

Field and Wave Electromagnetic

Chapter7

The time varying fields and Maxwell's equation

Introduction (1)

❖ Time static fields

1) Electrostatic

$$\nabla \times \vec{E} = 0, \quad \nabla \cdot \vec{D} = \rho, \quad \vec{D} = \epsilon \vec{E}$$

2) Magnetostatic

$$\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{H} = \vec{J}, \quad \vec{H} = \frac{1}{\mu} \vec{B}$$

note) \vec{E} and \vec{D} are not related to \vec{B} and \vec{H} for time static cases

Example)

A static \vec{E} field in a conducting medium \Rightarrow steady current. ($\vec{J} = \sigma \vec{E}$)

\Rightarrow give rises to a static magnetic field: Ampere's law. But \vec{E} field can be completely determined from the static electric charge or potential distributions

\Rightarrow magnetic field is a consequence

Introduction (2)

❖ Time varying fields

☞ \vec{E} and \vec{D} are properly related to \vec{B} and \vec{H}

1) modify $\nabla \times \vec{E}$ equation \rightarrow fundamental postulate leading to Faraday's law

2) then modify the $\nabla \times \vec{H}$ equation to be consistent with the equation of continuity

cf) $\nabla \cdot \vec{J} = 0$ for static. but $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ for time varying

3) $\nabla \cdot \vec{D} = \rho$ and $\nabla \cdot \vec{B} = 0$ never changes.

Faraday's Law

- ✓ Michael Faraday \Rightarrow 1831, experimental law \Rightarrow postulate
- ✓ Definition : the quantitative relationship between the induced emf and the rate of change of flux linkage
- ❖ Fundamental postulate for Electromagnetic Induction

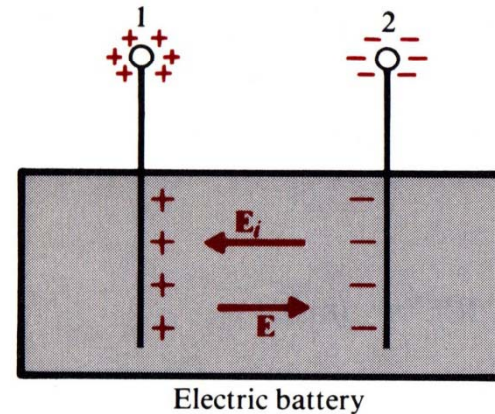
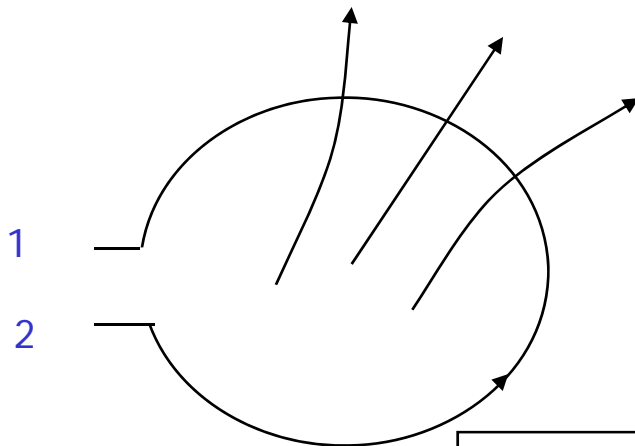
$$\boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}} \Rightarrow \text{Non-conservative field cannot be expressed}$$

as the gradient of a scalar potential

$$\boxed{\oint_c \vec{E} \cdot d\vec{l} = -\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}}$$

A Stationary Circuit in a Time Varying Magnetic Field (1)

$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \cdot d\vec{s} \quad (\because \frac{\partial}{\partial t} \rightarrow \frac{d}{dt}, \text{ since stationary } \frac{\partial}{\partial t} \vec{s} = 0)$$



Right hand rule (counter clock wise)

① $emf, v = \int_1^2 \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$, Assume $\frac{d\Phi}{dt} > 0$

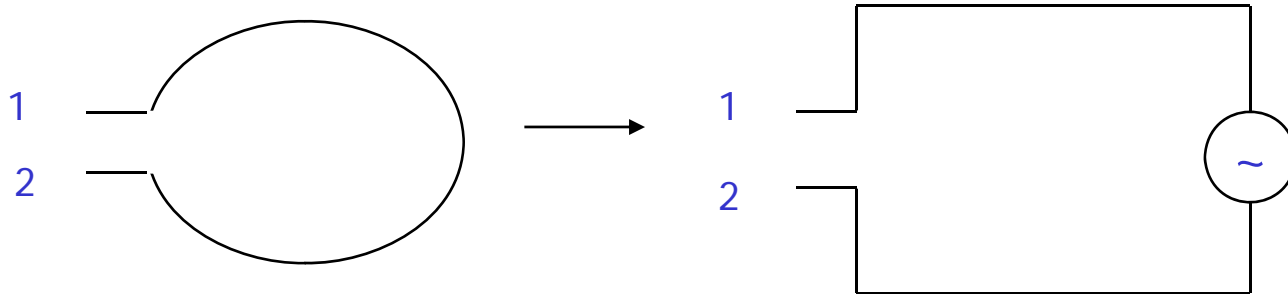
$\Rightarrow v < 0 \Rightarrow$ driving current to flow in the direction of clock wise

② potential difference of gap between terminal 1 and 2

assume $V_{12} = -\int_2^1 \vec{E} \cdot d\vec{l} < 0 \quad \because V_2 > V_1$

A Stationary Circuit in a Time Varying Magnetic Field (2)

Define $v = \oint_C \vec{E} \cdot d\vec{l}$: emf induced in circuit with contour C



v : electromotive force driving current in the direction of right hand rule

❖ Meaning of contour integral

Field between the terminal in the gap

$$\oint_{\text{right hand}} \vec{E} \cdot d\vec{l} = \int_1^2 \vec{E} \cdot d\vec{l} \quad (\because \text{inside contour } \vec{E} = 0) = -\int_2^1 \vec{E} \cdot d\vec{l} = V_1 - V_2 = V_{12}$$

\Rightarrow can be replaced with voltage source. But polarity of $v = V_{12}$ depends on the change of the flux linkage

A Stationary Circuit in a Time Varying Magnetic Field (3)

e.g) $\frac{\partial \vec{B}}{\partial t} > 0$ then v (current is in the direction of left hand rule) *i.e* $V_{12} < 0$

Define $\Phi = \int_S \vec{B} \cdot \vec{ds}$: magnetic flux crossing surface S [Wb]

then $v = -\frac{d\Phi}{dt} \Rightarrow$ This is valid even in the absense of a physical closed circuit

note The emf induced in a stationary loop caused by a time-varying magnetic field is a transformer emf

