -\frac{\rho}{\varepsilon}$$

② Assume that both $V_1$ and $V_2$ satisfy the same boundary condition on $S_1, S_2, \ldots, S_n$

# Uniqueness of Electrostatic Solution

③ Define

$$V_d = V_1 - V_2, \quad \text{then from ①,} \quad \nabla^2 V_d = 0$$

④ On conducting boundaries, the potentials are specified and $V_d = 0$

⑤ $\nabla \cdot (f\vec{A}) = f\nabla \cdot \vec{A} + \vec{A} \cdot \nabla f$

$$\rightarrow \nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + (\nabla V_d) \cdot (\nabla V_d)$$

$$\therefore \int_\tau \nabla \cdot (V_d \nabla V_d) dv = \int_\tau |\nabla V_d|^2 \, dv = \oint_s (V_d \nabla V_d) \cdot \hat{n} ds,$$

when $\hat{n}$ denote the unit normal outward from $\tau$. $(S_0, S_1, S_2, \ldots, S_n)$

⑥ over the conducting boundaries, $V_d = 0$

$V_1, V_2 \propto \frac{1}{R}, \quad \nabla V_d \propto \frac{1}{R^2} \quad \rightarrow \quad V_d \nabla V_d \propto \frac{1}{R^3},$ but surface area $S_0 \propto R^2$.

$$\therefore \lim_{R\rightarrow\infty} \oint_s (V_d \nabla V_d) \cdot \hat{n} ds = 0 \qquad \therefore \int_\tau |\nabla V_d|^2 \, dv = 0$$

# Uniqueness of Electrostatic Solution

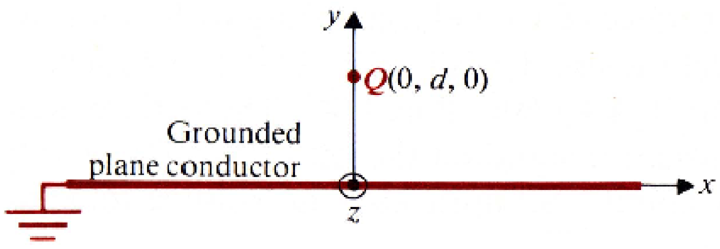
⑦ $|\nabla V_d| \geq 0$ for everywhere

$\therefore \int_{\tau} |\nabla V_d|^2 \, dv = 0 \quad \text{only for } \nabla V_d = 0$

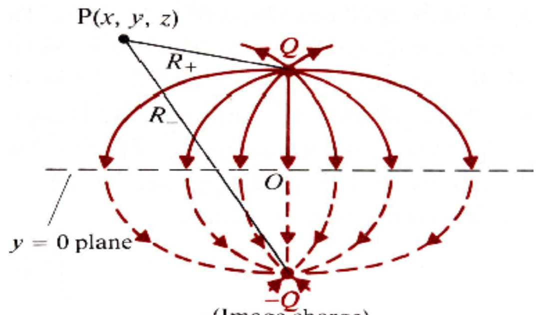
in other words $V_d$ has the same value at all points in τ as well as on the conducting surface $\rightarrow V_d = 0$ throughout the volume $\tau$.

$\therefore V_1 = V_2$ and there is only one possible solution.

# Method of images




(a) Physical arrangement.




(b) Image charge and field lines.

✓ $\nabla^2 V = \dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$ for $y > 0$ except at the point charge.

The solution $V(x, y, z)$ should satisfy the following conditions

① $V(x, 0, z) = 0$

② $V \to \dfrac{Q}{4\pi\varepsilon_0 R}$ as $R \to 0$ where $R$ is the distance to $Q$

③ At point P $(x \to \pm\infty, y \to \pm\infty, or \;\; z \to \pm\infty)$, the potential $V \to 0$

④ The potential function is even with respect to the $x$ and $z$ coordinates

$$V(x, y, z) = V(-x, y, z), \qquad V(x, y, z) = V(x, y, -z)$$

$\to$ very difficult to satisfy all these condition

# Method of images

- From figure (b),

$$V(x,y,z)=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_+}-\frac{1}{R_-}\right) \quad \text{for } y>0$$

where, $R_+=[x^2+(y-d)^2+z^2]^{1/2}$, $R_-=[x^2+(y+d)^2+z^2]^{1/2}$

satisfy all the conditions mentioned above.

