Class Handout: Chapter 2 Second-Order Systems

2006 Fall

- Analytic function, smooth function

  We consider

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

where f_1 and f_2 are smooth.

* "vector field" $f(x)$

I. Qualitative Behavior of Linear Systems

$$\dot{x} = Ax, \qquad A \in \mathbb{R}^{2\times 2}$$

Let

$$A = MJ_rM^{-1}$$

where J_r is a real Jordan matrix. That is,

$$J_r = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \text{or} \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}, \text{or} \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

where $k = 0$ or 1.

For more on real Jordan matrix, refer to `http://algebra.math.ust.hk/eigen/02_complex/exercise2_answer.shtml`.

If A has a zero eigenvalue, then the equilibrium is a set (so, we will treat the case later).

Case 1. Real e.v. $\lambda_1 \neq \lambda_2 \neq 0$.

$$z = M^{-1}x, \qquad \dot{z} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} z$$

Then,

$$z_2(t) = \left(z_{20}/z_{10}^{\frac{\lambda_2}{\lambda_1}} \right) z_1(t)^{\frac{\lambda_2}{\lambda_1}}$$

Case 2. Complex e.v. $\lambda_{1,2} = \alpha \pm j\beta$

In z,

$$\dot{z}_1 = \alpha z_1 - \beta z_2$$
$$\dot{z}_2 = \beta z_1 + \alpha z_2.$$

With $r := \sqrt{z_1^2 + z_2^2}$ and $\theta := \tan^{-1} z_2/z_1$, we have

$$\dot{r} = \alpha r, \qquad \dot{\theta} = \beta.$$

Then,

$$r(t) = r_0 e^{\alpha t}, \qquad \theta(t) = \theta_0 + \beta t.$$

Case 3. $\lambda_1 = \lambda_2 = \lambda \neq 0$

In z,

$$\dot{z}_1 = \lambda z_1 + k z_2, \qquad \dot{z}_2 = \lambda z_2.$$

Then,

$$\begin{aligned} z_2(t) &= e^{\lambda t} z_{20} \\ z_1(t) &= e^{\lambda t} z_{10} + \int_0^t e^{\lambda(t-\tau)} k e^{\lambda \tau} z_{20} d\tau \\ &= e^{\lambda t} z_{10} + kte^{\lambda t} z_{20}. \end{aligned}$$

Case 4.

$$J_r = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

* Perturbation: $\dot{x} = (A + \Delta A)x$

Since e.v. is continuous to its parameters, saddle, node, and focus are robust to a small perturbation. However, the center is not robust, e.g.,

$$\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}.$$

II. Multiple Equilibria

For example, consider a pendulum equation with friction:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -10 \sin x_1 - x_2$$

where x_1: position (angle), x_2: angular velocity. See Figure 2.16.

