Class Handout: Chapter 3 Fundamental Properties

2006 Fall

* Lipschitz condition

I. Existence and Uniqueness of Solutions to Initial Value Problems

Theorem 3.1 (Local Existence and Uniqueness)

$$\dot{x} = f(t, x)$$

$f(t, x)$: piecewise continuous in t and satisfies

$$\|f(t, x) - f(t, y)\| \le L\|x - y\|, \qquad \forall x, y \in \bar{B}_r(x_0),\ \forall t \in [t_0, t_1]$$

with some $L > 0$, $r > 0$ and some $x_0 \in \mathbb{R}^n$.

Then, $\exists \delta > 0$ s.t. the equation

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

has a *unique* sol. over $[t_0, t_0 + \delta]$.

Note.

A point x^* satisfying $x = T(x)$ is called "fixed point of the mapping T."

One way to find x^*: Let $x_2 = T(x_1)$, $\cdots$, $x_{k+1} = T(x_k)$. If $T(\cdot)$ is *contracting mapping*, then it is known that $x_k \to x^*$. (Ex. $T(x) = 0.5x$, $x^* = 0$.) In fact, when T is contracting, x^* exists and is unique.

This idea can also be applied to a vector x that is not only in the Euclidean space, but also in the Banach space (e.g., a set of functions $x(t)$ having a certain properties). Roughly stated, a solution of $\dot{x} = f(t, x)$, $x(t_0) = x_0$ is $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$. Let

$$P(x(\cdot)) \text{ or } P(x) := x_0 + \int_{t_0}^{t} f(s, x(s))ds.$$

And, show that $P(\cdot)$ is a contraction. Then, we are done.

However, in order to do that, we need to study properties of the set of functions.

(Linear) Vector Space $\mathcal{X}$ over the field $\mathcal{R}$

Components: set $\mathcal{X}$, field $\mathcal{R}$, + operation, scalar multiplication, 0 element, 1 element

Properties: ... (see other book) ... $x + y \in \mathcal{X}$, $cx \in \mathcal{X}$, ...

Normed linear space: Linear vector space with a norm $\|\cdot\|$

See the book.

Convergence: A sequence $\{x_k\} \in \mathcal{X}$ converges to $x \in \mathcal{X}$ if $\|x_k - x\| \to 0$ as $k \to \infty$.

Closed Set: A set $S \subset \mathcal{X}$ is closed if and only if every convergent sequence with elements in S has its limit in S.

Cauchy Sequence: A sequence such that

$$\|x_k - x_m\| \to 0 \qquad \text{as} \qquad k, m \to \infty.$$

Every convergent seq. is Cauchy, but not vice versa.

Banach Space: Complete Normed Linear Space. ('Complete' means that every Cauchy seq. in $\mathcal{X}$ converges to a vector in $\mathcal{X}$.)

Example B.1

- The set $\mathcal{X}$: $C[a,b]$ = the set of all continuous functions $f : [a,b] \to \mathbb{R}^n$, which is a vector space on $\mathbb{R}$. ($x + y$ is defined by $(x+y)(t) = x(t) + y(t)$, cx is by $(cx)(t) = cx(t)$, zero/one function is $f(t) = 0/1$.)
- Norm: $\|x\|_C := \max_{t \in [a,b]} \|x(t)\|$.
- Completeness: for a given Cauchy seq. $\{x_k\}$, we need to show that there exists x in $C[a,b]$ and $\|x_k - x\|_C \to 0$ as $k \to \infty$.

If we fix the time t in $[a,b]$, each vector $x_k(t)$, $x_m(t)$ satisfies

$$\|x_k(t) - x_m(t)\| \le \|x_k - x_m\|_C \to 0 \qquad \text{as} \qquad k, m \to \infty$$

So $\{x_k(t)\}$ is a Cauchy seq. in $\mathbb{R}^n$. The set $\mathbb{R}^n$ is known to be complete. So, at each t, $\exists x(t)$ s.t. $x_k(t) \to x(t)$.

