CF of r.v. $F_X(x) = P(X \leq x)$

(a) $F_X(-\infty) = 0, \quad F_X(\infty) = 1$

(b) $\forall x' \geq x, \quad F_X(x') \geq F_X(x)$

(c) $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

pdf $f_X(x) = \dfrac{dF_X(x)}{dx} = \lim_{\Delta x \to 0} \dfrac{P(x < X \leq x + \Delta x)}{\Delta x}$

(a) $f_X(x) \geq 0$

(b) $F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx$

(c) $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

(d) $$P(X \in B) = \int_B f_X(x)\,dx.$$

$$EX = \int x f_X(x)\,dx. \quad \left(= \sum x\, p_X(x) \right)$$

$$E\underbrace{g(X)}_{Y} = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx.$$

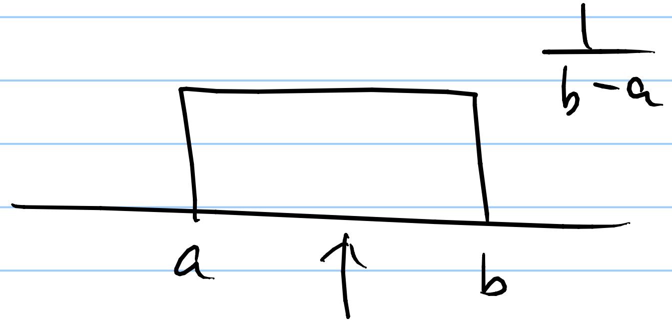
$$E(a_0 + a_1 X + a_2 X^2 + \cdots) = a_0 + a_1 EX + a_2 EX^2 + \cdots$$

$$EX = \mu_X$$

$$\mathrm{Var}(X) = E(X-\mu_X)^2 = EX^2 - \mu_X^2$$

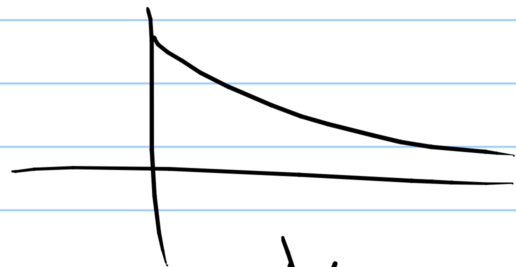
$$\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$$

