Field and Wave Electromagnetic

Chapter.5

Steady Electric Currents

Introduction

- ✓ Steady currents
  - Conduction current
  - Electrolytic current
  - Convection current

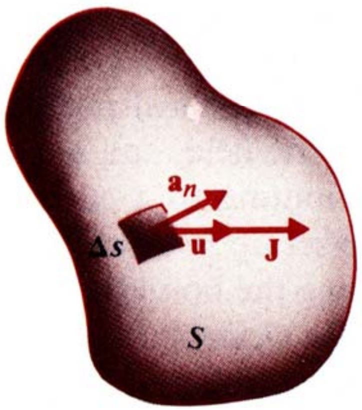
Current Density and Ohm's Law

N : number of charge carrier / unit volume

$\vec{u}$: velocity

① During time interval Δt, distance of charge carrier movement : $\vec{u}\,\Delta t$

② The amount of charge passing through the surface area Δs



$$\Delta Q = Nq\vec{u} \bullet \hat{n} \Delta s \Delta t$$

$$\therefore\ \Delta I = \frac{\Delta Q}{\Delta t} = Nq\vec{u} \bullet \hat{n} \Delta s = \vec{J} \cdot \overrightarrow{\Delta s}$$

and $\vec{J} = Nq\vec{u}$,where $\rho = Nq$: volume charge density

$$I = \int_s \vec{J} \cdot \overrightarrow{ds}$$

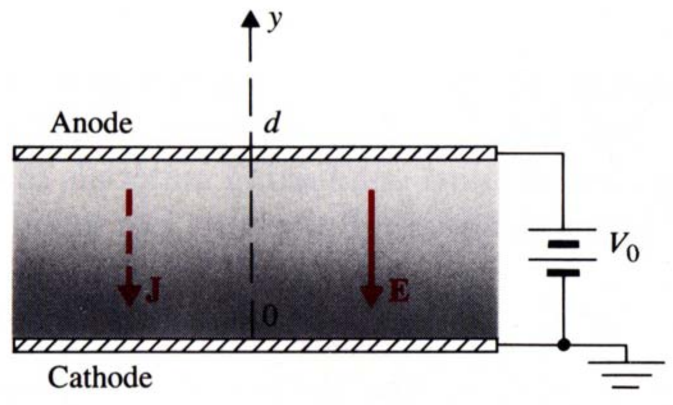
cf) $Nq = \rho$

then, $\vec{J} = \rho\vec{u}$: Convection Current density

Ex 5-1) Space-charge-limited vacuum diode

- Electrons are emitted from a cathode.
- $V = 0$ at cathode.
- Electrons are collected by an anode.
- $V = V_0$ at anode.
- Electrons at cathode has a zero initial velocity.

Find the relation between $\vec{J}$ and V_0.



Ex 5-1) Space-charge-limited vacuum diode

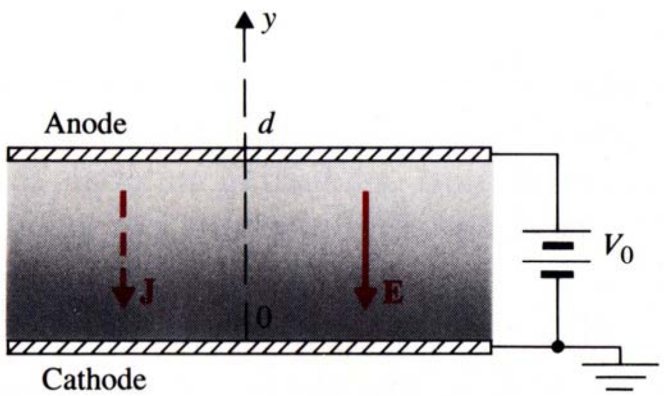
sol)

① $\vec{E}(0) = \hat{y}E_y(0) = -\hat{y}\left.\frac{dV(y)}{dy}\right|_{y=0} = 0$

② In the steady state, the current density is cc
and independent of y

$$\vec{J} = -\hat{y}J = \hat{y}\rho(y)u(y)$$

where, $\rho(y) < 0$ ($\because$ electrons)



③ $\vec{u}(y)$ and $\vec{E}(y)$ is governed by Newton's law.

$$m \cdot \frac{du(y)}{dt} = -eE(y) = e\frac{dV(y)}{dy}$$

$m = 9.11\times10^{-31}$ (kg), $e = -1.6\times10^{-19}$ (C)