Ex 7-1) A Circular Loop of N Turns of Conducting Wire

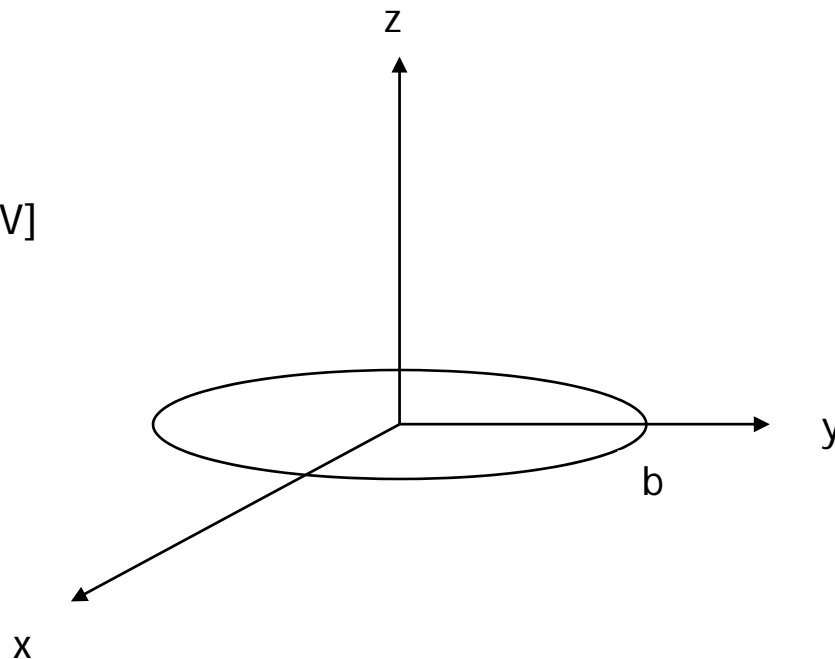
A circular loop of N turns, $\vec{B} = \hat{z}B_0 \cos\left(\frac{\pi r}{2b}\right) \sin \omega t$

Find the emf induced in the loop

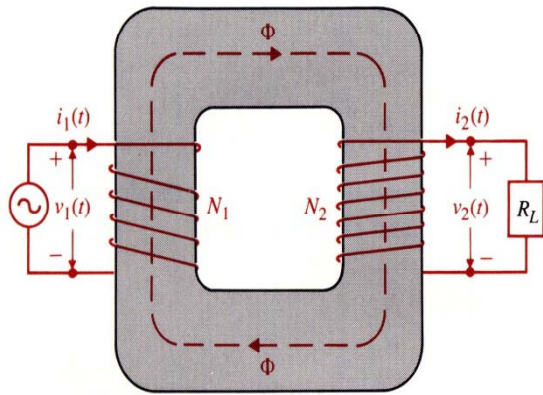
$$\begin{aligned} \text{sol) each turn } \Phi &= \int_S \vec{B} \cdot \vec{ds} = \int_0^b (\hat{z}B_0 \cos \frac{\pi r}{2b} \sin \omega t) \cdot (\hat{z}2\pi r dr) \quad \text{cf) } \int_0^{2\pi} d\phi = 2\pi \\ &= \frac{8b^2}{\pi} \left(\frac{\pi}{2} - 1\right) B_0 \sin \omega t \end{aligned}$$

$$\therefore \text{N-turns} \Rightarrow N\Phi$$

$$\therefore v = -N \frac{d\Phi}{dt} = -\frac{8Nb^2}{\pi} \left(\frac{\pi}{2} - 1\right) B_0 \cos \omega t \quad [\text{V}]$$



Transformers (1)



$$\text{mmf} \downarrow$$

$$\sum_j N_j I_j = \sum_k \mathfrak{R}_k \Phi_k$$

$N_1, N_2, i_1, i_2 \Rightarrow$ number of turns and the currents

\mathfrak{R} : the reluctance of the magnetic circuit

$$\therefore N_1 i_1 - N_2 i_2 = \mathfrak{R} \Phi$$

(where $N_1 i_1$: mmf in the positive direction, $N_2 i_2$: mmf in the negative direction)

$$\mathfrak{R} = \frac{l}{\mu S}$$

$$\therefore N_1 i_1 - N_2 i_2 = \frac{l}{\mu S} \Phi$$

Transformers (2)

a) Ideal transformer

$$\mu \rightarrow \infty, N_1 i_1 = N_2 i_2 \Rightarrow \boxed{\frac{i_1}{i_2} = \frac{N_2}{N_1}}$$

cf) Faraday's law

➤ $v_1 = N_1 \frac{d\Phi}{dt}$ (No negative sign, careful of sign of flux Φ)

➤ $v_2 = N_2 \frac{d\Phi}{dt}$ (But flux is in the reverse direction) $\therefore \boxed{\frac{v_1}{v_2} = \frac{N_1}{N_2}}$

➤ effective load seen by the source connected to primary winding

$$(R_1)_{eff} = \frac{v_1}{i_1} = \frac{\left(\frac{N_1}{N_2}\right) v_2}{\left(\frac{N_2}{N_1}\right) i_2} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

\therefore Impedance transformation $(Z_1)_{eff} = \left(\frac{N_1}{N_2}\right)^2 Z_L$

Transformers (3)

b) Real transformer

$$N_1 i_1 - N_2 i_2 = \frac{l}{\mu s} \Phi$$

$$\Rightarrow \Lambda_1 = N_1 \Phi = \frac{\mu s}{l} (N_1^2 i_1 - N_1 N_2 i_2), \quad \Lambda_2 = N_2 \Phi = \frac{\mu s}{l} (N_1 N_2 i_1 - N_2^2 i_2)$$

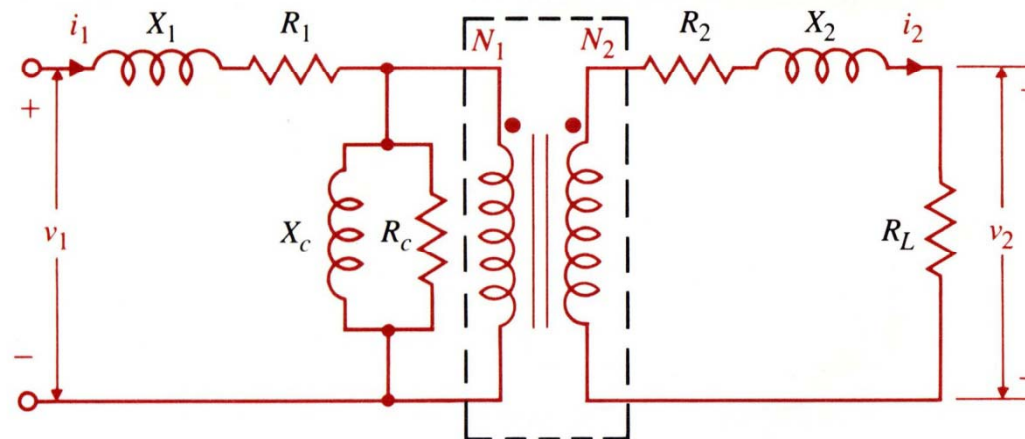
$$\therefore v_1 = L_1 \frac{di_1}{dt} - L_{12} \frac{di_2}{dt}, \quad v_2 = L_{12} \frac{di_1}{dt} - L_2 \frac{di_2}{dt}$$

$$\text{(where } L_1 = \frac{\mu s}{l} N_1^2, \quad L_2 = \frac{\mu s}{l} N_2^2, \quad L_{12} = \frac{\mu s}{l} N_1 N_2 \text{)}$$