Is $x(t)$ continuous and $\|x_k - x\|_C \to 0$? To see this, we first show that the convergence $x_k(t) \to x(t)$ is uniform in $t \in [a,b]$. Indeed, given $\epsilon > 0$, $\exists N$ s.t. $\|x_k - x_m\|_C < \epsilon/2$ for $k, m > N$. Then, for $k > N$,

$$\begin{aligned} \|x_k(t) - x(t)\| &\le \|x_k(t) - x_m(t)\| + \|x_m(t) - x(t)\| \\ &\le \|x_k - x_m\|_C + \|x_m(t) - x(t)\| \end{aligned}$$

With sufficiently large m (which may depend on t), each term can be made less than $\epsilon/2$. Thus, $\|x_k(t) - x(t)\| < \epsilon$ for $k > N$ regardless of t (Uniform Convergence). Therefore, $\|x_k - x\|_C \to 0$ as $k \to \infty$.

Now, consider

$$\|x(t+\delta) - x(t)\| \le \|x(t+\delta) - x_k(t+\delta)\| + \|x_k(t+\delta) - x_k(t)\| + \|x_k(t) - x(t)\|$$

Then, for a given $\epsilon > 0$, choose N s.t. $\|x_k(t) - x(t)\| < \epsilon/3$ for $k > N$ (by uniform convergence). Fix any such k and choose δ s.t. $\|x_k(t+\delta) - x_k(t)\| < \epsilon/3$. Then, we have

$$\|x(t+\delta) - x(t)\| \le \epsilon$$

with the δ (Continuity).

Theorem B.1 (Contraction Mapping Theorem)

S: a closed subset of a Banach space $\mathcal{X}$

T: a mapping that maps S into S, and

$$\|T(x) - T(y)\| \le \rho \|x - y\|, \qquad \forall x, y \in S, \quad 0 \le \rho < 1$$

Then,$\exists$ a unique vector $x^* \in S$ satisfying $x^* = T(x^*)$.

Proof. Let $x_1 \in S$ and $x_{k+1} = T(x_k)$. Prove that $\{x_k\}$ is Cauchy sequence.

Because $\mathcal{X}$ is a Banach space, there exists $x^* \in \mathcal{X}$ such that $x_k \to x^*$ as $k \to \infty$. Also, since S is closed, $x^* \in S$. Prove that $x^* = T(x^*)$.

Finally, prove that x^* is the unique fixed point of T in S.

Proof of Theorem 3.1. Study of the existence and uniqueness of the solution $x(t)$ of $\dot{x} = f(t, x)$ with $x(t_0) = x_0$ is equivalent to the study of the existence and uniqueness of

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) ds. \tag{1}$$

Let the right-hand side be a *mapping* of the continuous function $x : [t_0, t_1] \to \mathbb{R}^n$, and denote it by $(Px)(t)$, i.e.,

$$x(t) = (Px)(t).$$

(Note that $(Px)(t)$ is continuous.) Since a solution of the above is a fixed point of the mapping P, we may use the contraction mapping theory. This requires defining a Banach space $\mathcal{X}$ and a closed set $S \subset \mathcal{X}$ s.t. P maps S into S and is a contraction over S.

$$\mathcal{X} := C[t_0, t_0 + \delta], \qquad \text{with} \quad \|x\|_C = \max_{t \in [t_0, t_0+\delta]} \|x(t)\|$$

$$S := \{x \in \mathcal{X} : \|x - x_0\|_C \le r\}$$

where r comes from the assumption, and $\delta > 0$ will be chosen so that $[t_0, t_0 + \delta] \subset [t_0, t_1]$.

Although P maps $\mathcal{X}$ into $\mathcal{X}$ by definition, prove that P maps S into S with $\delta \leq r/(Lr+h)$ where $h = \max_{t \in [t_0, t_1]} \|f(t, x_0)\|$.

Prove that P is a constraction mapping over S with $\delta < 1/L$.

This proves that there exists a unique solution $x(t)$ in S. Is this also a unique solution in $\mathcal{X}$ (which is our original question)? For this, prove that any solution of (1) in $\mathcal{X}$ lies in S, which means that uniqueness of the solution in S implies uniqueness in $\mathcal{X}$.

Theorem 3.2 (Global Existence and Uniqueness)

$$\dot{x} = f(t, x)$$

$f(t, x)$: piecewise continuous in t and satisfy

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|, \qquad \forall x, y \in \mathbb{R}^n, \ \forall t \in [t_0, t_1]$$

with some $L > 0$.

Then, the equation

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0$$

has a *unique* sol. over $[t_0, t_1]$.

- This theorem implies that the solution exists on $[t_0, \infty)$ if the Lipschitz condition holds for $t \in [t_0, \infty)$. How?