$\therefore$ This is a solution of this problems by uniquiness theorem.

$$\vec{E}=-\nabla V \quad \text{for } y>0$$

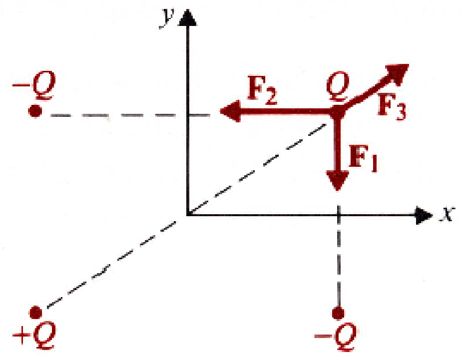
# Method of images

✓ $\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3$

$$\vec{F}_1 = -\hat{y}\frac{Q^2}{4\pi\varepsilon_0 (2d_2)^2}$$

$$\vec{F}_2 = -\hat{x}\frac{Q^2}{4\pi\varepsilon_0 (2d_1)^2}$$

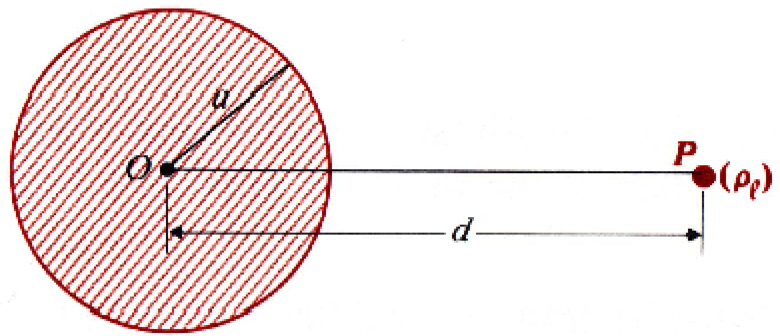
$$\vec{F}_3 = \frac{Q^2}{4\pi\varepsilon_0 [(2d_1)^2 + (2d_2)^2]^{3/2}}(\hat{x}2d_1 + \hat{y}2d_2)$$



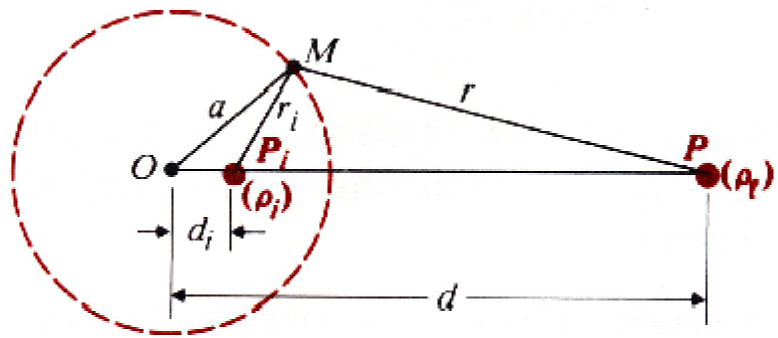

$$\therefore \vec{F} = \frac{Q^2}{16\pi\varepsilon_0}\left\{\hat{x}\left[\frac{d_1}{(d_1^2 + d_2^2)^{3/2}} - \frac{1}{d_1^2}\right] + \hat{y}\left[\frac{d_2}{(d_1^2 + d_2^2)^{3/2}} - \frac{1}{d_2^2}\right]\right\}$$

<u>Note</u> : Explain whether the force is in the direction to the conductor or not?

# Line Charge and Parallel Conducting Cylinder




(a) Line charge and parallel conducting cylinder.




(b) Line charge and its image.

① The image must be a parallel line charge inside the cylinder in order to make the cylindrical surface at $r = a$ an equipotential surface.

② Because of symmetry w.r.t. the line OP, the image line charge must lie somewhere along OP.

$\rightarrow$ At a point $P_i$ with a distance $d_i$ from the axis.

③ Two unknowns; $\rho_i, d_i$

# Line Charge and Parallel Conducting Cylinder

④ Assume $\boxed{\rho_i = -\rho_l}$ $\Rightarrow$ intelligent guess.

$\rightarrow$ proceed to check if this fails to satisfy the B.C. or not.

⑤ If this satisfy all B.C.'s, then it is the only solution by the uniqueness theorem.

⑥ Electric field intensity and potential at a distance $r$ from a line charge $\rho_l$

$$\vec{E} = \frac{\rho_l}{2\pi\varepsilon_0 r}\hat{r}$$

$$V = -\int_{r_0}^{r} E_r dr = -\frac{\rho_l}{2\pi\varepsilon_0}\int_{r_0}^{r}\frac{1}{r}dr = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r}$$

note) reference point for zero potential $r_0$ can not be $\infty$, since potential is finite.