- For an ideal transformer \Rightarrow No leakage flux $\therefore L_{12} = \sqrt{L_1 L_2}$
- For a real transformer $\therefore L_{12} = k \sqrt{L_1 L_2}$, $k < 1$ (k : coefficient of coupling)

Transformers (4)

❖ Equivalent circuit



R_1, R_2 : winding resistance

X_1, X_2 : leakage inductive reactance

R_c : power loss due to hysteresis and eddy current

X_c : nonlinear inductive reactance due to the nonlinear magnetization behavior of the ferromagnetic core

A Moving Conductor in a Static Magnetic Field

$$\vec{F}_m = q\vec{u} \times \vec{B}$$

→ Charge Separation

→ Coulomb force of an attraction

→ \vec{F}_m and \vec{F}_e will balance each other to be in equilibrium.

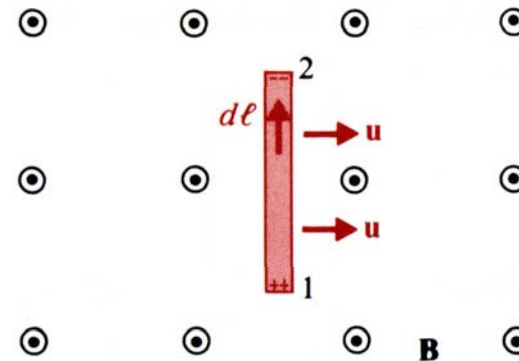
∴ Magnetic force per unit charge

$$\frac{\vec{F}_m}{q} = \vec{u} \times \vec{B}, \quad V = -\int \vec{E} \cdot d\vec{l}, \quad \vec{E} = -\frac{\vec{F}_m}{q}$$

$$\therefore V_{21} = \int_1^2 (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

The emf generated around the closed loop is

$$V' = \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l} \rightarrow \text{flux cutting emf}$$



Ex 6-5) A Metal Bar Sliding Over Conducting Rails

$$\bar{B} = \hat{z}B_0, \text{ constant } \bar{u}$$

$$\begin{aligned} \text{a) } V_0 &= V_1 - V_2 = \oint_C (\bar{u} \times \bar{B}) \cdot d\bar{l} \\ &= \int_{2'}^{1'} (\hat{x}u \times \hat{z}B_0) \cdot (\hat{y}dl) \\ &= -uB_0h \end{aligned}$$

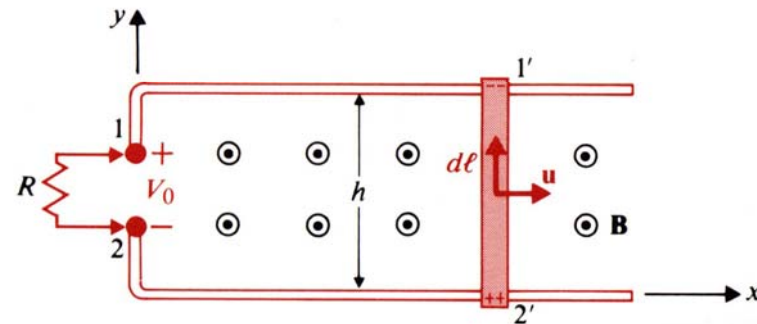
$$\text{b) } I = \frac{V_0}{R}, \quad P_l = I^2R = \frac{(uB_0h)^2}{R}$$

c) mechanical power

$$\bar{F}_m = I \int_{2'}^{1'} d\bar{l} \times \bar{B} = -\hat{x}IB_0h$$

(I : negative direction to $d\bar{l}$)

$$\therefore P_m = \bar{F} \cdot \bar{u} = -\bar{F}_m \cdot \bar{u} = \frac{u^2 B_0^2 h^2}{R}$$



A Moving Circuit in a Time Varying Magnetic Field (1)

$$\vec{F}_m = q(\vec{E} + \vec{u} \times \vec{B})$$

To an observer moving with C,

the force on q can be interpreted as caused by an electric field \vec{E}' ,

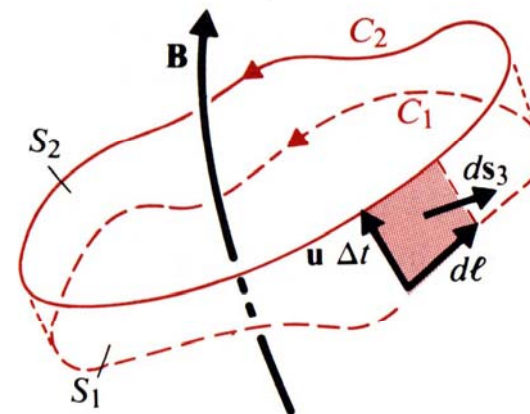
$$\vec{E}' = \vec{E} + \vec{u} \times \vec{B}$$

$$\oint_C \vec{E}' \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \oint_C (\vec{u} \times \vec{B}) \cdot d\vec{l} \rightarrow \text{General form of Faraday law}$$

the emf induced
in the moving
frame of reference

motional emf
due to the motion
of the circuit in \vec{B}

transformer emf
due to the time
variation



A Moving Circuit in a Time Varying Magnetic Field (2)

The time rate of change of magnetic flux,

$$\begin{aligned}\frac{d\Phi}{dt} &= \frac{d}{dt} \int_S \bar{B} \cdot d\bar{s} \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\int_{S_2} \bar{B}(t + \Delta t) \cdot d\bar{s}_2 - \int_{S_1} \bar{B}(t) \cdot d\bar{s}_1 \right]\end{aligned}$$

$$\text{cf) } \bar{B}(t + \Delta t) = \bar{B}(t) + \frac{\partial \bar{B}(t)}{\partial t} \Delta t + H.O.T. \quad \text{Taylor's series}$$

$$\therefore \frac{d}{dt} \int_S \bar{B} \cdot d\bar{s} = \int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} + \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\int_{S_2} \bar{B} \cdot d\bar{s}_2 - \int_{S_1} \bar{B} \cdot d\bar{s}_1 + H.O.T. \right]$$

assuming side surface S_3 as the area swept out by the conductor in time Δt

$$d\bar{S}_3 = d\bar{l} \times \bar{u} \Delta t$$

from divergence theorem

$$\begin{aligned}\int_V \nabla \cdot \bar{B} \, dv &= \int_{S_2} \bar{B} \cdot d\bar{s}_2 - \int_{S_1} \bar{B} \cdot d\bar{s}_1 + \int_{S_3} \bar{B} \cdot d\bar{s}_3 \\ \therefore \int_{S_2} \bar{B} \cdot d\bar{s}_2 - \int_{S_1} \bar{B} \cdot d\bar{s}_1 &= -\Delta t \oint_C (\bar{u} \times \bar{B}) \cdot d\bar{l} \\ \therefore \frac{d}{dt} \int_S \bar{B} \cdot d\bar{s} &= \int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s} - \oint_C (\bar{u} \times \bar{B}) \cdot d\bar{l} \\ \therefore V' &= \oint_C \tilde{E}' \cdot d\bar{l} = -\frac{d}{dt} \int_S \bar{B} \cdot d\bar{s} = -\frac{d\Phi}{dt}\end{aligned}$$