- $\mathrm{Unif}(a,b)$

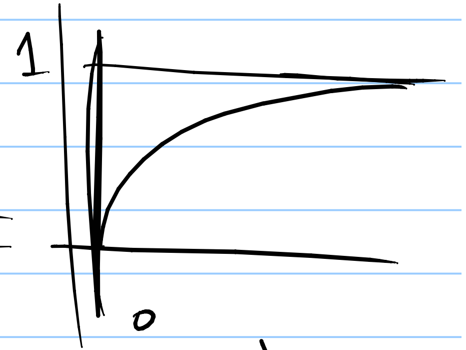


$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

- $\mathrm{Exp}(\lambda)$, $\lambda > 0$



$\lambda e^{-\lambda x}$



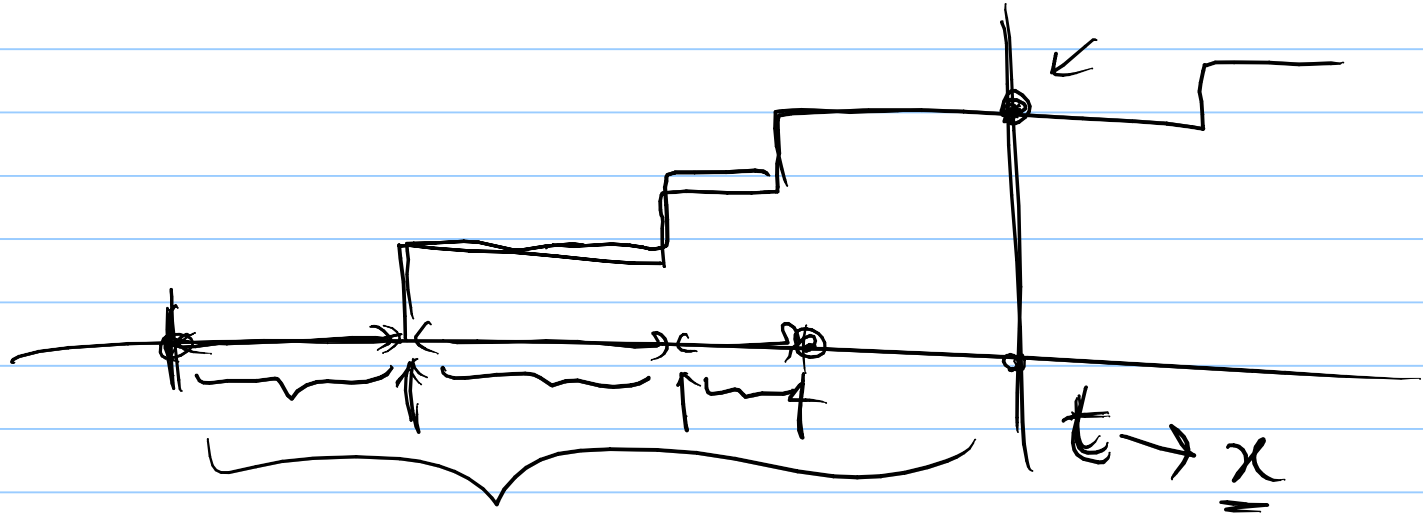
$1 - e^{-\lambda x}$

$$EX = \frac{1}{\lambda} \qquad \mathrm{Var}(X) = \frac{1}{\lambda^2}$$

∘ Erlang $Er(n, \lambda)$, $\lambda > 0$

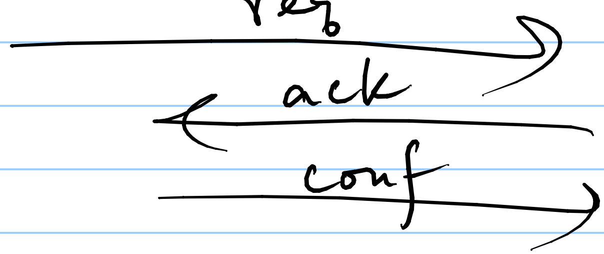
$$f_X(x) = \begin{cases} \lambda^n x^{n-1} e^{-\lambda x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$

$$EX = \frac{n}{\lambda} \qquad Var(X) = \frac{n}{\lambda^2}$$

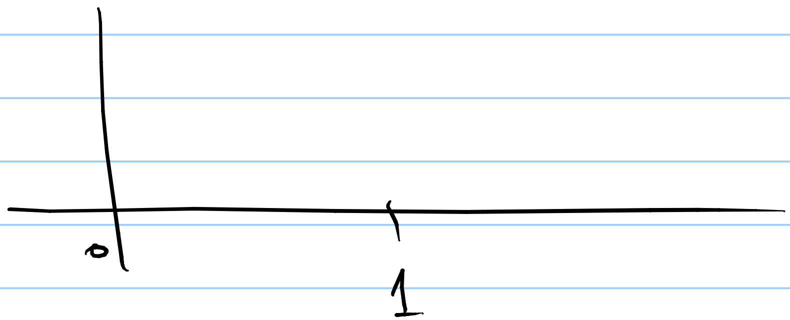


$X_1 \sim \exp(\lambda)$, $X_2 \sim \exp(\lambda)$, $\cdots$

$$\underline{X} = \underline{X_1} + \underline{X_2} + \cdots + \underline{X_n}$$



handshake



λ: rate of arrival

x: time

$$p_K(k) = \frac{(\lambda x)^k}{k!} e^{-\lambda x}.$$

$\underline{\lambda x}$

X_1, X_2, X_3 ind $\exp(\lambda)$

$$\{\underline{K \geq n}\} = \{\underbrace{\sum_{i=1}^{n} X_i \leq x}\} = \underline{F_X(x)}$$