Proof. In the proof of Theorem 3.1, the δ is chosen as

$$\delta \leq \min\left\{t_1 - t_0, \frac{r}{Lr + h}, \frac{\rho}{L}\right\}, \qquad \rho < 1$$

where $h = \max_{t \in [t_0, t_1]} \|f(t, x_0)\|$.

We note that, for the given initial condition, h is determined, and, since f satisfies global Lipschitz condition, the constant r can be taken arbitrarily large so that $\frac{r}{Lr+h} \geq \frac{\rho}{L}$.

Then, δ is determined by

$$\delta \leq \min\left\{t_1 - t_0, \frac{\rho}{L}\right\}.$$

If $\delta = t_1 - t_0$, the proof is done. If $\delta = \frac{\rho}{L} < t_1 - t_0$, then, choose a new $\delta \leq \frac{\rho}{L}$ so that $[t_0, t_1]$ is divided by δ. Finally, by applying the local existence and uniqueness theorem repeatedly, we get the result. (Each time of application, the function h and r may need to be re-calculated because h depends on the initial condition.)

Theorem 3.3 $f(t, x)$: piecewise continuous in t, locally Lipschitz in x for all $t \in [t_0, \infty)$ and all x in a domain $D \subset \mathbb{R}^n$.
W: a compact subset of D and $x_0 \in W$.
Every sol. of $\dot{x} = f(t, x)$ with $x(t_0) = x_0$ lies in W.

Then, $\exists$ a unique sol. on $t \in [t_0, \infty)$.

II. Continuous Dependence on Initial Conditions and Parameters

Gronwall-Bellman's Lemma

If a continuous $y : [a, b] \to \mathbb{R}$ satisfies

$$y(t) \leq \lambda(t) + \int_a^t \mu(s)y(s)ds, \qquad a \leq t \leq b,$$

$\lambda : [a, b] \to \mathbb{R}$ is continuous,

$\mu : [a, b] \to \mathbb{R}$ is continuous and nonnegative,

then,

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\left[\int_s^t \mu(\tau)d\tau\right] ds.$$

In particular, if $\lambda(t) = \lambda$, then,

$$y(t) \leq \lambda \exp\left[\int_a^t \mu(\tau)d\tau\right].$$

If, in addition, $\mu(t) = \mu \geq 0$, then

$$y(t) \leq \lambda \exp[\mu(t - a)].$$

Proof. Let $z(t) := \int_a^t \mu(s)y(s)ds$ and $v(t) := z(t) + \lambda(t) - y(t) \geq 0$. Then,

$$\dot{z}(t) = \mu(t)y(t) = \mu(t)z(t) + \mu(t)\lambda(t) - \mu(t)v(t).$$

Let the state transition matrix

$$\phi(t, s) = \exp\left[\int_s^t \mu(\tau)d\tau\right].$$

Then, since $z(a) = 0$,

$$z(t) = \int_a^t \phi(t, s)[\mu(s)\lambda(s) - \mu(s)v(s)]ds \leq \int_a^t \phi(t, s)\mu(s)\lambda(s)ds.$$

Since

$$y(t) \leq \lambda(t) + z(t) \leq \lambda(t) + \int_a^t \phi(t, s)\mu(s)\lambda(s)ds,$$

the proof is done.

If $\lambda(t) = \lambda$, we have

$$\begin{aligned}
y(t) &\leq \lambda + z(t) \leq \lambda + \lambda \int_a^t \mu(s) \exp\left[\int_s^t \mu(\tau)d\tau\right] ds \\
&= \lambda - \lambda \int_a^t \frac{d}{ds}\left\{\exp\left[\int_s^t \mu(\tau)d\tau\right]\right\} ds \\
&= \lambda - \lambda \left\{\exp\left[\int_s^t \mu(\tau)d\tau\right]\right\}\Bigg|_{s=a}^{s=t} \\
&= \lambda - \lambda \left(1 - \exp\left[\int_a^t \mu(\tau)d\tau\right]\right)
\end{aligned}$$

and, in addition, if $\mu(t) = \mu$, we have

$$
\begin{aligned}
y(t) \le \lambda + z(t) &\le \lambda \exp\left[\int_a^t \mu(\tau) d\tau\right] \\
&= \lambda \exp\left[\mu(t-a)\right].
\end{aligned}
$$

□

Theorem 3.4

$f(t,x)$: piecewise continuous in t, Lipschitz (with Lipschitz constant L) in x, on $[t_0, t_1] \times W$ where $W \subset \mathbb{R}^n$ is an open connected set.