Maxwell's Equation (1)

static	Time varying
$\nabla \times \vec{E} = 0$ $\nabla \cdot \vec{D} = \rho, \quad \vec{D} = \epsilon \vec{E}$	$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ $\nabla \cdot \vec{D} = \rho$
$\nabla \times \vec{H} = \vec{J}$ $\nabla \cdot \vec{B} = 0, \quad \vec{B} = \mu \vec{H}$	$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ $\nabla \cdot \vec{B} = 0$

Maxwell's Equation (2)

Note ① Continuity equation

$$\nabla \cdot \vec{J} = 0: \text{ for steady state current}$$

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}: \text{ time varying current}$$

② Vector identity

$$\nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J} \Rightarrow \text{contradiction } \nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\therefore \nabla \cdot (\nabla \times \vec{H}) = 0 = \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = \nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right), \text{ where } \rho = \nabla \cdot \vec{D}$$

$$\nabla \times \vec{H} = \vec{J} + \left(\frac{\partial \vec{D}}{\partial t} \right)$$

Displacement current density. [A/m²]
Time varying electric field and induced magnetic field → coupling

Cf) Lorentz force equation, $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$

Integral Form of Maxwell's Equation

Cf) Differential form \rightarrow Point function

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \Rightarrow \text{Apply Stokes's theorem over open surface } S \text{ with contour } C$$

$$\int_S (\nabla \times \vec{E}) \cdot \vec{ds} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds}$$

$$\textcircled{1} \oint_C \vec{E} \cdot d\vec{l} = -\oint_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} = -\frac{d\Phi}{dt} : \text{Faraday's law}$$

$$\textcircled{2} \oint_C \vec{H} \cdot d\vec{l} = I + \oint_S \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds} : \text{Ampere's circuital law}$$

$$\textcircled{3} \nabla \cdot \vec{D} = \rho \Rightarrow \oint_S \vec{D} \cdot \vec{ds} = Q : \text{Gauss law}$$

$$\textcircled{4} \nabla \cdot \vec{B} = 0 \Rightarrow \oint_S \vec{B} \cdot \vec{ds} = 0 : \text{No isolated magnetic charge}$$

Ex. 7-5

- (a) Displacement current = conduction current
 1 conduction current \rightarrow current on the wire.
 Apply circuit theorem

$$i_c = C_1 \frac{dv_c}{dt} = C_1 V_0 \omega \cos \omega t$$

- 2 Displacement current. Reminding

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

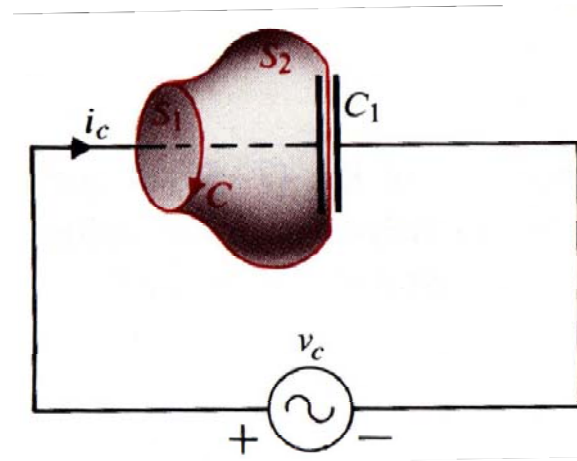
Assuming the area A , plate separation d , permittivity μ , then $C_1 = \epsilon \frac{A}{d}$

Assume \vec{E} is uniform in the dielectric (ignoring fringing effects)

then

$$\vec{E} = \frac{v_c}{d}, \quad \vec{D} = \epsilon \vec{E} = \epsilon \frac{V_0}{d} \sin \omega t$$

$$i_D = \int_A \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds} = \epsilon \frac{A}{d} V_0 \omega \cos \omega t = C_1 V_0 \omega \cos \omega t = i_c$$



Ex. 7-5

(b) Magnetic field intensity reminding Ampere's law

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}, \quad \oint_C \vec{H} \cdot d\vec{l} = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{s}$$

① surface S1 with ring C

② surface S2 with ring C

① $\vec{D} = 0, \quad \oint_C \vec{H} \cdot d\vec{l} = 2\pi r H_\phi$

(Symmetry around the wire along the contour C) \Rightarrow constant H_ϕ

$$I = \int_S \vec{J} \cdot d\vec{s} = i_c = C_1 V_0 \omega \cos \omega t$$

② no conduction current, but displacement current

$$I = i_d, \quad \therefore H_\phi = \frac{C_1 V_0}{2\pi r} \omega \cos \omega t$$

Potential Functions (1)

❖ Vector magnetic potential, \vec{A}

$$\vec{B} = \nabla \times \vec{A} \text{ (Solenoidal nature of } \vec{B}\text{)}$$

$$\text{Vector identity } \nabla \cdot \vec{B} = 0, \nabla \cdot (\nabla \times \vec{A}) = 0$$

▷ Recall Faraday's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \therefore \nabla \times \vec{E} = -\frac{\partial}{\partial t} (\nabla \times \vec{A}) \Rightarrow \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$$

Curl free

▷ Vector identity $\nabla \times (\nabla V) = 0$

and reminding $\vec{E} = -\nabla V$ for electromagnetics

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V \text{ for time varying i.e) } \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \text{ [V / m]}$$

Potential Functions (2)

Cf) Static $\rightarrow \frac{\partial \vec{A}}{\partial t} = 0 \quad \therefore \vec{E} = -\nabla V$

Time varying \vec{E} is induced by charge distribution ρ and time varying magnetic field \rightarrow time varying current, \vec{J}

▷ \vec{B} also depends on $\vec{A} \quad \therefore \vec{E}, \vec{B}$ are coupled

▷ $V = \frac{1}{4\pi\epsilon_0} \int_{v'} \frac{\rho}{R} dv', \quad \vec{A} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\vec{J}}{R} dv' \quad :$ From the static condition

These are solution of poisson equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

▷ The time-retardation effects associated with the finite velocity of propagation is neglected

Potential Functions (3)

Quasi-static fields

- ρ and \vec{J} vary slowly with time
- the range of interest R is small compared to the wavelength
- cf) Frequency is high, and R is large compared to wavelength
: time-retardation effect must be included.