$\uparrow$ er(n, λ)

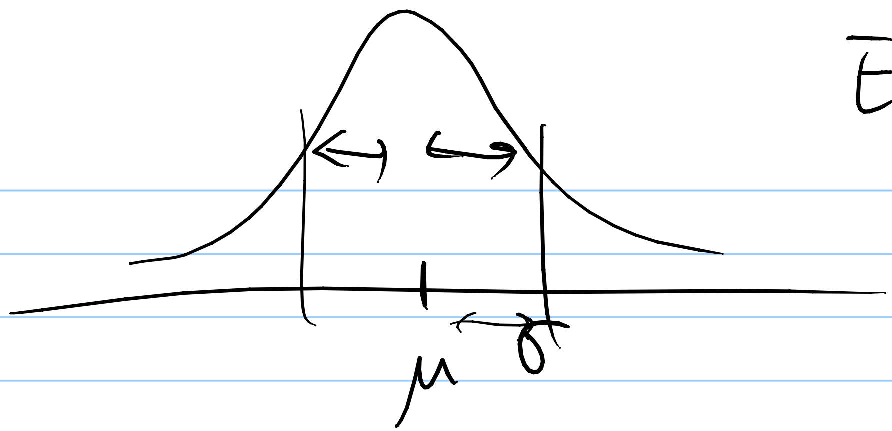
$$= \underline{1 - F_K(n-1)}$$

$\uparrow$ Poiss(λx)

⊛ Gaussian r.v. $(\underline{\mu}, \underline{\sigma^2})$

Normal r.v.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

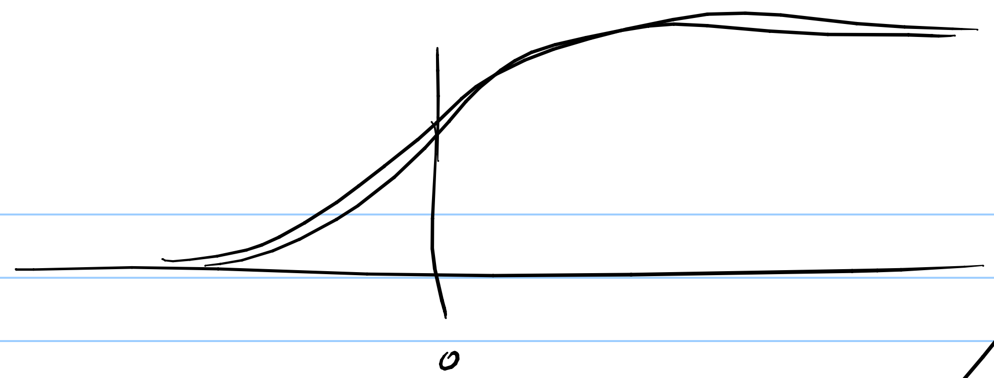


$$EX = \mu \qquad \mathrm{Var}(X) = \sigma^2$$

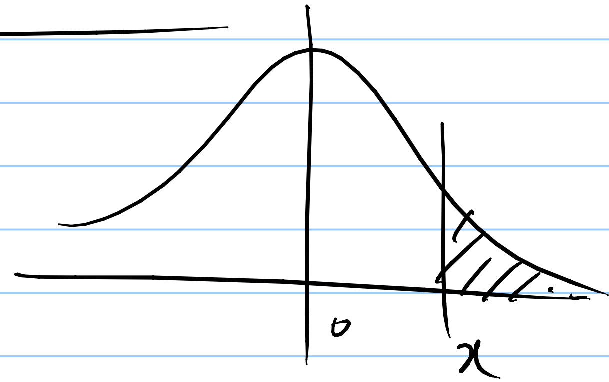
$$\underset{(\mu,\sigma^2)}{X \text{ Gauss}} \Rightarrow \underset{(a\mu+b,\ a^2\sigma^2)}{aX+b \text{ Gauss}}.$$

