Field and Wave Electromagnetic

Chapter 6

Static Magnetic field

Introduction

- Experimental law
  - A small test charge in an electric field $\vec{E}$

    $\Rightarrow$ Electric force , $\vec{F_e} = q\vec{E}\ [N]$
  - A small test charge in motion in a magnetic field $\vec{H}$

    $\Rightarrow$ Magnetic force, $\vec{F_m} = q\vec{u} \times \vec{B}\ [N]$

    where, $\vec{u}$: velocity of test charge, $\vec{B}$: Magnetic flux density $[\frac{Wb}{m^2}$ or $T]$

    $cf)$ 1 tesla $= 10^4\, gauss$, weber = volt•second
  - Total electromagnetic force $= \vec{F_e} + \vec{F_m}$

    $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})\ [N]$: Lorentz's force equation

    $\Rightarrow$ Fundamental postulate of electromagnetic model

Introduction

- ✓ Two postulate
  - $\nabla \bullet \vec{B}$ $\Rightarrow$ Solenoidal characteristic($\nabla \bullet \vec{B} = 0$) $\Rightarrow$ vector magnetic potential
    $\Rightarrow$ vector poisson's equation
  - Biot-Savart law(vector potential)
  - $\nabla \times \vec{B} = \mu_0 \vec{J} \Rightarrow$ Ampere's circuital law
  - Magnetization vector $\vec{M}$
    cf) polarization $\vec{P}$
  - Magnetic field intensity $\vec{H}$ and permeability
    cf) permittivity
  - Boundary condition

Fundamental postulates of magnetics in free space

$\nabla \bullet \vec{B} = 0$

$\nabla \times \vec{B} = \mu_0 \vec{J}$, $\mu_0 = 4\pi \times 10^{-7}$: permeability in free space, $\vec{J}$: current density

$cf)\nabla \bullet (\nabla \times \vec{B}) = \mu_0 \nabla \bullet \vec{J} = 0$ $\quad i.e)\ \nabla \bullet \vec{J} = 0,\ \vec{J}$ is steady current (If $\nabla \bullet \vec{J} \neq 0,$ then ?)

1. $\nabla \bullet \vec{B} = 0 \Rightarrow \oint_S \vec{B} \bullet \vec{ds} = 0$: no magnetic flow source

   $i.e)$ the magnetic flux lines always close upon themselves

2. $\nabla \times \vec{B} = \mu_0 \vec{J} \Rightarrow \int_S (\nabla \times \vec{B}) \bullet \vec{ds} = \mu_0 \int_S \vec{J} \bullet \vec{ds}$ where S is the an open surface.

   $\boxed{\oint_c \vec{B} \bullet \vec{dl} = \mu_0 I}$ → follow the right hand rule

Ampere's circuital law

$\Rightarrow$ the circulation of the magnetic flux density in free space around any closed path is equal to μ_0 times the total current flowing through the surface bounded by the path

Ex 6-1) Magnetic flux density of an infinitely long circular conductor

✓ steady current, $I \Rightarrow J = \dfrac{I}{\pi b^2}$ (uniformly flow)

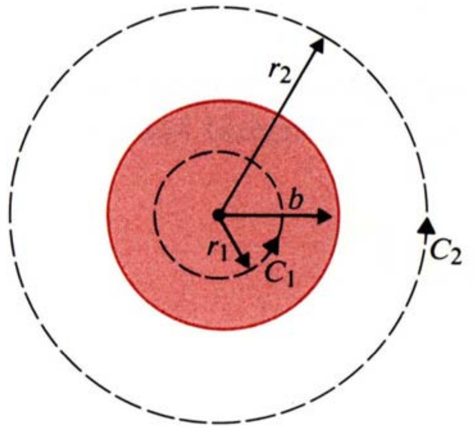
sol) ➢ Cylindrical symmetry

➢ I flows along z

➢ $\vec{B} = B_\phi \hat{\phi}$

➢ assume circular path

➢ remember the right hand rule



$$a)\ r<b,\quad \vec{B}=B_\phi\hat{\phi},\ \vec{dl}=rd\phi\hat{\phi},\quad \oint_{C_1}\vec{B}\bullet\vec{dl}=\int_0^{2\pi}B_\phi r d\phi = 2\pi r B_\phi$$

$$\mu_0\int_S \vec{J}\bullet\vec{ds}=\mu_0\int_0^r\int_0^{2\pi}\frac{I}{\pi b^2}\hat{z}\bullet rdrd\phi\hat{z}=\mu_0\frac{I}{\pi b^2}\int_0^r\int_0^{2\pi}rdrd\phi=\mu_0\frac{I}{\pi b^2}\left(\frac{1}{2}r^2 2\pi\right)=\mu_0 I\left(\frac{r}{b}\right)^2$$