Ex 5-1) Space-charge-limited vacuum diode

$$m\frac{du}{dt} = m\frac{du}{dy}\frac{dy}{dt} = mu\frac{du}{dy} = \frac{d}{dy}\left(\frac{1}{2}mu^2\right)$$

$$\therefore \quad \frac{d}{dy}\left(\frac{1}{2}mu^2\right) = e\frac{dV}{dy}$$

Integrating both side with a given boundary condition , $u(0)=0, V(0)=0$ at $y=0$

$$\frac{1}{2}mu^2 = eV$$

$$u = \left(\frac{2e}{m}V\right)^{\frac{1}{2}}$$: analogy to free fall motion in the gravitational field

Ex 5-1) Space-charge-limited vacuum diode

To find $V(y)$, we have to solve Poisson's equation with ρ expressed in terms of $V(y)$

$$J = -\rho u, \qquad (\rho < 0)$$

$$\rho = -\frac{J}{u} = -J\sqrt{\frac{m}{2e}}V^{-\frac{1}{2}}$$

✓ Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\therefore \frac{d^2V}{dy^2} = -\frac{\rho}{\varepsilon_0} = \frac{J}{\varepsilon_0}\sqrt{\frac{m}{2e}}V^{-1/2}$$

Ex 5-1) Space-charge-limited vacuum diode

cf)
$$\begin{cases} \dfrac{d^2V}{dy^2}\cdot\left(2\dfrac{dV}{dy}\right) = K\cdot V^{-\frac{1}{2}}\cdot 2\dfrac{dV}{dy} \\ \dfrac{d}{dy}\left[\left(\dfrac{dV}{dy}\right)^2\right] = \dfrac{d}{dy}\left[K\cdot 4V^{\frac{1}{2}}\right] \\ \therefore \left(\dfrac{dV}{dy}\right)^2 = 4K\cdot V^{\frac{1}{2}} + c \end{cases}$$

$$\therefore \left(\frac{dV}{dy}\right)^2 = \frac{4J}{\varepsilon_0}\sqrt{\frac{m}{2e}}V^{\frac{1}{2}} + c$$

Ex 5-1) Space-charge-limited vacuum diode

B.C. at $y=0,\ V=0,\ \dfrac{dV}{dy}=0$ (zero initial velocity)

$$\therefore\quad c=0$$

$$\therefore\quad \frac{dV}{dy}=2\sqrt{\frac{J}{\varepsilon_0}}\cdot\left(\frac{m}{2e}\right)^{\frac{1}{4}}V^{\frac{1}{4}}$$

$$\therefore\quad V^{-\frac{1}{4}}dV=2\sqrt{\frac{J}{\varepsilon_0}}\left(\frac{m}{2e}\right)^{\frac{1}{4}}dy$$

$$\therefore\frac{4}{3}V_0^{\frac{3}{4}}=2\sqrt{\frac{J}{\varepsilon_0}}\left(\frac{m}{2e}\right)^{\frac{1}{4}}d.$$ B.C. $\begin{cases}\text{at } y=0,\ V=0\\ \text{at } y=d,\ V=V_0\end{cases}$

$$\therefore J=\frac{4\varepsilon_0}{9d^2}\sqrt{\frac{2e}{m}}V_0^{\frac{3}{2}}$$ $\Rightarrow$ Child Langmuir law

Current Density and Ohm's Law

✓ In case of conduction currents.

- electrons, holes

$$J = \sum_i N_i q_i \vec{u_i}$$

- Atoms remain neutral
- Average drift velocity is directly proportional to the electric field intensity
- metallic conductors

$$\vec{u} = -\mu_e \vec{E}$$: velocity, μ_e : mobility ($m^2/V \cdot S$)

Current Density and Ohm's Law

cf) μ_e : 3.2×10^{-3} for Copper

: 1.4×10^{-4} for Al

: 5.2×10^{-3} for Silver

$\vec{J} = -\rho_e \mu_e \vec{E}$, where $\rho_e = -Ne$: charge density of drifting electrons.

Point function ← $\boxed{J = \sigma E}$, $\boxed{\sigma = -\rho_e \mu_e}$ → conductivity for metallic conductor

cf) $\boxed{\sigma = -\rho_e \mu_e + \rho_h \mu_h} \Rightarrow$ for semiconductor

cf) In general $\mu_e \neq \mu_h$, for germanium, $\mu_e = 0.38$, $\mu_h = 0.18$

(electrons, holes)

for silicon, $\mu_e = 0.12$, $\mu_h = 0.03$

Current Density and Ohm's Law

- Conductivity
  - Isotropic materials obeying, $\vec{J} = \sigma \vec{E}$: ohmic media
  - Conductivity

$$\sigma : \left(\text{A}/\left(\text{V}\cdot\text{m}\right)\right) \text{ or Simens per meter } \left(\text{S/m}\right)$$

$$\text{cf) } \sigma \text{ is } \begin{cases} \text{for copper : } 5.80\times10^{7}\left(\text{S/m}\right) \\ \text{for germanium : } 2.2 \left(\text{S/m}\right) \\ \text{for silicon : } 16\times10^{-3}\left(\text{S/m}\right) \\ \text{for Hard rubber : } 10^{-15}\left(\text{S/m}\right) \end{cases}$$