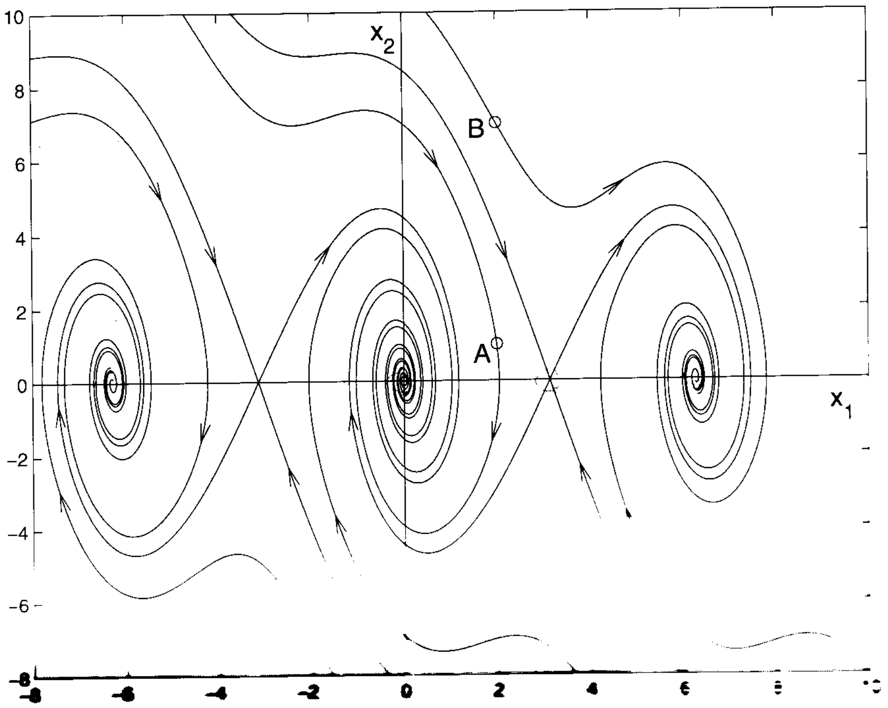

Fig. 1. Figure 2.16

III. Qualitative Behavior Near Equilibrium Points

Consider

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$

in which (p_1, P_2) is an equilibrium. Then, by Taylor series expansion, we have:

$$\dot{y} = Ay$$

Local behavior can be determined when the linearization is

- stable/unstable node with distinct eigenvalues
- stable/unstable focus
- saddle.

Example: The pendulum case:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -10\cos x_1 & -1 \end{bmatrix}.$$

Then,

$$J_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -10 & -1 \end{bmatrix}, \qquad \text{e.v.: } -0.5 \pm j3.12$$
$$J_{(\pi,0)} = \begin{bmatrix} 0 & 1 \\ 10 & -1 \end{bmatrix}, \qquad \text{e.v.: } -3.7, 2.7$$

"hyperbolic" equilibrium

Example: Consider

$$\dot{x}_1 = -x_2 - \mu x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 - \mu x_2(x_1^2 + x_2^2).$$

The origin is the center, but it resembles an unstable focus when $\mu < 0$. (See the book for the detail.)

IV. Limit Cycles

* Oscillation / Nontrivial periodic solution / Periodic orbit / Closed trajectory

* Harmonic oscillator / Linear oscillator is not structurally stable.

* Nonlinear oscillator / Isolated closed orbit / Limit cycle

Van der Pol equation:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2$$

* Jump phenomenon / Relaxation oscillation

* Stable/unstable limit cycle

V. Numerical Construction of Phase Portraits

In Matlab, try `quiver`.

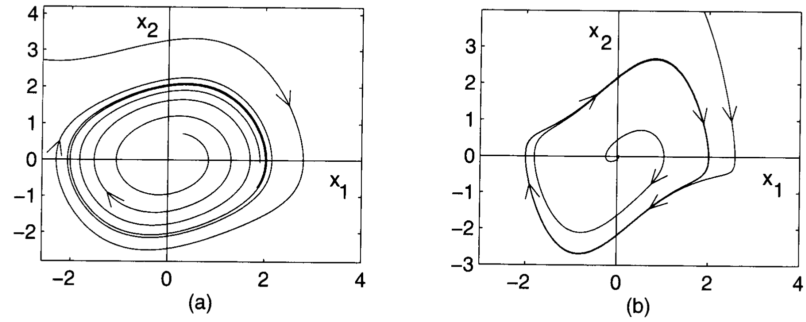

Figure 2.19: Phase portraits of the Van der Pol oscillator: (a) $\varepsilon = 0.2$; (b) $\varepsilon = 1.0$.