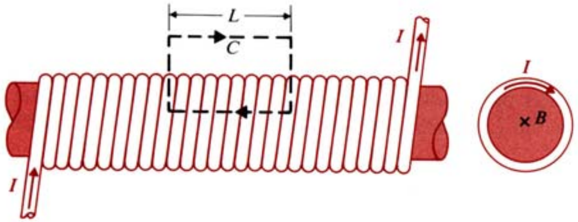
$$\therefore 2\pi r B_\phi=\mu_0 I\left(\frac{r}{b}\right)^2 \rightarrow B_\phi=\frac{\mu_0}{2\pi}\frac{r}{b^2}I \qquad \therefore \vec{B}=B_\phi\hat{\phi}=\frac{\mu_0 I}{2\pi b^2}r\hat{\phi}\ ,\ \text{for } r<b$$

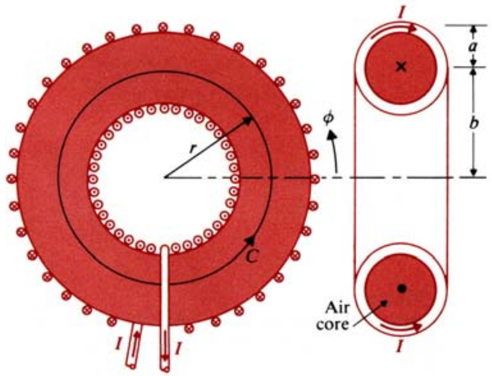
$$b)\ r>b,\quad \oint_{C_1}\vec{B}\bullet\vec{dl}=2\pi r B_\phi,\quad \mu_0\int_S\vec{J}\bullet\vec{ds}=\mu_0 I$$

$$\therefore 2\pi r B_\phi=\mu_0 I \rightarrow B_\phi=\frac{\mu_0 I}{2\pi r} \qquad \therefore \vec{B}=B_\phi\hat{\phi}=\frac{\mu_0 I}{2\pi r}\hat{\phi}\ ,\ \text{for } r>b$$

Ex 6-3) A current-carrying long solenoid

✓ Solenoid (infinitely long solenoid, n turns/unit length,current I)





sol a) no field outside solenoid $\Rightarrow$ Ampere's law

$\rightarrow$ no current inside closed path $\rightarrow \vec{B} = 0.$

symmetry, $\vec{B}$ inside is parallel($||$) to axis

Ampere's law $BL = \mu_0 nLI \rightarrow \therefore B = \mu_0 nI$

b) A special case of toroid

➢ inside toroid,

$$\oint \vec{B} \bullet \vec{dl} = 2\pi r B_\phi = \mu_0 NI \begin{cases} B_\phi = \dfrac{\mu_0 NI}{2\pi r}, & \text{for } b-a<r<b+a \\ B_\phi = 0, & \text{for } r<b-a \text{ and } r>b+a \end{cases}$$

if $b = \infty$, then B_ϕ will be constant inside toroid. $B_\phi = \dfrac{\mu_0 NI}{2\pi b} = \mu_0 \dfrac{N}{2\pi b} I = \mu_0 nI$

vector magnetic potential

① $\nabla \bullet \vec{B} = 0 \Rightarrow \vec{B}$ is solenoidal

② $\nabla \bullet (\nabla \times \vec{A}) = 0$, from vector identity

③ $\vec{A}$: vector magnetic potential $[\frac{Wb}{m}]$ $cf)\ \vec{E} = -\nabla V$,analogy $\vec{B} = \nabla \times \vec{A}$

☞ $\vec{A}$ can be defined by its curl and divergence, then $\nabla \bullet \vec{A}$?