Let $y(t)$ and $z(t)$ be the solutions of

$$\dot{y} = f(t,y), \qquad y(t_0) = y_0$$

and

$$\dot{z} = f(t,z) + g(t,z), \qquad z(t_0) = z_0$$

such that $y(t), z(t) \in W$ for all $t \in [t_0, t_1]$.

If

$$\|g(t,x)\| \le \mu, \qquad \forall (t,x) \in [t_0, t_1] \times W$$

for some $\mu > 0$, then,

$$\|y(t) - z(t)\| \le \|y_0 - z_0\| \exp[L(t-t_0)] + \frac{\mu}{L}\{\exp[L(t-t_0)] - 1\}, \qquad \forall t \in [t_0, t_1].$$

Proof.

$$
\begin{aligned}
y(t) &= y_0 + \int_{t_0}^t f(s, y(s)) ds \\
z(t) &= z_0 + \int_{t_0}^t [f(s, z(s)) + g(s, z(s))] ds \\
\|y(t) - z(t)\| &\le \|y_0 - z_0\| + \int_{t_0}^t \|f(s, y(s)) - f(s, z(s))\| ds + \int_{t_0}^t \|g(s, z(s))\| ds \\
&\le \gamma + \mu(t - t_0) + \int_{t_0}^t L\|y(s) - z(s)\| ds
\end{aligned}
$$

where $\gamma = \|y_0 - z_0\|$. By Gronwall-Bellman inequality,

$$
\begin{aligned}
\|y(t) - z(t)\| &\le \gamma + \mu(t-t_0) + \int_{t_0}^t L[\gamma + \mu(s - t_0)] \exp[L(t-s)] ds \\
&\le \gamma + \mu(t-t_0) - \gamma - \mu(t-t_0) + \gamma \exp[L(t-t_0)] + \int_{t_0}^t \mu \exp[L(t-s)] ds \\
&= \gamma \exp[L(t-t_0)] + \frac{\mu}{L}\{\exp[L(t-t_0)] - 1\}
\end{aligned}
$$

Theorem 3.5

$f(t,x,\lambda)$: continuous in (t,x,λ), locally Lipschitz in x uniformly in t and λ, on $[t_0,t_1] \times D \times \bar{B}_c(\lambda_0)$ where $D \subset \mathbb{R}^n$ is an open connected set.

Let $y(t,\lambda_0)$ be a solution of $\dot{x} = f(t,x,\lambda_0)$ with $y(t_0,\lambda_0) = y_0 \in D$.

Suppose $y(t,\lambda_0)$ is defined and $\in D$ for $t \in [t_0,t_1]$.

Then, given $\epsilon > 0$, $\exists \delta > 0$ s.t. if

$$\|z_0 - y_0\| < \delta \qquad \text{and} \qquad \|\lambda - \lambda_0\| < \delta,$$

there is a unique solution $z(t,\lambda)$ of $\dot{x} = f(t,x,\lambda)$ with $z(t,\lambda) = z_0$, defined on $[t_0,t_1]$, and $z(t,\lambda)$ satisfies

$$\|z(t,\lambda) - y(t,\lambda_0)\| < \epsilon, \qquad \forall t \in [t_0,t_1].$$

Proof. $y(t,\lambda_0)$ is bounded on $[t_0,t_1]$ because it is continuous in t and $[t_0,t_1]$ is compact.

Let a tube U be $\{(t,x) : t \in [t_0,t_1], \|x - y(t,\lambda_0)\| \le \epsilon\}$ so that $U \subset [t_0,t_1] \times D$. This is always possible by reducing ϵ when necessary. (Why?)

In U, $f(t,x,\lambda)$ is Lipschitz in x with L.

By continuity of f, for any $\alpha > 0$, $\exists \beta > 0$ (with $\beta < c$) s.t.

$$\|f(t,x,\lambda) - f(t,x,\lambda_0)\| < \alpha, \qquad \forall (t,x) \in U,\ \forall \lambda \in B_\beta(\lambda_0)$$

Take $\alpha < \epsilon$ and $\|z_0 - y_0\| < \alpha$.