From the equations

$$\vec{B} = \nabla \times \vec{A}, \quad \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \quad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (\vec{B} = \mu \vec{H}, \quad \vec{D} = \epsilon \vec{E})$$

$$\nabla \times \nabla \times \vec{A} = \mu \vec{J} + \mu \epsilon \frac{\partial}{\partial t} \left(-\nabla V - \frac{\partial \vec{A}}{\partial t} \right)$$

Recalling vector identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Potential Functions (4)

$$\therefore \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} - \nabla \left(\mu \epsilon \frac{\partial V}{\partial t} \right) - \mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2}$$

$$\nabla^2 \vec{A} - \mu \epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} + \nabla \left(\nabla \cdot \vec{A} + \mu \epsilon \frac{\partial V}{\partial t} \right)$$

- we only designated $\nabla \times \vec{A} = \vec{B}$ but we are free to choose $\nabla \cdot \vec{A}$
- vector \vec{A} will be specified by giving $\nabla \times \vec{A}$ and $\nabla \cdot \vec{A}$

- let $\boxed{\nabla \cdot \vec{A} + \mu \epsilon \frac{\partial V}{\partial t} = 0}$, for static $\frac{\partial V}{\partial t} = 0 \therefore \nabla \cdot \vec{A} = 0$

Lorentz gauge for potentials

Potential Functions (5)

cf) For static, $\frac{\partial V}{\partial t} = 0$, $\frac{\partial \vec{A}}{\partial t} = 0$

$$\therefore \nabla \cdot \vec{A} = 0$$

then vector poisson equation

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

- Then nonhomogeneous wave equation for vector potential becomes

$$\boxed{\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}} : \text{Vector potential wave equation}$$

Potential Functions(6)

∇ Scalar potential wave equation

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}, \quad \nabla \cdot \vec{D} = \rho \Rightarrow -\nabla \cdot \epsilon \left(\nabla V + \frac{\partial \vec{A}}{\partial t} \right) = \rho$$

$$\therefore \nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon}, \quad \nabla \cdot \vec{A} = -\mu\epsilon \frac{\partial V}{\partial t}$$

$$\therefore \boxed{\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}}$$

Boundary Condition (1)

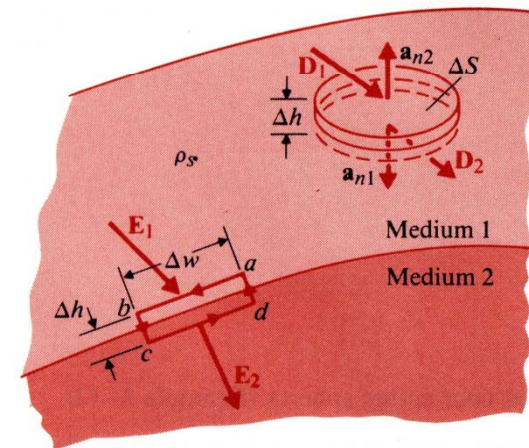
- Electric field's boundary condition

$$\oint_c \vec{E} \cdot d\vec{l} = - \int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \quad \dots \textcircled{1} \quad \oint_s \vec{D} \cdot d\vec{s} = \int_v \rho dv \quad \dots \textcircled{2}$$

From equation ①

$$\int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} \rightarrow 0 \text{ when } \Delta h \rightarrow 0, \text{ since area } S \rightarrow 0$$

$$\therefore \boxed{E_{1t} = E_{2t}} \quad (E_{1t} \Delta w - E_{2t} \Delta w = 0)$$



Boundary Condition (2)

From equation ②

$$\oint_s \vec{D} \cdot \vec{ds} = (\vec{D}_1 \cdot \hat{n}_2 + \vec{D}_2 \cdot \hat{n}_1) \Delta S = \hat{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) \Delta S = \rho_s \Delta S$$

$$\therefore \boxed{\hat{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s}, \quad D_{2n} - D_{1n} = \rho_s$$

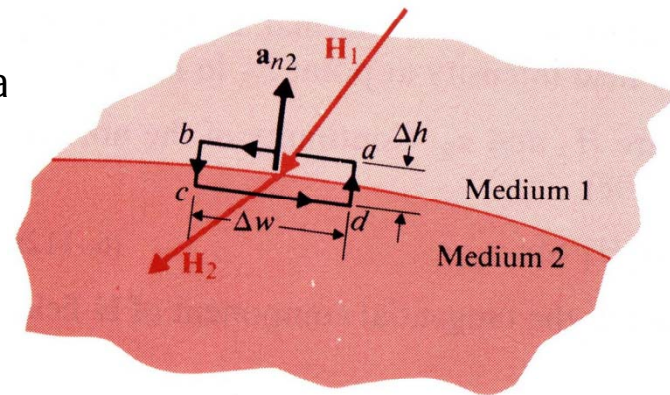
Magnetic field's boundary conditions

$$\oint_c \vec{H} \cdot \vec{dl} = \int_s \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \vec{ds}$$

$$\therefore \vec{H}_1 \cdot \vec{\Delta w} + \vec{H}_2 \cdot (-\vec{\Delta w}) = J_{sn} \Delta w, \quad H_{1t} - H_{2t} = J_{sn}$$

$$i.e) \hat{n}_2 \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$$

$$cf) \hat{n}_2 \text{ \& } \vec{J}_s \text{ are perpendicular to ea}$$



Boundary Condition (3)

note)

The tangential component of the \vec{H} field is discontinuous across an interface where a free surface current exists

if both media have finite conductivity, currents are defined by volume current density

→ surface currents do not exist

→ $H_{1t} = H_{2t}$

i.e) discontinuous only for interface with an ideal perfect conductor or superconductor.

$$\nabla \cdot \vec{B} = 0 \quad \therefore \boxed{B_{1n} = B_{2n}}$$

Interface Between Two Lossless Linear Media

Linear media \rightarrow permittivity : ε , permeability : μ

Lossless $\rightarrow \sigma=0$

Assume, at interface, no free charge $\rightarrow \rho_s=0$

no surface currents $\rightarrow J_s=0$

$$E_{1t} = E_{2t} \Rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\varepsilon_1}{\varepsilon_2}, \quad H_{1t} = H_{2t} \Rightarrow \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2}$$

$$D_{1n} = D_{2n} \Rightarrow \varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}, \quad B_{1n} = B_{2n} \Rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

Interface between a Dielectric and Perfect Conductor (1)