$\Rightarrow$ We have a freedom to choose $\nabla \bullet \vec{A}$ for our convenience

$cf)$ $\vec{E}$ and $\vec{B}$ are defined by both curl and divergence

④ what is left? $\Rightarrow \nabla \times \vec{B} = \mu_0 \vec{J}$

☞ $\nabla \times \nabla \times \vec{A} = \mu_0 \vec{J}$, where $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \bullet \vec{A}) - \nabla^2 \vec{A}$ or $\nabla^2 \vec{A} = \nabla(\nabla \bullet \vec{A}) - \nabla \times \nabla \times \vec{A}$

$cf)\ \nabla^2 \vec{A} = \hat{x}\nabla^2 A_x + \hat{y}\nabla^2 A_y + \hat{z}\nabla^2 A_z$, but not true for other coordinate systems

vector magnetic potential

Coulomb gauge or Coulomb condition

✓ $\nabla(\nabla\bullet\vec{A})-\nabla^2\vec{A}=\mu_0\vec{J}$

choose $\boxed{\nabla\bullet\vec{A}=0}$ to simplify the equation

$cf)$ Lorentz gauge(Lorentz condition), $\boxed{\nabla\bullet\vec{A}+\mu\varepsilon\frac{\partial V}{\partial t}=0}$

$\therefore \boxed{\nabla^2\vec{A}=-\mu_0\vec{J}} \Rightarrow$ vector poisson's equation

- In cartesian coordinates,

$$\nabla^2 A_x=-\mu_0 J_x,\ \nabla^2 A_y=-\mu_0 J_y,\ \nabla^2 A_z=-\mu_0 J_z$$

$\Rightarrow$ The same form as the scalar poisson's equations,

$$\nabla^2 V=-\frac{\rho}{\varepsilon_0} \Rightarrow V=\frac{1}{4\pi\varepsilon_0}\int_{V'}\frac{\rho}{R}dv'$$

$cf)$ ρ should be known. If ρ is not known, but B.C. is known,
then we should select poisson's equation.

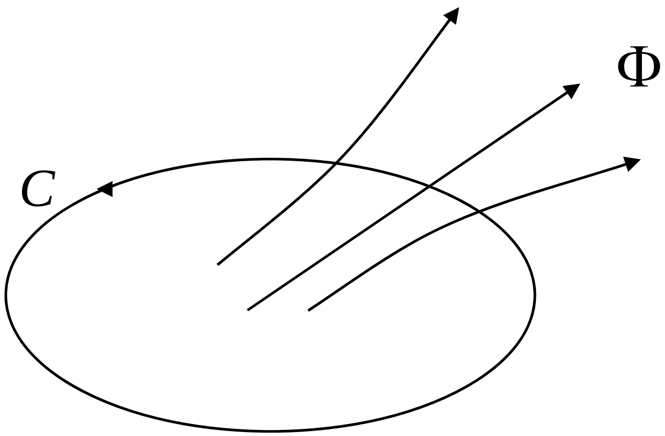
vector magnetic potential

- Using the analogy,

$$A_x = \frac{\mu_0}{4\pi}\int_{V'} \frac{J_x}{R} dv' \qquad \therefore\ \vec{A} = \frac{\mu_0}{4\pi}\int_{V'} \frac{\vec{J}}{R} dv' \ [\frac{Wb}{m}]$$

- ✓ Magnetic flux, Φ

$$\Phi = \int_S \vec{B} \bullet \vec{ds} = \int_S (\nabla \times \vec{A}) \bullet \vec{ds} = \oint_C \vec{A} \bullet \vec{dl} \quad [Wb]$$



The Biot-Savart law and application

- ✓ Assume constant cross sectional area S, $\vec{J}dv' = JS\vec{dl'} = I\vec{dl'}$
  - ➢ $\vec{A} = \dfrac{\mu_0 I}{4\pi}\oint_{C'}\dfrac{\vec{dl'}}{R}$, since the current I must flow in a closed path

w.r.t the space coordinate of the field point (observation point)

w.r.t source coordinate

- ➢ $\vec{B} = \nabla\times\vec{A} = \nabla\times\dfrac{\mu_0 I}{4\pi}\oint_{C'}\dfrac{\vec{dl'}}{R} = \dfrac{\mu_0 I}{4\pi}\oint_{C'}\nabla\times\dfrac{\vec{dl'}}{R}$, $\quad cf)\ \nabla\times(f\vec{G}) = f\nabla\times\vec{G} + (\nabla f)\times\vec{G}$

