Class Handout: Chapter 1

2006 Fall

I. System and the State

- Mathematical system:
  - differential equation: $\dot{x} = f(x, u)$, $y = h(x, u)$, $x(0) = x_0$
  - difference equation: $x(i+1) = f(x(i), u(i))$, $y(i) = h(x(i), u(i))$, $x(0) = x_0$
  - ODE / PDE
- State and the state-space
- Solution (trajectory) of the system
- Phase portrait

A. Linear system

- It has the linear property.
- For each initial condition, there exists one and only one solution.
- The solution is defined for all time.
- The only possible attractor is the origin or an entire vector subspace.

B. Equilibrium

For $\dot{x} = f(x)$, we say x_0 an equilibrium if

$$x_0 \in \mathbb{R}^n, \qquad f(x_0) = 0.$$

Exercise 1. Consider a pendulum

$$MR^2\ddot{\theta} + b\dot{\theta} + MgR\sin\theta = 0.$$

In a state-space form (with $x_1 = \theta$, $x_2 = \dot{\theta}$), it is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{b}{MR^2}x_2 - \frac{g}{R}\sin x_1. \end{aligned} \tag{1}$$

Now find equilibrium points.

In most cases, we consider an equilibrium as the origin. Why?

* Set of equilibrium / invariant set / limit cycle

* Isolated equilibrium points

C. Existence of solution

A general nonlinear system of the form

$$\dot{x} = f(x), \qquad x(0) = x_0$$

may not have any solution. For example, the system

$$\dot{x} = 1 - 2\mathrm{sgn}(x)$$

where the function $\mathrm{sgn}(\cdot)$ is defined by

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

does not have any solution. (Why?)

In a very general form, we can recall the Caratheodory conditions for guaranteeing the existence of solution. Consider the following ODE

$$\dot{x} = f(x,t), \qquad x(0) = x_0,$$

or, equivalently, the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds$$

where it is the Lebesgue integral that is being used. Then, a solution exists if the following are satisfied:

1. $f(x,t)$ is continuous in x for each fixed t,
2. $f(x,t)$ is measurable in t for each fixed x,
3. for each compact set $\mathcal{U} = \mathcal{X} \times [a,b] \subset \mathbb{R}^n \times \mathbb{R}$, there exists a function $m_{\mathcal{U}} : [a,b] \to \mathbb{R}_{\geq 0}$, Lebesgue integrable on $[a,b]$, such that $|f(x,t)| \leq m_{\mathcal{U}}(t)$ for all $(x,t) \in \mathcal{U}$.

* This condition is quite general, so in almost all cases, the existence of solution would not be a matter. For example, when $f(x,t)$ is continuous with respect to their argument, then the existence of solution follows. However, when $f(x,t)$ is discontinuous w.r.t. the state x, then we need to be more careful (and this frequently happens when one uses sliding-mode control).

D. Uniqueness of the solution

Consider the system

$$\dot{x} = |x|^{1/3}, \qquad x(0) = 0.$$

This system has infinitely many solutions:

$$\phi(t,0) = \begin{cases} 0, & t \in [0, t_1] \\ (2(t-t_1)/3)^{3/2}, & t \geq t_1 \end{cases}$$

with any $t_1 \geq 0$. (Why?)

Locally and globally Lipschitz function.

For the system $\dot{x} = f(x)$, if $f(\cdot)$ is locally Lipschitz, then for each initial condition x_0, there exists $T > 0$ and one and only one solution $\phi(\cdot, x_0)$ on $[0, T]$.

When $f(\cdot)$ is globally Lipschitz, the solution exists for all $t \in [0, \infty)$.

Exercise 2. Prove that, if a function is continuously differentiable, then it is locally Lipschitz.

E. Finite Escape Time

Consider the system

$$\dot{x} = -x^2.$$

Its unique solution starting at $x = -1$ is

$$\phi(t, 1) = \frac{1}{t-1}$$

which is defined only for $t \in [0, 1)$.

One sufficient condition to rule out the finite escape time is global Lipschitzness of $f(\cdot)$.

F. Multiple Isolated Equilibria

Linear systems do not have multiple isolated equilibria. On the other hand, consider

$$\dot{x} = \sin(x),$$

which has many isolated equilibria.

G. Isolated Periodic Solution (Limit Cycle)

If $\phi(t, x)$ is a periodic solution to a linear system, then, for all λ, $\phi(t, \lambda x) = \lambda\phi(t, x)$ is another periodic solution, i.e., the periodic solutions cannot be isolated.

For nonlinear system, consider

$$\dot{x}_1 = x_2 + x_1(1 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = -x_1 + x_2(1 - x_1^2 - x_2^2)$$

which can be converted, by defining r and θ with $x_1 = r\sin(\theta)$ and $x_2 = r\cos(\theta)$, to

$$\dot{r} = r - r^3$$
$$\dot{\theta} = 1.$$

This system has the limit cycle. Check this by sketching its phase portrait.

H. Bounded Region of Attraction

Discuss this with the above example.

I. Asymptotic Stability without Exponential Stability

Consider

$$\dot{x} = -x^3$$

which is globally asymptotically stable. This can be verified by looking at its solution

$$\phi(t,x) = \mathrm{sgn}(x)\sqrt{\frac{x^2}{1+2x^2t}}.$$

The solution decays like $\frac{1}{\sqrt{1+t}}$ rather than exponential.

J. Chaos and Bifurcation