# Line Charge and Parallel Conducting Cylinder

⑦ The potential at a point on or outside the cylindrical surface:

superposition of contributions by $\rho_l$ and $\rho_i$, at a point $M$

$$V_M = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r} - \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_0}{r_i} = \frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_i}{r}$$

where reference point for zero potential is at the same distance from $\rho_l$ and $\rho_i$

⑧ equipotential surface are specified by

$$\frac{r_i}{r} = \text{constant}$$

if an equipotential surface is to coinside with the cylindrical surface ($\overline{\text{OM}}=a$)

# Line Charge and Parallel Conducting Cylinder

cf) If $\frac{d_i}{a}$ and $\frac{a}{d}$ are constant and $\angle$MOP$_i$ common, then $\frac{r_i}{r}$=constant.

Since $\Delta$OMP and $\Delta$OMP$_i$ is similar,

$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{constant} \quad (\because a,\ d,\ d_i \text{ fixed})$$

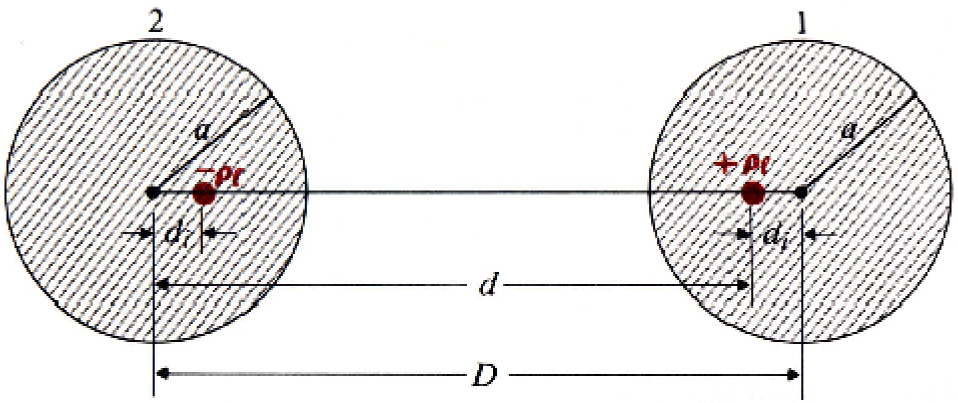
$\frac{r_i}{r}$ can be constant over the cylindrical surface then, $P_i = \frac{a}{d} \rightarrow$ Inverse point

cf) As the point M moves along the cylindrical surface, $r$ and $r_i$ changes but their ratio is constant

cf) By symmetry, parallel cylindrical surface surrounding the original line charge $\rho_l$ with radius $a$ and its axis at a distance $d_i$ to the right of $P$ is also equipotential surface.

# Ex 4-4) Cross section of two-wire transmission line

✓ Determine the capacitance per unit length between two long parallel, circular conducting wires of radius $a$. the axes of the wire are seperated by a distance $D$.



# Ex 4-4) Cross section of two-wire transmission line

① Conducting surface $\rightarrow$ equipotential surfaces

$\therefore$ equipotential surfaces can be generated by two parallel line charge

$+\rho_l, \ -\rho_l$ seperated by a distance $(D-2d_i)=d-d_i$.

② Potential on 1

$$V_1=-\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d}$$ positive quantity.

Potential on 2

$$V_2=\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{a}{d}$$ negative quantity. because (a<d)

(cf $V_2=\frac{\rho_l}{2\pi\varepsilon_0}\ln\frac{r_i}{r}$) , $\frac{r_i}{r}=\frac{a}{d}=const.$

if $r_0$ is at the same distance from both $\rho_l$ and $-\rho_l$

# Ex 4-4) Cross section of two-wire transmission line

$$\therefore\ V_{12} = \frac{\rho_l}{\pi\varepsilon_0}\ln\frac{d}{a}$$

$Q$ per unit length = $\rho_l$

$$\therefore\ C \text{ (per unit length)} = \frac{Q}{V_{12}} = \frac{\pi\varepsilon_0}{\ln(d/a)}\ [F/m]$$

$$d = D - d_i = D - \frac{a^2}{d},$$

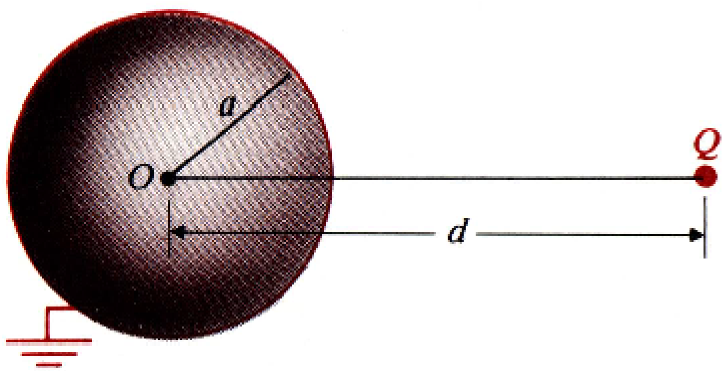
$$d = \frac{1}{2}(D \pm \sqrt{D^2 - 4a^2})$$