$$= \frac{\mu_0 I}{4\pi}\oint_{C'}\cancel{\frac{1}{R}\underbrace{\left(\nabla\times\vec{dl'}\right)}_{=0}} + \left(\nabla\frac{1}{R}\right)\times\vec{dl'} = \frac{\mu_0 I}{4\pi}\oint_{C'}\left(\nabla\frac{1}{R}\right)\times\vec{dl'}, \qquad cf)\nabla\left(\frac{1}{R}\right) = -\frac{\vec{R}}{R^3} = -\hat{R}\frac{1}{R^2}$$

$$\therefore\ \vec{B} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{-\hat{R}}{R^2}\times\vec{dl'} \Rightarrow \boxed{\vec{B} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{\vec{dl'}\times\hat{R}}{R^2}\ [T]\ \text{ : Biot-Savart law}}$$

$$cf)\ \vec{B} = \oint_{C'} d\vec{B} \quad \text{and} \quad d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{\vec{dl'}\times\hat{R}}{R^2}$$

Biot-Savart law : A formula to determine $\vec{B}$ caused by a current I in a closed path C'

The Biot-Savart law and application

Comment

- $\vec{B}$ can be found from both Biot-Savart law or Ampere's law
- but Ampere's law can only be used in case that we have a path over which $\vec{B}$ is constant(by some symmetry)

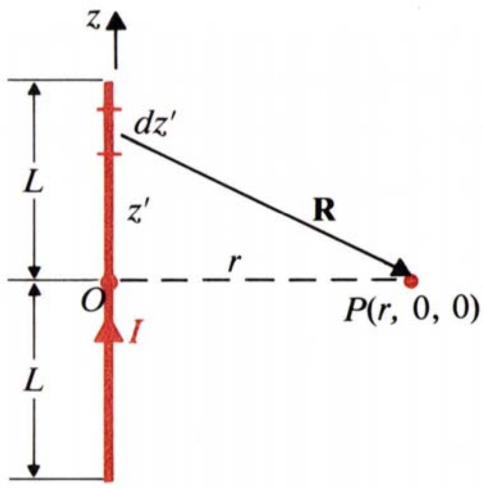
Ex 6-4) two different methods

sol $a)$ $\vec{A} = \hat{z}\frac{\mu_0 I}{4\pi}\int_{-L}^{L}\frac{dz'}{\sqrt{(z')^2+r^2}} = \hat{z}\frac{\mu_0 I}{4\pi}\ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}$

$$\vec{B} = \nabla\times\vec{A} = \nabla\times(\hat{z}A_z) = \hat{r}\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \hat{\phi}\frac{\partial A_z}{\partial r},$$

$\frac{\partial}{\partial\phi} = 0$ due to cylindrical symmetry

$$\therefore \vec{B} = -\hat{\phi}\frac{\partial}{\partial r}[\frac{\mu_0 I}{4\pi}\ln\frac{\sqrt{L^2+r^2}+L}{\sqrt{L^2+r^2}-L}] = \hat{\phi}\frac{\mu_0 IL}{2\pi r\sqrt{L^2+r^2}}$$



$b)$ Biot-Savart law

$$\vec{R} = \hat{r}r - \hat{z}z',\ \ \vec{dl}'\times\vec{R} = \hat{z}dz'\times(\hat{r}r - \hat{z}z') = \hat{\phi}rdz'$$

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{rdz'}{(r^2+z'^2)^{3/2}}\hat{\phi} \qquad \therefore \vec{B} = \int d\vec{B} = \hat{\phi}\frac{\mu_0 IL}{2\pi r\sqrt{L^2+r^2}}$$

cf) $\int_{-L}^{L}\frac{rdz'}{(z'^2+r^2)^{\frac{3}{2}}}$, (sinθ=$\frac{z'}{\sqrt{z'^2+r^2}}$, tan$\theta$=$\frac{z'}{r}$, $z' = r\tan\theta \rightarrow dz' = r\sec^2\theta d\theta, 1+\tan^2\theta = \sec^2\theta$)

$$= \int\frac{r^2\sec^2\theta d\theta}{r^3\sec^3\theta} = \int\frac{1}{r}\frac{1}{\sec\theta}d\theta = \int\frac{1}{r}\cos\theta d\theta = \frac{1}{r}\sin\theta\Big|_{-\tan^{-1}\left(\frac{L}{r}\right)}^{\tan^{-1}\left(\frac{L}{r}\right)} = \frac{1}{r}\left(\frac{2L}{\sqrt{r^2+L^2}}\right)$$