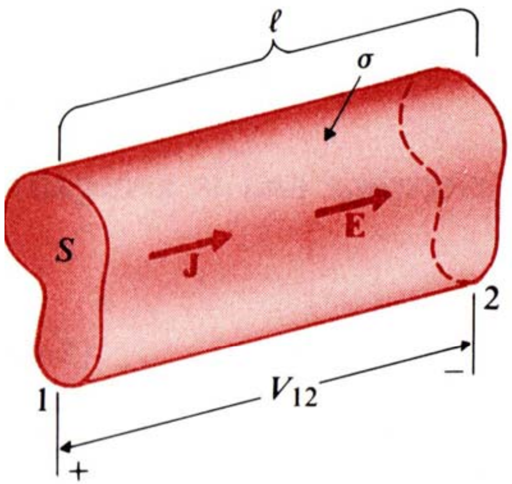
Current Density and Ohm's Law

✓ Ohm's law

$$\vec{J} = \sigma \vec{E} \Rightarrow$$ point form of Ohm's law

cf) $V_{12} = RI \Rightarrow$ Not a point relation

- Assume a piece of homogeneous conducting material with conductivity σ, length l, uniform crosssection S.



$$V_{12} = El \quad \Rightarrow \quad E = \frac{V_{12}}{l}$$

$$I = \int_s \vec{J} \cdot \vec{ds} = JS, \quad J = \frac{I}{S}$$

$$\frac{I}{S} = \sigma \cdot \frac{V_{12}}{l}$$

$$\therefore \; V_{12} = \left(\frac{l}{\sigma S} \right) I = R I$$

$$R = \frac{l}{\sigma S} = \rho \cdot \frac{l}{S}$$: Resistance

Electromotive Force and Kirchhoff's Voltage Law

✓ $\oint_C \vec{E}\cdot d\vec{l} = 0 \Rightarrow$ Static E-field is irrotational (i.e. Conservative)

For an ohmic material, $\vec{J} = \sigma\vec{E}$

$\therefore \oint_C \frac{1}{\sigma}\vec{J}\cdot d\vec{l} = 0$: Steady current cannot be maintained in the same direction in a closed circuit by an electrostatic field.

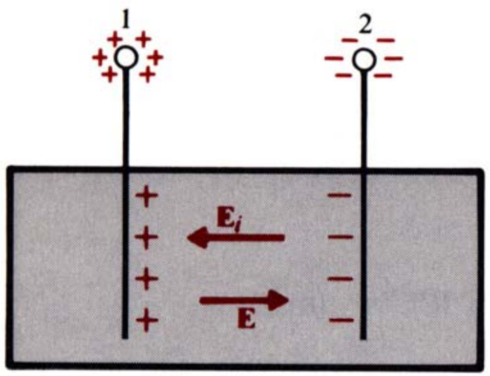
- To have steady current, nonconservative field should supply the energy which will be dissipated by collision
- The source of nonconservative field : chemical energy, generators.
- Impressed electric field intensity

$\Downarrow$

driving forces for charge carrier

$\Downarrow$

Equivalent impressed electric field intensity $\vec{E}_i$



Electric battery

Electromotive Force and Kirchhoff's Voltage Law

$\vec{E}_i$ is produced by chemical action

$(E_i$: impressed electric field intensity$)$

$\vec{E}_i = -\vec{E}$ $\because$ (No current flows in the open-circuited battery

$\Rightarrow$ the net force acting on the charge carriers must vanish

$$\int_2^1 \vec{E}_i \cdot \vec{dl} = -\underbrace{\int_2^1 \vec{E} \cdot \vec{dl}}_{\text{Inside the source}} = \mathcal{V}$$: electromotive force

$\Rightarrow$ driving forces for charge carrier

✓ Electro motive force : The line integral of the impressed field intensity $\vec{E}_i$ from the negative to positive electrode inside the battery