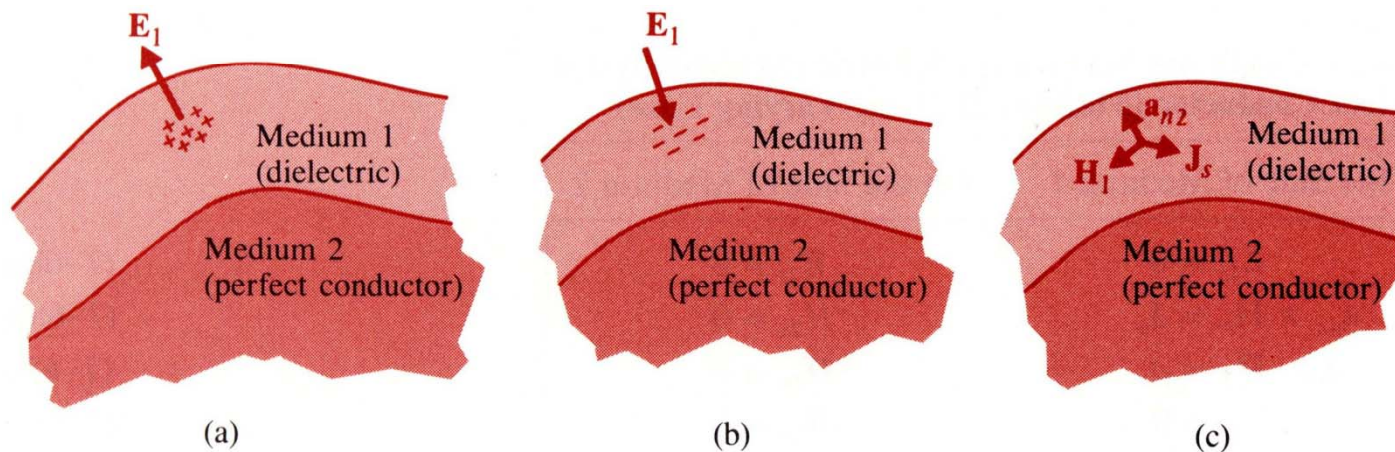
Good conductor \rightarrow perfect conductor

Interior of perfect conductor (surface charge only) : \vec{E}

$(\vec{E}, \vec{D}) \Rightarrow (\vec{B}, \vec{H})$ are zero in the interior of a conductor

cf) In static case, \vec{E} , \vec{D} may be zero, but \vec{H} , \vec{B} may not be zero.

$$\vec{E}_2 = 0, \vec{H}_2 = 0, \vec{D}_2 = 0, \vec{B}_2 = 0$$



Interface between a Dielectric and Perfect Conductor (2)

$$\textcircled{1} E_{1t} = 0, E_{2t} = 0$$

$$\textcircled{2} \hat{n}_2 \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s, H_{2t} = 0$$

$$H_{1t} = J_{sn}, \quad \text{if } J_{sn} = 0 \rightarrow H_{1t} = 0$$

$$\textcircled{3} \hat{n}_2 \cdot (\vec{D}_1 - \vec{D}_2) = \rho_s, D_{2n} = 0, D_{1n} = \rho_s$$

$$\textcircled{4} B_{1n} = 0, B_{2n} = 0$$

note) \hat{n}_2 : outward normal from medium2

At an interface between a dielectric and a perfect conductor

: \vec{E}_1 is normal to and points away from(into) the conductor surface

$$|\vec{E}_1| = E_{1n} = \frac{\rho_s}{\epsilon_1}$$

: \vec{H}_1 is tangential to the interface with a magnitude of

$$|\vec{H}_1| = |\vec{H}_{1t}| = |\vec{J}_s|$$

cf) direction $\hat{n}_2 \times (\vec{H}_1 - \vec{H}_2) = \vec{J}_s$

Wave Equation and Their Solutions (1)

$$\text{Wave equation : } \nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

Solution :

- Assume an elemental point charge at time t , $\rho(t)\Delta v'$ located at the origin of the coordinates.
- Spherical coordinates.
- V depends only on R . and t because of spherical symmetry.
(No dependence on ϕ)
- Except at the origin,

$$\frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = 0$$

Wave Equation and Their Solutions (2)

New variable

$$V(R, t) = \frac{1}{R} U(R, t)$$

$$R^2 \frac{\partial}{\partial R} \left(\frac{1}{R} U(R, t) \right) = R^2 \left[-\frac{1}{R^2} U + \frac{1}{R} \frac{\partial U}{\partial R} \right] = -U + R \frac{\partial U}{\partial R}$$

$$\frac{1}{R^2} \frac{\partial}{\partial R} \left[-U + R \frac{\partial U}{\partial R} \right] = \frac{1}{R^2} \left[-\frac{\partial U}{\partial R} + \frac{\partial U}{\partial R} + R \frac{\partial^2 U}{\partial R^2} \right]$$

$$\therefore \frac{1}{R} \frac{\partial^2 U}{\partial R^2} - \mu\epsilon \frac{1}{R} \frac{\partial^2 U}{\partial t^2} = 0, \quad \text{i.e.)} \quad \frac{\partial^2 U(R, t)}{\partial R^2} - \mu\epsilon \frac{\partial^2 U(R, t)}{\partial t^2} = 0$$

Any function of $(t - R\sqrt{\mu\epsilon})$ or of $(t + R\sqrt{\mu\epsilon})$ will satisfy the differential equation

$f(t - R\sqrt{\mu\epsilon})$ is a wave equation which travels away from the origin

Wave Equation and Their Solutions (3)

$f(t + R\sqrt{\mu\varepsilon})$ is a wave equation which travels to the origin \rightarrow physical nonsense

$$\therefore U(R, t) = f(t - R\sqrt{\mu\varepsilon})$$

the function at $R + \Delta R$ at a later $t + \Delta t$.

$$U(R + \Delta R, t + \Delta t) = f\left[t + \Delta t - (R + \Delta R)\sqrt{\mu\varepsilon}\right] = f(t - R\sqrt{\mu\varepsilon})$$

$$\text{if } \Delta t = \Delta R\sqrt{\mu\varepsilon}.$$

$$\frac{\Delta R}{\Delta t} = \frac{1}{\sqrt{\mu\varepsilon}} \Rightarrow \lim_{\Delta t \rightarrow 0} \frac{\Delta R}{\Delta t} = u = \frac{1}{\sqrt{\mu\varepsilon}} : \text{velocity of propagation}$$

$$V(R, t) = \frac{1}{R} f\left(t - \frac{R}{u}\right)$$

Determine $f\left(t - \frac{R}{u}\right)$

Wave Equation and Their Solutions (4)

Recall potential function induced by a static point charge $\rho(t)\Delta v$ at the origin

$$\boxed{\Delta V(R) = \frac{\rho(t)\Delta v'}{4\pi\epsilon R}}, \quad \Delta f\left(t - \frac{R}{u}\right) = \frac{\rho\left(t - \frac{R}{u}\right)\Delta v'}{4\pi\epsilon}$$

$$\therefore \boxed{V(R, t) = \frac{1}{4\pi\epsilon} \int_{v'} \frac{\rho\left(t - \frac{R}{u}\right)}{R} dv'} : \text{Retarded scalar potential}$$

Scalar potential at a distance R from the source at time t

→ Depends on the value of charge distribution at an earlier time $\left(t - \frac{R}{u}\right)$