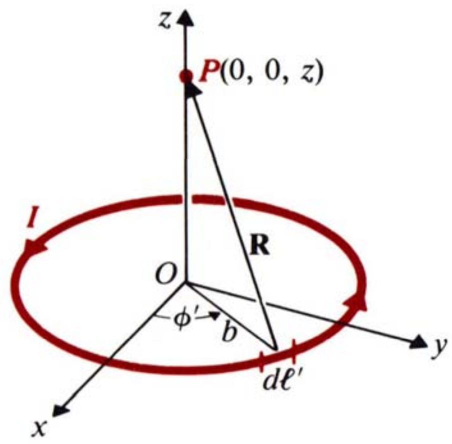
Ex 6-6) A circular loop carrying current I

sol) $\overrightarrow{dl}' = \hat{\phi} b d\phi', \ \vec{R} = \hat{z} z - \hat{r} b, \ R = \sqrt{z^2 + b^2}$

$\overrightarrow{dl}' \times \vec{R} = \hat{\phi} b d\phi' \times (\hat{z} z - \hat{r} b) = \hat{r} bzd\phi' + \hat{z} b^2 d\phi'$

by symmetry, $\hat{r}$ component contributeed by diametrically opposite to dl' elements cancel each other

$$\therefore \ \vec{B} = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \hat{z} \frac{b^2 d\phi'}{(z^2 + b^2)^{\frac{3}{2}}} = \hat{z} \frac{\mu_0 I b^2}{2(z^2 + b^2)^{\frac{3}{2}}}$$



The Magnetic Dipole

✓ Find the magnetic flux density at a distant point of a small circular loop of radius b that carries current I (a magnetic dipole)

① $R \gg b$

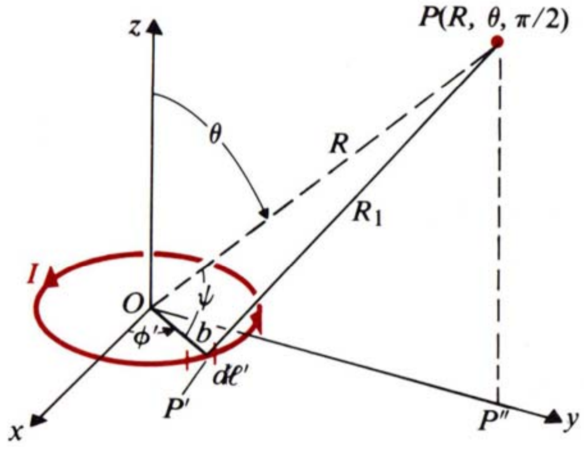
② $\vec{A} \rightarrow \vec{B} = \nabla \times \vec{A}$

③ $\vec{A} = \dfrac{\mu_0 I}{4\pi} \oint_{C'} \dfrac{\vec{dl}'}{R_1}$

④ By symmetry, $\vec{B}$ is independent of ϕ of the field point

$\Rightarrow$ consider $\vec{P}(R, \theta, \dfrac{\pi}{2})$ for the convenience

cf) $\hat{\phi}' \neq \hat{\phi}$



The Magnetic Dipole

⑤ at $\vec{P}, \hat{\phi} = -\hat{x}$ and $\vec{dl}' = (-\hat{x}\sin\phi' + \hat{y}\cos\phi')bd\phi'$

$cf)\ \left|\vec{dl}'\right| = bd\phi,\quad \widehat{\phi}' = -\hat{x}\sin\phi' + \hat{y}\cos\phi' \qquad \therefore \vec{dl}' = \widehat{\phi}'\left|\vec{dl}'\right|$

⑥ for every $I\vec{dl}'$, for $-\frac{\pi}{2} < \phi' < \frac{\pi}{2}$

there is another symmetrically located differential current element on the other side of the y-axis

$\Rightarrow$ equal amount of contribution to $\vec{A}$ in the $-\hat{x}$.