$$\int_a^b \underline{f_X(x)}\,dx = \underline{P(a < X \le b)}$$

$$\Phi(x) = \underset{(0,1)}{F_X(x)} = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\,du$$



$$\underset{\sim}{Q}(x) = 1 - \underset{\sim}{\Phi}(x)$$



$$\underline{\underline{N}}(\mu, \sigma^2)$$

$$\underset{\sim}{F_X(x)} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(u-\mu)^2}{2\sigma^2}} \, du$$

$$v = \frac{u-\mu}{\sigma} \qquad dv = \frac{du}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi\cancel{\sigma^2}}} \int_{-\infty}^{\frac{x-\mu}{\sigma}} e^{-\frac{v^2}{2}} \, dv \cdot \cancel{\sigma}$$

$$= \underline{\Phi\left(\frac{x-\mu}{\sigma}\right)} = 1 - Q\left(\frac{x-\mu}{\sigma}\right)$$

$$P(a < X \le b) = F_X(b) - F_X(a)$$

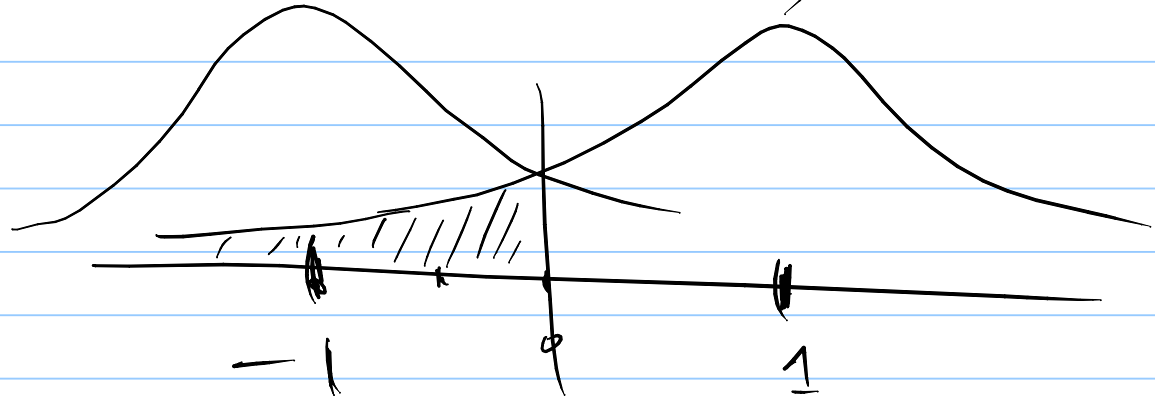
$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

$$= Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$$

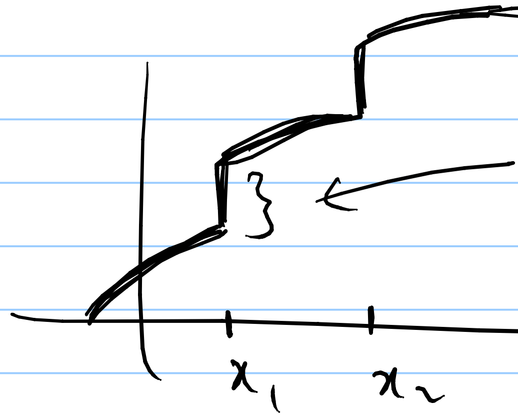
$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-u^2) \, du$$

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-u^2) \, du$$

$$f = 2Q(\sqrt{2}x)$$



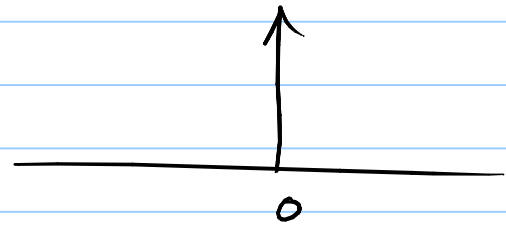
Mixed r.v.