Retarded vector potential

$$\vec{A}(R, t) = \frac{\mu}{4\pi} \int_{v'} \frac{\vec{J}\left(t - \frac{R}{u}\right)}{R} dv'$$

Source Free Wave Equation

If the wave is in a simple (linear, isotropic and homogeneous)
non conducting medium. i.e) ϵ, μ ($\sigma=0$)

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}, \quad \nabla \times \vec{H} = \epsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = 0, \quad \nabla \cdot \vec{H} = 0$$

From vector identity

$$\nabla \times \nabla \times \vec{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \vec{H}) = -\mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$cf) \vec{A} \times \vec{B} \times \vec{C} = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) = \vec{B}(\vec{A} \cdot \vec{C}) - (\vec{A} \cdot \vec{B})\vec{C}$$

$$\nabla(\nabla \cdot \vec{E}) - (\nabla \cdot \nabla)\vec{E} = -\nabla^2 \vec{E}$$

$$\therefore \nabla^2 \vec{E} - \mu \epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \text{ if } \mu \epsilon = \frac{1}{u^2}$$

$$\boxed{\nabla^2 \vec{E} - \frac{1}{u^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0, \quad \nabla^2 \vec{H} - \frac{1}{u^2} \frac{\partial^2 \vec{H}}{\partial t^2} = 0} : \text{Homogeneous vector wave equation}$$

Time Harmonic Fields

Maxwell's equations

- linear differential equations
- sinusoidal time variation of source functions at given frequency
- \vec{E} , \vec{H} are sinusoidal with the same frequency

Time harmonic \rightarrow steady state sinusoidal

Phasors : Amplitude and phase information

\rightarrow independent of time

cf) $e^{j\omega t}$: time dependent factor

Time Harmonic Electromagnetics (1)

Vector phasors of field vectors : depend on space coordinates

$$\vec{E}(x, y, z; t) = \text{Re} \left[\vec{E}(x, y, z) e^{j\omega t} \right],$$

where $\vec{E}(x, y, z)$: vector phasor : complex quantity

$$\frac{\partial}{\partial t} \vec{E}(x, y, z; t) = \text{Re} \left[j\omega \vec{E}(x, y, z) e^{j\omega t} \right]$$

where $j\omega \vec{E}(x, y, z)$: vector phasor

$$\int \vec{E}(x, y, z; t) dt = \text{Re} \left[\frac{\vec{E}(x, y, z)}{j\omega} e^{j\omega t} \right]$$

where $\frac{\vec{E}(x, y, z)}{j\omega}$: vector phasor

$$\text{i.e.) } \frac{\partial}{\partial t} \Rightarrow j\omega, \quad \frac{\partial^2}{\partial t^2} \Rightarrow (j\omega)^2, \quad \int \Rightarrow \frac{1}{j\omega}$$

Time Harmonic Electromagnetics (2)

Maxwell's equations

Vector field phasors (\vec{E} , \vec{H})

Source phasors (ρ , \vec{J}), Simple (linear, isotropic and homogeneous) media

$$\left. \begin{aligned} \nabla \times \vec{E} &= -j\omega\mu\vec{H}, \quad \nabla \times \vec{H} = \vec{J} + j\omega\varepsilon\vec{E} \\ \nabla \cdot \vec{E} &= \frac{\rho}{\varepsilon}, \quad \nabla \cdot \vec{H} = 0 \end{aligned} \right\} \text{Assuming } e^{j\omega t}$$

Time harmonic wave equation for V and \vec{A}

$$\left. \begin{aligned} \nabla^2 V + k^2 V &= -\frac{\rho}{\varepsilon} \\ \nabla^2 \vec{A} + k^2 \vec{A} &= -\mu\vec{J} \end{aligned} \right\} \begin{aligned} &\text{Non-homogeneous helmholtz's equations} \\ &\text{where } k^2 = \omega^2\mu\varepsilon, \quad k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{u} : \text{wave-number} \end{aligned}$$

Time Harmonic Electromagnetics (3)

$$cf) \nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} \quad (\nabla \cdot \vec{A} + \mu\epsilon \frac{\partial V}{\partial t} = 0 \Rightarrow \nabla \cdot \vec{A} + j\omega\mu\epsilon V = 0)$$

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\nabla^2 \vec{H} - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Phasor solution

$$V(R, t) = \frac{1}{4\pi\epsilon} \int_{v'} \frac{\rho e^{j\omega(t-\frac{R}{u})}}{R} dv' \Rightarrow V(R) e^{j\omega t} = \frac{1}{4\pi\epsilon} \int_{v'} \frac{\rho e^{-j\frac{\omega}{u}R}}{R} dv' \cdot e^{j\omega t}$$

$$\begin{cases} V(R) = \frac{1}{4\pi\epsilon} \int_{v'} \frac{\rho e^{-jkR}}{R} dv' \quad [\text{V}] \\ \vec{A}(R) = \frac{\mu}{4\pi} \int_{v'} \frac{\vec{J} e^{-jkR}}{R} dv' \quad [\text{Wb/m}] \end{cases}$$

Expressions for the retarded scalar and vector potentials due to time harmonic sources

Time Harmonic Electromagnetics (4)

cf) $e^{-jkR} = 1 - jkR - \frac{k^2 R^2}{2} + \dots$: Taylor series expansion.

$$k = \frac{\omega}{u} = \frac{2\pi f}{u} = \frac{2\pi}{\lambda}, \quad u = f\lambda$$

if $kR = 2\pi \frac{R}{\lambda} \ll 1 \Rightarrow e^{-jkR} = 1$, then $V(R) = \frac{1}{4\pi\epsilon} \int_{v'} \frac{\rho}{R} dv' \Rightarrow$ static potential

Procedure for determining the electric and magnetic fields due to time harmonic charge and current distributions

1. Find phasors $V(R)$ and $A(R)$

2. Find phasors $\vec{E}(R) = -\nabla V - j\omega\vec{A}$ cf) $\vec{E} = -\nabla V - \frac{\partial\vec{A}}{\partial t}$

$$\vec{B}(R) = \nabla \times \vec{A}$$

3. Find instantaneous $\vec{E}(R, t)$

Source-free Fields in Simple Media (1)

Source free fields in simple media

$$\begin{cases} \nabla \times \vec{E} = -j\omega\mu\vec{H} \\ \nabla \times \vec{H} = j\omega\varepsilon\vec{E} \\ \nabla \cdot \vec{E} = 0 \\ \nabla \cdot \vec{H} = 0 \end{cases}$$

$$\begin{cases} \nabla^2 \vec{E} + k^2 \vec{E} = 0 \\ \nabla^2 \vec{H} + k^2 \vec{H} = 0 \text{ and } k^2 = \omega^2 \mu \varepsilon \end{cases}$$