but opposite contribution to $\vec{A}$ in the $\hat{y}$

$$\therefore \vec{A} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{\vec{dl}'}{R_1} = -\hat{x}\frac{\mu_0 I}{4\pi}\int_0^{2\pi}\frac{b\sin\phi'}{R_1}d\phi' = \hat{\phi}\frac{\mu_0 Ib}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin\phi'}{R_1}d\phi'$$

The Magnetic Dipole

⑦ cosine 제 2법칙

$R_1^{\,2} = R^2 + b^2 - 2bR\cos\psi$

cf) $R\cos\psi =$ projection of R on $OP' =$ projection of OP'' on $OP' = R\sin\theta\sin\phi'$

$$\therefore R_1^{\,2} = R^2 + b^2 - 2bR\sin\theta\sin\phi' \qquad \therefore \frac{1}{R_1} = \frac{1}{R}\left(1 + \frac{b^2}{R^2} - \frac{2b}{R}\sin\theta\sin\phi'\right)^{-\frac{1}{2}}$$

cf) $R \gg b, \dfrac{b^2}{R^2}$ is the second order of very small value $\left(\dfrac{b}{R}\right)$, $\dfrac{b^2}{R^2}$ will be ignored

☞ Taylor series's expansion

$$\frac{1}{R_1} \cong \frac{1}{R}\left(1 - \frac{2b}{R}\sin\theta\sin\phi'\right)^{-\frac{1}{2}} \cong \frac{1}{R}\left(1 + \frac{b}{R}\sin\theta\sin\phi'\right)$$

The Magnetic Dipole

⑧ $\vec{A} = \hat{\phi}\dfrac{\mu_0 Ib}{2\pi R}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+\dfrac{b}{R}\sin\theta\sin\phi')\sin\phi' d\phi'$

$cf)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\phi' d\phi' = 0,\qquad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\phi' d\phi' = \dfrac{1}{2}\pi \qquad \therefore \vec{A} = \hat{\phi}\dfrac{\mu_0 Ib^2}{4R^2}\sin\theta$

⑨ $$\vec{B} = \nabla\times\vec{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{R} & R\hat{\theta} & R\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ A_R & RA_\theta & R\sin\theta A_\phi \end{vmatrix} = \frac{\mu_0 Ib^2}{4R^2\sin\theta}\begin{vmatrix} \hat{R} & R\hat{\theta} & R\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ 0 & 0 & \dfrac{\sin^2\theta}{R} \end{vmatrix}$$

The Magnetic Dipole

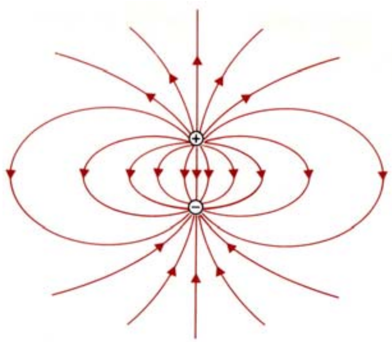
$$\vec{B} = \frac{\mu_0 I b^2}{4R^2 \sin\theta}(\hat{R}\frac{2\sin\theta\cos\theta}{R} + R\hat{\theta}\frac{\sin^2\theta}{R^2})$$

$$\boxed{\vec{B} = \frac{\mu_0 I b^2}{4R^3}(\hat{R}2\cos\theta + \hat{\theta}\sin\theta)}$$

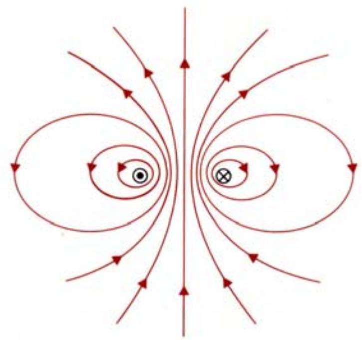
cf) $\hat{R}$ component is not cancelled

cf) Electric dipole and $\vec{E}$ field

☞ $\vec{E} = \dfrac{p}{4\pi\varepsilon_0 R^3}(\hat{R}2\cos\theta + \hat{\theta}\sin\theta)$ and $V = \dfrac{\vec{p}\bullet\hat{R}}{4\pi\varepsilon_0 R^2}$,where $\vec{E} = -\nabla V$, $\vec{p} = q\vec{d}$



(a) Electric dipole.



(b) Magnetic dipole.