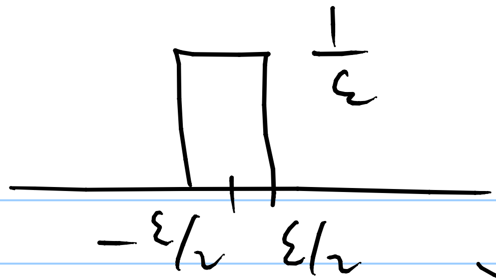


$P(X = x_1) > 0$

Dirac Delta $\delta(x)$

$$= \lim_{\varepsilon \to 0} d_\varepsilon(x)$$





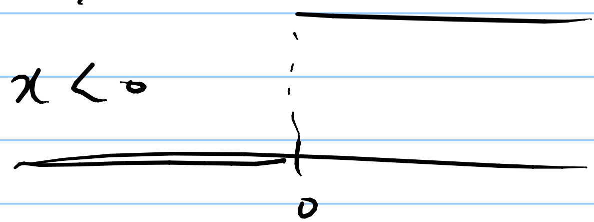
$$\int_{-\infty}^{\infty}\delta(x)\,dx = \int_{-\Delta}^{\Delta}\delta(x)\,dx = 1 \qquad \Delta > 0$$

$$\int_{x_0-\Delta}^{x_0+\Delta}\delta(x-x_0)\,dx = 1$$

$$\int_{-\infty \to x_0-\Delta}^{\infty \to x_0+\Delta} \underline{g(x)}\,\delta(x-\underline{x_0})\,dx = g(x_0)$$

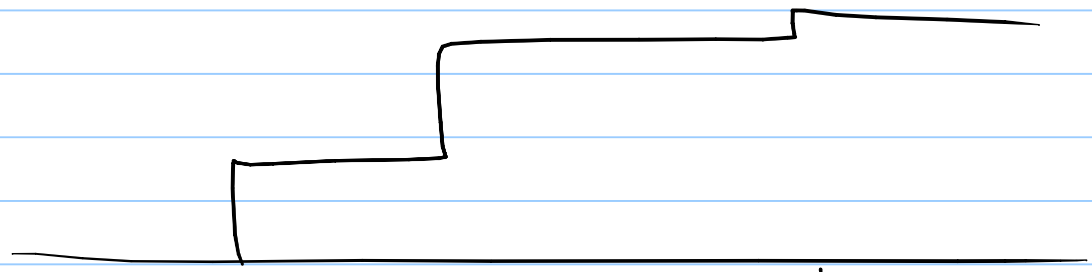
sifting property

$$\int_{-\infty}^{x}\delta(u)\,du = U(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