Electromotive Force and Kirchhoff's Voltage Law

✓ Conservative electrostatic field

$$\oint_C \vec{E}\cdot\vec{dl} = \int_1^2 \vec{E}\cdot\vec{dl} + \int_2^1 \vec{E}\cdot\vec{dl} = 0$$

outside the source   inside the source

$$\mathcal{V} = \int_1^2 \vec{E}\cdot\vec{dl} = V_{12} = V_1 - V_2$$: voltage rise

outside the source

With a resistor connecting two terminals, the total electric field intensity must $(\vec{E}+\vec{E}_i)$ be used in the point form of Ohm's law

$$\vec{J} = \sigma(\vec{E}+\vec{E}_i),$$

$\vec{E}_i$ exist inside the battery only, $\vec{E}$ exist both inside and outside the battery

Electromotive Force and Kirchhoff's Voltage Law

$$\vec{E} + \vec{E}_i = \frac{J}{\sigma}, \quad \mathcal{V} = \oint_C \left(\vec{E} + \vec{E}_i\right) \cdot \vec{dl} = \oint_C \frac{1}{\sigma} \vec{J} \cdot \vec{dl} = RI$$

cf) no source of nonconservative field $\oint_C \frac{1}{\sigma} \vec{J} \cdot \vec{dl} = 0$

$$\sum_j \mathcal{V}_j = \sum_k R_k I_k$$: Kirchhoff's voltage law

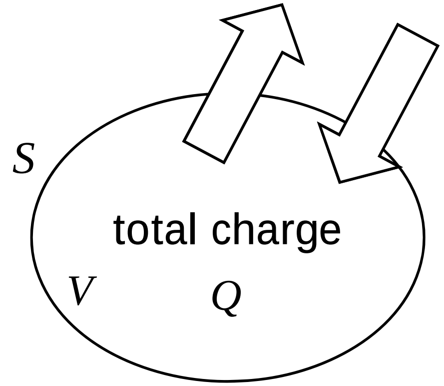
→ Around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebric sum of the voltage drops across the resistances.

Equation of Continuity and Kirchhoff's current law

✓ The principle of conservation of charge

$$I = \oint_S \vec{J} \cdot \vec{ds} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho \, dv$$

total outward flux $\Rightarrow$ Decrease in total charge inside



➢ Applying divergence theorem, and assuming the stationary volume

$$\int_V \nabla \cdot \vec{J} \, dv = -\int_V \frac{\partial \rho}{\partial t} dv,$$

(taking partial derivative in the integral since ρ may be a function of time as well as space)

$$\therefore \boxed{\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}}$$: continuity equation $\Rightarrow$ point relationship

Steady current $\Rightarrow$ Charge density does not vary with time : i.e. $\frac{\partial \rho}{\partial t} = 0$

$$\therefore \; \nabla \cdot \vec{J} = 0, \quad I_{out} = I_{in}$$

Equation of Continuity and Kirchhoff's current law

In other words,

Steady current = Divergenceless = Solenoidal

$\Rightarrow$ The field lines (or stream lines) of steady current close upon themselves.

i.e. $\oint_S \vec{J} \cdot \vec{ds} = 0 \Rightarrow \sum_j I_j = 0 \Rightarrow$ Kirchoff's current law

cf) Relaxation time

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}, \quad \text{and} \quad \vec{J} = \sigma \vec{E}$$

$$\sigma \nabla \cdot \vec{E} = -\frac{\partial \rho}{\partial t}, \quad \text{and} \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \quad \text{in a simple medium}$$

$$\therefore \quad \frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon} \rho = 0$$

$$\therefore \quad \rho = \rho_0 e^{-\left(\frac{\sigma}{\varepsilon}\right)t},$$

where $\rho_0 =$ initial charge density at $t = 0$, relaxation time $\tau = \frac{\varepsilon}{\sigma}$

ex) copper : σ = 5.80×10^7, $\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}$, $\tau = 1.52 \times 10^{-19}$

Power Dissipation and Joule's law

✓ Macroscopic observation

$\vec{E} \Rightarrow$ drift motion of conduction electrons $\Rightarrow$ electrons collide with atoms

$\Rightarrow$ vibration of lattice

i.e. Electric field energy $\Rightarrow$ Thermal vibration

The work Δw by $\vec{E}$ on a charge q

$$\Delta w = q\vec{E} \cdot \Delta \vec{l}$$

$\therefore$ Power $p = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = q\vec{E} \cdot \vec{u}$ where, $\vec{u}$: drift velocity

Total power in a volume dv

$$dP = \sum_i p_i = \vec{E} \cdot \left(\sum_i N_i q_i \vec{u}_i \right) dv = \vec{E} \cdot \vec{J}\, dv$$

or $\frac{dP}{dv} = \vec{E} \cdot \vec{J}$ $\left(\text{W/m}^3\right)$: Power density under steady current conditions