The Magnetic Dipole

- ☞ $\vec{A} = \hat{\phi}\dfrac{\mu_0 (I\pi b^2)}{4\pi R^2}\sin\theta$
- ☞ $\vec{A} = \dfrac{\mu_0 \vec{m} \times \hat{R}}{4\pi R^2}$, where $\vec{m} = \hat{z}I\pi b^2 = \hat{z}IS = \hat{z}m$ $\Rightarrow$ magnetic dipole moment
- ☞ $\vec{B} = \dfrac{\mu_0 m}{4\pi R^3}(\hat{R}2\cos\theta + \hat{\theta}\sin\theta)$

Scalar Magnetic Potential

✓ $\nabla \times \vec{B} = \mu_0 \vec{J}$

$\Rightarrow$ for source free region, $\vec{J} = 0 \quad \therefore \nabla \times \vec{B} = 0,\ \ \vec{B}$ is curl-free

$cf)\ \nabla \times \vec{E} = 0$: electric scalar potential

$\Rightarrow \vec{B} = -\mu_0 \nabla V_m$ where V_m is scalar magnetic potential

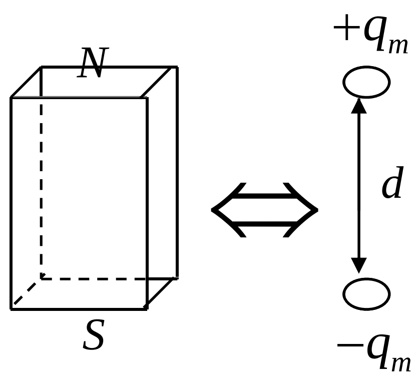
analogy to electric potential, $V_{m2} - V_{m1} = -\int_{P_1}^{P_2} \frac{1}{\mu_0} \vec{B} \bullet \vec{dl}$

if there were magnetic charge with a volume density ρ_m (fictitious)

in a volume V', then $V_m = \frac{1}{4\pi} \int_{V'} \frac{\rho_m}{R} dv'$ $\quad$ $note)\ \rho_m$: fictitious

Scalar Magnetic Potential

cf) Small bar magnet : Magnet field line is the same as that of magnetic dipole



☞ $\vec{m} = q_m \vec{d} = \widehat{a_n} IS$, then

☞ $V_m = \dfrac{\vec{m} \cdot \hat{R}}{4\pi R^2}$, $\vec{B} = -\mu_0 \nabla V_m$

⇒ These analogy approaches can be used only for the region of source free

⇒ If $\nabla \times \vec{B} \neq 0$, then V_m is not single valued ⇒ depending on the integral path

⇒ Vector potential and circulating current is better for the study of magnetic fields in magnetic materials

Magnetization and equivalent current density

① orbitting electrons $\Rightarrow$ circulating currents

$\rightarrow$ magnetic dipole

② spinning nuclei(electron) $\Rightarrow$ magnetic dipole

③ if external $\vec{B}=0,$ then atomic dipole have random orientations

$\Rightarrow$ no net magnetic moment .

cf) exception for permanent magnets

④ external $\vec{B}\Rightarrow$

☞ alignment of magnetic moment due to spinning electrons

☞ change in the orbit motion of electrons

Magnetization and equivalent current density

⑤ $\overrightarrow{m_k}$: magnetic dipole moment of an atom

$$\overrightarrow{M} = \lim_{\Delta V \to 0} \frac{\sum_{k=1}^{n\Delta v} \overrightarrow{m_k}}{\Delta v} \quad [A/m]$$, where n : atoms per unit volume,

$\overrightarrow{M}$: magnetization vector

$\Rightarrow$ volume density of magnetization

⑥ magnetic dipole moment $d\overrightarrow{m}$ of an elemental volume, dv' is $d\overrightarrow{m} = \overrightarrow{M} dv'$, then

$$d\overrightarrow{A} = \frac{\mu_0 \overrightarrow{M} \times \widehat{R}}{4\pi R^2} dv'$$