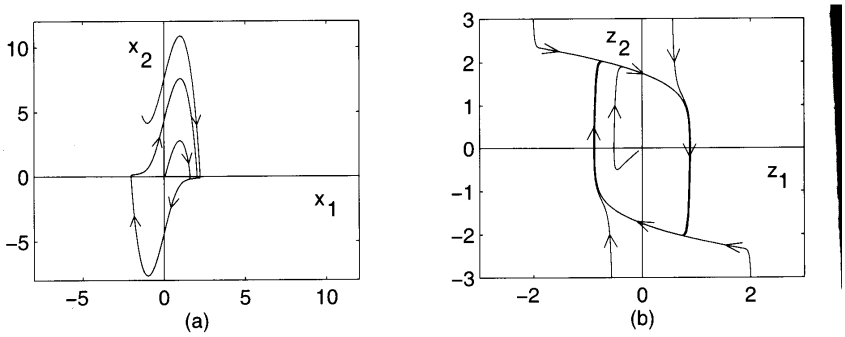

Figure 2.20: Phase portrait of the Van der Pol oscillator with $\varepsilon = 5.0$: (a) in x_1–$\boldsymbol{x}$ plane; (b) in z_1–z_2 plane.

VI. Existence of Periodic Orbits

Consider

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^2$$

where $f(\cdot)$ is continuously differentiable.

Lemma 1. Poincare-Bendixson Criterion *Consider a closed bounded region M s.t.*

- *M contains no equilibrium, or contains only one equilibrium at which the Jacobian has eigenvalues with positive real parts,*
- *every trajectory starting in M stays in M for all future time.*

Then, M contains a periodic solution.

Example 1.(Example 2.8) Consider

$$\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2).$$

Apply the Poincare-Bendixson Criterion to determine the existence of periodic solution. (Hint: use $V(x) = x_1^2 + x_2^2$.)

Lemma 2. Bendixson Criterion *If, on a simply connected (="no holes") region D, the quantity*

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

is not identically zero and does not change sign, then the system has no periodic solution within D.

Example 2. (Example 2.10) Consider

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= ax_1 + bx_2 - x_1^2 x_2 - x_1^3\end{aligned}$$

and $D = \mathbb{R}^2$. Determine b so that there's no periodic solution.

* Index of an equilibrium:

- Index of a node, a focus, or a center is +1.
- Index of a saddle is -1.
- Index of a closed orbit is +1.
- Index of a closed curve not encircling any equilibrium is 0.
- Index of a closed curve is equal to the sum of the indices of the equilibrium within it.

Lemma 3. Index Theorem *Inside any periodic orbit γ, there must be at least one equilibrium.*

Suppose the equilibrium points inside γ are hyperbolic, then if N is the number of nodes and foci and S is the number of saddles, it must be that $N - S = 1$.

Example 3. Consider

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= x_1 + x_2 - 2x_1 x_2.\end{aligned}$$

There are two equilibria at $(0, 0)$ and $(1, 1)$. At each, we have

$$J_{(0,0)} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad \text{which is a saddle,}$$

$$J_{(1,1)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad \text{which is a stable focus.}$$

Therefore, the only possibility is that the periodic orbit encircles the point $(1, 1)$.

VII. Bifurcation

- $\dot{x} = \mu - x^2$: saddle-node bifurcation
- $\dot{x} = \mu x - x^2$: transcritical bifurcation
- $\dot{x} = \mu x - x^3$: supercritical pitchfork bifurcation
- $\dot{x} = \mu x + x^3$: subcritical pitchfork bifurcation
- Supercritical Hopf bifurcation

$$\begin{aligned} \dot{x}_1 &= x_1(\mu - x_1^2 - x_2^2) - x_2 \\ \dot{x}_2 &= x_2(\mu - x_1^2 - x_2^2) + x_1 \end{aligned}$$

that is,

$$\dot{r} = \mu r - r^3, \qquad \dot{\theta} = 1.$$

- Subcritical Hopf bifurcation

$$\begin{aligned} \dot{x}_1 &= x_1(\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2) - x_2 \\ \dot{x}_2 &= x_2(\mu + (x_1^2 + x_2^2) - (x_1^2 + x_2^2)^2) + x_1 \end{aligned}$$

that is,

$$\dot{r} = \mu r + r^3 - r^5, \qquad \dot{\theta} = 1.$$

* global bifurcation

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \mu x_2 + x_1 - x_1^2 + x_1 x_2 \end{aligned}$$

See Fig. 2.32.

"Homoclinic orbit"