For a given volume v

$$P = \int_V \vec{E} \cdot \vec{J}\, dv \quad \text{(W)} \quad : \text{Joule's law}$$

Power Dissipation and Joule's law

cf) Special case

Conductor : Constant cross section

$dv = ds\,dl,\ dl$ measured in the direction $\vec{J}$

$$P = \int_L E\,dl \int_s J\,ds = VI = I^2 R \text{ (W)}$$

cf) $V = -\int E\,dl$

note: Justification of $\vec{E}$ in a conductor

① Voltage rise : Source of nonconservative field

② Not P.E.C : relaxation time

③ Steady current driven by non conservation source $\Rightarrow -\dfrac{\partial \rho}{\partial t} = 0$

④ resistance $\Rightarrow$ finite , P.E.C $\Rightarrow \sigma = \infty$

Boundary Conditions for Current Density

For steady current,

$\nabla \cdot \vec{J} = 0, \ \oint_S \vec{J} \cdot \vec{ds} = 0$	: continuity equation
$\nabla \times \left(\dfrac{\vec{J}}{\sigma} \right) = 0, \ \oint_C \dfrac{1}{\sigma} \vec{J} \cdot \vec{dl} = 0$	: conservative electrostatic field

① The normal component of a divergenceless vector field is continuous

$$\therefore J_{1n} = J_{2n}$$

② The tangential component of a curl-free vector field is continuous across an interface

$$\frac{J_{1t}}{\sigma_1} = \frac{J_{2t}}{\sigma_2} \quad \text{or} \quad \boxed{\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}}$$

③ Boundary conditions between two different lossy dielectrics

(permitivities ε_1 and ε_2, finite conductivity σ_1 and σ_2)

- The tangential component of the electric field

$$E_{2t} = E_{1t} \qquad \left(\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2} \right)$$

Boundary Conditions for Current Density

- Normal component of the electric field

$$J_{1n} = J_{2n} \Rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n}$$

$$D_{1n} - D_{2n} = \rho_s$$

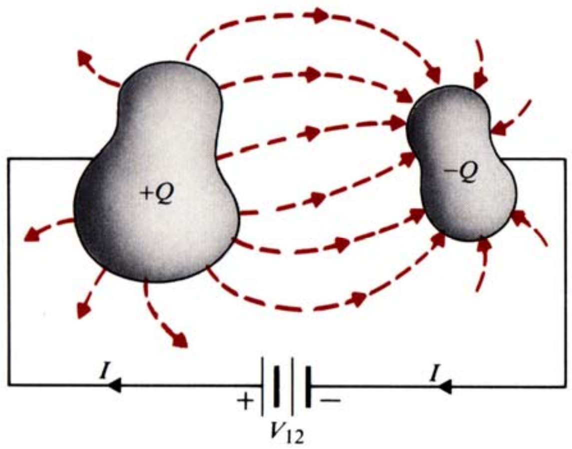
(with the reference with normal $\widehat{n_2}$ from medium 2 to medium 1)

$$\Rightarrow \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$$

$$\therefore \rho_s = \left(\varepsilon_1 \frac{\sigma_2}{\sigma_1} - \varepsilon_2 \right) E_{2n} = \left(\varepsilon_1 - \varepsilon_2 \frac{\sigma_1}{\sigma_2} \right) E_{1n}$$

if $\frac{\sigma_2}{\sigma_1} = \frac{\varepsilon_2}{\varepsilon_1}$, then $\rho_S = 0$

Resistance Calculation



✓ $C = \frac{Q}{V} = \frac{\oint_S \vec{D}\cdot\vec{ds}}{-\int_L \vec{E}\cdot\vec{dl}} = \frac{\oint \varepsilon\vec{E}\cdot\vec{ds}}{-\int_L \vec{E}\cdot\vec{dl}}$

When the dielectric medium is lossy, a current will flow from the positive to the negative conductor.

$\Rightarrow \vec{J} = \sigma\vec{E}$ and $\vec{J}$ and $\vec{E}$ are in the same direction

$$\therefore R = \frac{V}{I} = \frac{-\int_L \vec{E}\cdot\vec{dl}}{\oint_S \vec{J}\cdot\vec{ds}} = \frac{-\int_L \vec{E}\cdot\vec{dl}}{\oint \sigma\vec{E}\cdot\vec{ds}} \Rightarrow RC = \frac{C}{G} = \frac{\varepsilon}{\sigma}$$