Magnetization and equivalent current density

$$cf)\ \nabla'\left(\frac{1}{R}\right)=\frac{\vec{R}}{R^3}=\frac{\hat{R}}{R^2},\quad \nabla\left(\frac{1}{R}\right)=-\frac{\vec{R}}{R^3}=-\frac{\hat{R}}{R^2}$$

$$d\vec{A}=\frac{\mu_0}{4\pi}\vec{M}\times\nabla'\left(\frac{1}{R}\right)dv'\Rightarrow\vec{A}=\int_{V'}d\vec{A}=\frac{\mu_0}{4\pi}\int_{V'}\vec{M}\times\nabla'\left(\frac{1}{R}\right)dv'$$

$$cf)\nabla'\times\left(\frac{\vec{M}}{R}\right)=\frac{1}{R}\nabla'\times\vec{M}+\nabla'\left(\frac{1}{R}\right)\times\vec{M}\qquad\therefore\vec{M}\times\nabla'\left(\frac{1}{R}\right)=\frac{1}{R}\nabla'\times\vec{M}-\nabla'\times\left(\frac{\vec{M}}{R}\right)$$

$$\vec{A}=\frac{\mu_0}{4\pi}\int_{V'}\frac{\nabla'\times M}{R}dv'-\frac{\mu_0}{4\pi}\int_{V'}\nabla'\times\left(\frac{\vec{M}}{R}\right)dv'$$

$$cf)\int_{V'}(\nabla'\times\vec{F})dv'=-\oint_S\vec{F}\times\vec{ds}'$$

Vector identity

proof) $\int_{V'}\nabla\bullet(\vec{F}\times\vec{C})dv=\oint_S(\vec{F}\times\vec{C})\vec{ds}$ and $\nabla\bullet(\vec{F}\times\vec{C})=\vec{C}\bullet(\nabla\times\vec{F})-\vec{F}\bullet\underbrace{\cancel{(\nabla\times\vec{C})}}_{=0}=\vec{C}\bullet(\nabla\times\vec{F})$

$$(\vec{F}\times\vec{C})\bullet\vec{ds}=\vec{ds}\bullet(\vec{F}\times\vec{C})=\vec{F}\bullet(\vec{C}\times\vec{ds})=\vec{C}\bullet(\vec{ds}\times\vec{F})=-\vec{C}\bullet(\vec{F}\times\vec{ds})$$

$$\therefore\vec{C}\bullet\int_{V'}(\nabla'\times\vec{F})dv'=-\vec{C}\bullet\oint_S\vec{F}\times\vec{ds}'\Rightarrow\int_{V'}(\nabla'\times\vec{F})dv'=-\oint_S\vec{F}\times\vec{ds}'$$

✓ $$\boxed{\vec{A}=\frac{\mu_0}{4\pi}\int_{V'}\frac{\nabla'\times M}{R}dv'+\frac{\mu_0}{4\pi}\oint_{S'}\frac{\vec{M}\times\vec{ds}'}{R}}$$

Magnetization and equivalent current density

- $\vec{J}_m = \nabla' \times \vec{M} \ [A/m^2]$

  : magnetization vector effect is equivalent to a volume current density

- $\vec{J}_{ms} = \vec{M} \times \hat{n}' \ [A/m]$

  $\hat{n}'$: the unit outward normal vector from ds' and S' is surface bounding the volume V'

✓ equivalent magnetization charge densities

- $dV_m = \dfrac{\vec{M} \bullet \vec{R}}{4\pi R^2} \Rightarrow$

$$V_m = \frac{1}{4\pi}\int_{V'} \frac{\vec{M} \bullet \vec{R}}{R^2} dv' = \frac{1}{4\pi}\int_{V'} \vec{M} \bullet \nabla' \left(\frac{1}{R}\right) dv' = \frac{1}{4\pi}\oint_{S'} \frac{\vec{M} \bullet \hat{n}'}{R} ds' + \frac{1}{4\pi}\int_{V'} \frac{-(\nabla' \bullet \vec{M})}{R} dv'$$

$\therefore \rho_{ms} = \vec{M} \bullet \hat{n}, \quad \rho_m = -\nabla \bullet \vec{M}$ analogy to $\rho_{PS} = \vec{P} \bullet \hat{n}, \quad \rho_P = -\nabla \bullet \vec{P}$

$$\boxed{cf)\nabla' \bullet \left(\frac{1}{R}\vec{M}\right) = \frac{1}{R}\nabla' \bullet \vec{M} + \vec{M} \bullet \nabla' \frac{1}{R}, \quad \therefore \vec{M} \bullet \nabla' \left(\frac{1}{R}\right) = \nabla' \bullet \left(\frac{1}{R}\vec{M}\right) - \frac{1}{R}\nabla' \bullet \vec{M}}$$