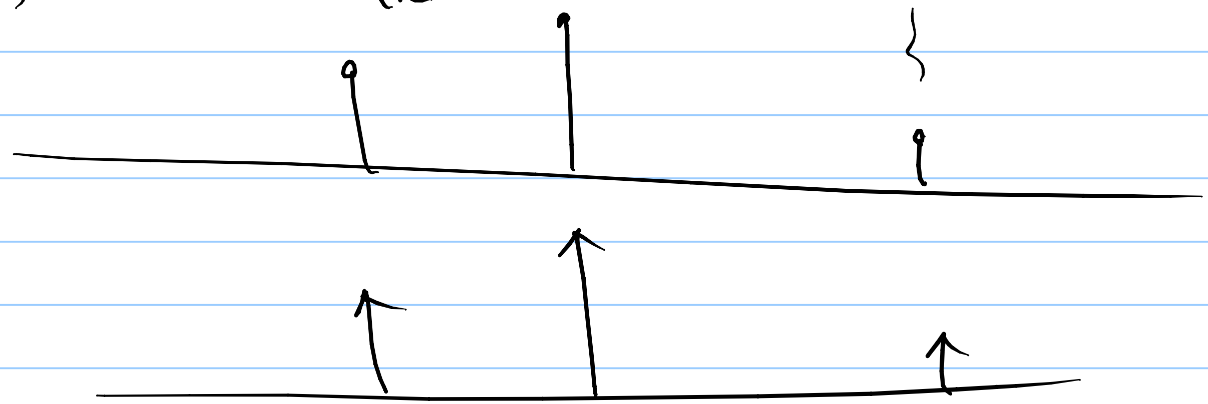


$$\delta(x) = \frac{d\,u(x)}{dx}.$$

Discr rv $F_X(x) = \sum_{x_i \in S_X} p_X(x_i)\, u(x-x_i)$



$$f_X(x) = \frac{dF_X(x)}{dx} = \sum p_X(x_i)\, \delta(x-x_i)$$



$$EX = \int x \underline{\underline{f_X(x)}}\,dx = \int x \left(\sum p_X(x_i)\,\delta(x-x_i) \right) dx$$

$$= \sum p_X(x_i) \underbrace{\int \underline{x}\,\delta(x-x_i)\,dx}_{x_i} = \sum x_i \underline{\underline{p_X(x_i)}}$$

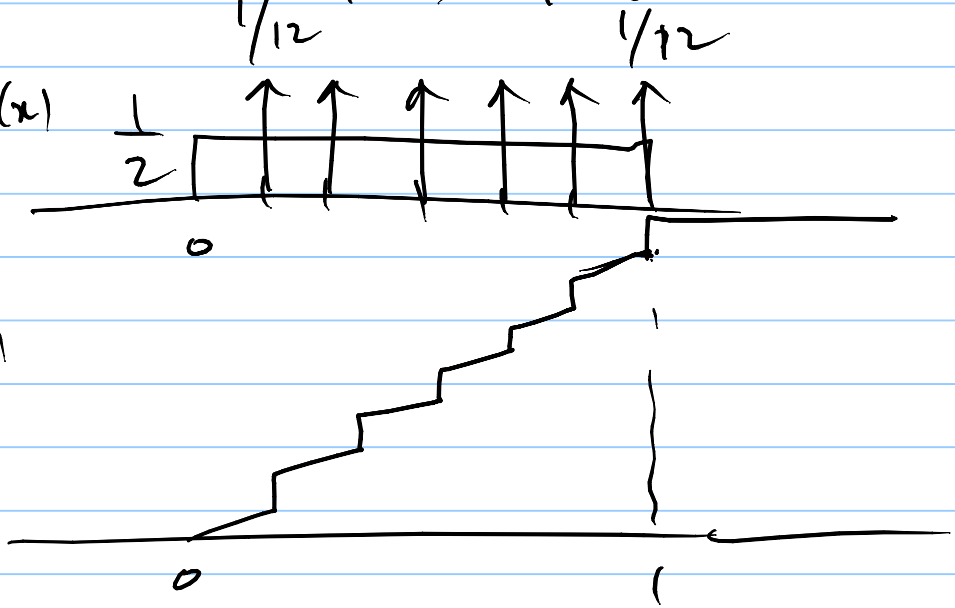
Example: Toss a coin

$\underline{\underline{H}} \rightarrow$ spin a wheel $\underline{[0,1]}$

$T \rightarrow$ toss a die $\underline{\left\{ \frac{1}{6}, \frac{2}{6}, \dots, \frac{6}{6} \right\}}$

X $f_X(x)$

$F_X(x)$



$$F_X(x) = a F_c(x) + (1-a) F_d(x)$$

$\frac{1}{2}$

$$f_X(x) = a f_c(x) + (1-a) f_d(x)$$