choose only +sign $(\because D, d \gg a)$

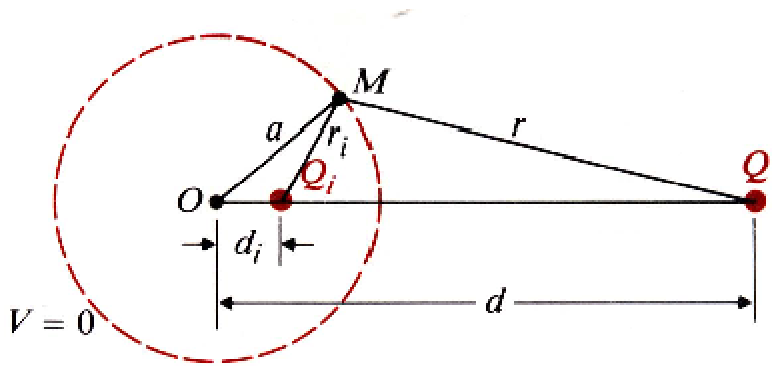
$$\therefore\ C = \frac{\pi\varepsilon_0}{\ln\left[(D/2a) + \sqrt{(D/2a)^2 - 1}\right]} = \frac{\pi\varepsilon_0}{\cosh^{-1}(D/2a)}$$

cf) $\ln\left[x + \sqrt{x^2 - 1}\right] = \cosh^{-1} x$

# Point Charge and Conducting Sphere




(a) Point charge and grounded conducting sphere.




(b) Point charge and its image.

① Considering symmetry, assume the image charge $Q$ (negative point charge) inside the sphere and on the line joining O and $Q$.
distance $d_i$ from origin O

② $Q_i \neq -Q$ to make the spherical surface $R = a$ a zero-potential surface.

③ both $d_i$ and $Q_i$ : unknowns

# Point Charge and Conducting Sphere

④ At an arbitrary point M on the equipotential surface

$$V_M = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{Q_i}{r_i}\right) = 0$$

$$\therefore \frac{r_i}{r} = -\frac{Q_i}{Q} = const, \qquad \therefore -\frac{Q_i}{Q} = \frac{a}{d}$$

$$\therefore \boxed{Q_i = -\frac{a}{d}Q, \quad d_i = \frac{a^2}{d}}$$

$\vec{E}$, and $V$ can be calculated from two point charge $Q$, $-\frac{a}{d}Q$

# Boundary value problem

✓ Develop a method for solving three-dimensional problems where the boundaries, over which the potential or its normal derivative is specified, coincide with the coordinate surfaces of an orthogonal, curvilinear system

✓ The method of seperation of variables

✓ Boundary value problems

① Dirichlet problems : the value of potential is specified everywhere on the boundaries

② Neumann problems :
the normal derivative of the potential is specified everywhere on the boundaries

③ Mixed boundary-value problems :
the potential is specified over the remaining ones.

# Cartesian coordinate

Laplace's equation for scalar electric potential V in Cartesian coordinates

✓ $$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

Assume $V(x,y,z)=X(x)Y(y)Z(z),$ where $X(x),Y(y),Z(z)$ are functions of only $x, y,$ and $z,$respectively. Then,

$$Y(y)Z(z)\frac{d^2X(x)}{dx^2}+X(x)Z(z)\frac{d^2Y(y)}{dy^2}+X(x)Y(y)\frac{d^2Z(z)}{dz^2}=0$$

- divided by $V(x,y,z),$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}+\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=0$$

$$\therefore \frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=-k_x^2 \qquad \because \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}$$ is independent of $x$

$$\frac{d^2X(x)}{dx^2}+k_x^2X(x)=0, \quad \frac{d^2Y(y)}{dy^2}+k_y^2Y(y)=0, \quad \frac{d^2Z(z)}{dz^2}+k_z^2Z(z)=0$$

$k_x, k_y, k_z$ : separation constant

# Cartesian coordinate

- Reminds
  - Possible solution of $X''(x) + k_x^2 X(x) = 0$

| $k_x^2$ | $k_x$ | $X(x)$ | Exponential forms† of $X(x)$ |
|---|---|---|---|
| 0 | 0 | $A_0 x + B_0$ | |
| + | $k$ | $A_1 \sin kx + B_1 \cos kx$ | $C_1 e^{jkx} + D_1 e^{-jkx}$ |
| − | $jk$ | $A_2 \sinh kx + B_2 \cosh kx$ | $C_2 e^{kx} + D_2 e^{-kx}$ |

- $A_0, B_0, A_1, B_1, A_2, B_2$ can be determined by the given boundary condition