Homogeneous vector Helmholtz's equation

Principle of duality : Source free Maxwell's equations in a simple media are invariant under the linear transformation

$$\vec{E}' = \eta \vec{H}, \quad \vec{H}' = -\frac{\vec{E}}{\eta}, \quad \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

Source-free Fields in Simple Media (2)

If simple medium is conducting i.e) $\sigma \neq 0 \Rightarrow \vec{J} = \sigma \vec{E}$

$$\therefore \nabla \times \vec{H} = (\sigma + j\omega\epsilon)\vec{E} = j\omega\left(\epsilon + \frac{\sigma}{j\omega}\right)\vec{E} = j\omega\epsilon_c\vec{E}$$

and $\boxed{\epsilon_c = \epsilon - j\frac{\sigma}{\omega} \text{ [F/m]}}$: complex permittivity

- cf) · out of phase polarization : power loss to overcome a fractional damping mechanism caused by the inertia the charged particle
- finite conductivity \rightarrow ohmic losses

Complex permittivity

$\epsilon_c = \epsilon' - j\epsilon'' \text{ [F/m]}$, where ϵ'' : out of phase polarization and finite conductivity

\Rightarrow equivalent conductivity $\boxed{\sigma = \omega\epsilon''}$ \leftarrow representing all losses

Source-free Fields in Simple Media (3)

Complex permeability : out of phase component of magnetization

$$\mu = \mu' - j\mu'', \text{ where } \mu' \gg \mu'' \text{ for ferromagnetic materials}$$

$$\therefore \mu = \mu'$$

Complex wavenumber

$$k_c = \omega\sqrt{\mu\epsilon_c} = \omega\sqrt{\mu(\epsilon' - j\epsilon'')} : \text{ in a lossy dielectric}$$

Loss tangent

$$\left\{ \begin{array}{l} \tan \delta_c = \frac{\epsilon''}{\epsilon'} \cong \frac{\sigma}{\omega\epsilon}, \text{ where } \delta_c : \text{ loss angle} \\ \text{loss tangent } \frac{\epsilon''}{\epsilon'} \end{array} \right.$$

Good conductor & Good insulator

$$\sigma \gg \omega\epsilon : \text{ Good conductor}$$

$$\sigma \ll \omega\epsilon : \text{ Good insulator}$$

Source-free Fields in Simple Media (4)

Cf) Electric hertz vector, $\vec{\pi}$

$$\textcircled{1} \quad \vec{A} = \mu\epsilon \frac{\partial \vec{\pi}}{\partial t}, \quad V = -\nabla \cdot \vec{\pi}$$

: combine the vector and scalar potential and satisfy the Lorentz condition

$$\left\{ \begin{array}{l} \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \\ \nabla \cdot \vec{A} + \mu\epsilon \frac{\partial V}{\partial t} = 0 \end{array} \right.$$

\textcircled{2} combine continuity equation with \vec{J} and ρ

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0, \quad \boxed{\vec{J} = \frac{\partial \vec{P}}{\partial t}, \quad \rho = -\nabla \cdot \vec{P}}$$

\textcircled{3} Single vector equation

$$\nabla^2 \vec{\pi} - \mu\epsilon \frac{\partial^2 \vec{\pi}}{\partial t^2} = -\frac{\vec{P}}{\epsilon}$$

$$\textcircled{4} \quad \vec{E} = \nabla(\nabla \cdot \vec{\pi}) - \mu\epsilon \frac{\partial^2 \vec{\pi}}{\partial t^2} = \nabla \times \nabla \times \vec{\pi} - \frac{\vec{P}}{\epsilon}$$

$$\vec{H} = \epsilon \nabla \times \frac{\partial \vec{\pi}}{\partial t}$$

The electromagnetic spectrum

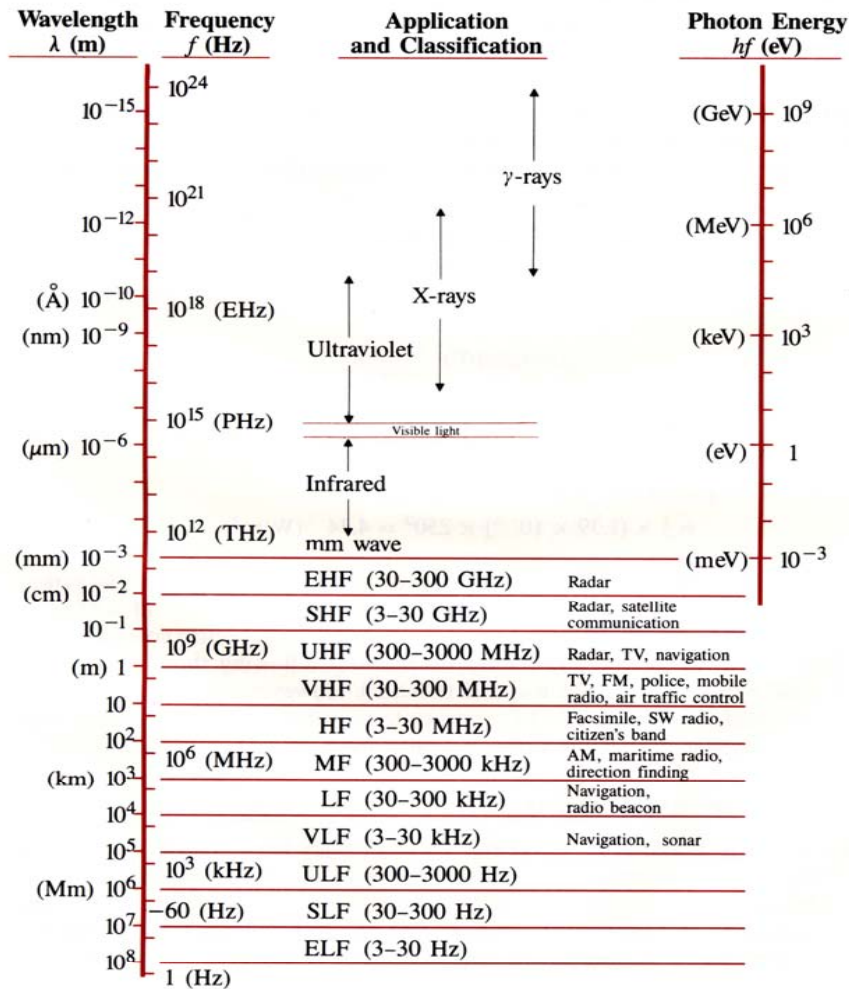


FIGURE 7-9 Spectrum of electromagnetic waves.

TABLE 7-5 Band Designations for Microwave Frequency Ranges

Old [†]	New	Frequency Ranges (GHz)
Ka	K	26.5-40
K	K	20-26.5
K	J	18-20
Ku	J	12.4-18
X	J	10-12.4
X	I	8-10
C	H	6-8
C	G	4-6
S	F	3-4
S	E	2-3
L	D	1-2
UHF	C	0.5-1

[†] Because the old band designations have been in wide use since the early days of radar, they are still in common use because of habit.