By local existence and uniqueness theorem, $\exists$ a unique sol. $z(t,\lambda)$ on $[t_0, t_0+\Delta]$ with some $\Delta > 0$. This sol. starts inside U, and as long as it remains in U, it can be extended. In fact, with a small enough α, it remains in U for all $t \in [t_0,t_1]$. To see this, let τ be the first time the solution leaves U (be careful..it will be better to say τ is the time when $z(t,\lambda)$ is at the boundary of U). On the time interval $[t_0,\tau]$, the conditions of Theorem 3.4 are satisfied with $\mu = \alpha$ because

$$\dot{z} = f(t,z,\lambda_0) + [f(t,z,\lambda) - f(t,z,\lambda_0)], \qquad z(t_0,\lambda) = z_0.$$

Thus, we have

$$\begin{aligned} \|z(t,\lambda) - y(t,\lambda_0)\| &\le \alpha \exp[L(t-t_0)] + \frac{\alpha}{L}\{\exp[L(t-t_0)] - 1\} \\ &< \alpha \left(1 + \frac{1}{L}\right) \exp[L(t-t_0)] \end{aligned}$$

Choose

$$\alpha \le \epsilon \frac{L \exp[-L(t_1 - t_0)]}{1+L}$$

then, the sol. $z(t,\lambda)$ cannot leave U during $[t_0,\tau]$. This idea confirms that $z(t,\lambda)$ cannot leave U during $[t_0,t_1]$. Taking $\delta = \min\{\alpha,\beta\}$ completes the proof.

III. Differentiability of Solutions and Sensitivity Equations

$$\dot{x} = f(t, x, \lambda_0), \qquad x(t_0) = x_0$$

where $f(t, x, \lambda)$ is C^0 in (t, x, λ), is C^1 in (x, λ), for all $(t, x, \lambda) \in [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^p$, which has a unique sol. $x(t, \lambda_0)$ over $[t_0, t_1]$.

Question: Inspect the effect of the parameter λ on the solution $x(t, \lambda)$. We suppose that the *sensitivity function*

$$S(t) := x_\lambda(t, \lambda) = \frac{\partial x}{\partial \lambda}(t, \lambda)$$

represents the effect of λ on the solution $x(t, \lambda)$. Can we calculate $S(t)$ at a given λ_0?

For $x_\lambda(t, \lambda_0)$ to exist, $x(t, \lambda)$ (as well as $x(t, \lambda_0)$) need to exist for λ close to λ_0. This is guaranteed by Theorem 3.5, because it guarantees that, for all λ sufficiently close to λ_0, the equation

$$\dot{x} = f(t, x, \lambda), \qquad x(t_0) = x_0$$

has a unique sol. $x(t, \lambda)$ over $[t_0, t_1]$ (that is close to $x(t, \lambda_0)$).

Then, how can we obtain $x_\lambda(t, \lambda_0)$? Since

$$x(t, \lambda) = x_0 + \int_{t_0}^{t} f(s, x(s, \lambda), \lambda) ds,$$

we have

$$x_\lambda(t, \lambda) = \int_{t_0}^{t} \left[\frac{\partial f}{\partial x}(s, x(s, \lambda), \lambda) x_\lambda(s, \lambda) + \frac{\partial f}{\partial \lambda}(s, x(s, \lambda), \lambda) \right] ds.$$

Idea: differentiate the above to obtain a differential equation for $S(t) = x_\lambda(t, \lambda_0)$.

$$\dot{S}(t) = A(t, \lambda_0) S(t) + B(t, \lambda_0), \qquad S(t_0) = 0$$

which is called *sensitivity equation*, where

$$A(t, \lambda) = \left. \frac{\partial f(t, x, \lambda)}{\partial x} \right|_{x = x(t, \lambda)}, \qquad B(t, \lambda) = \left. \frac{\partial f(t, x, \lambda)}{\partial \lambda} \right|_{x = x(t, \lambda)}$$

- Sensitivity functions provide first-order estimates of the effect of parameter variations on solutions
- To get $S(t)$, we need to
  1. solve the nominal solution $x(t, \lambda)$ first,
  2. and then, solve the sensitivity equation for $S(t)$

or, solve them at the same time by the equation (3.7).

- (MATLAB) Differential equation for a matrix $S(t)$:
- (Discussion) About the initial condition x_0. In Example 3.7, "... This pattern is consistent when we solve for other initial states."

IV. Comparison Principle

- Gronwall-Bellman inequality
- Comparison principle

Two tools for computing bounds on the solution $x(t)$ for $\dot{x} = f(t, x)$ without computing the solution itself. Frequently, we will arrive at a scalar *differential inequality* of the type

$$\dot{v}(t) \leq f(t, v(t)).$$

Then, by the comparison principle, you will conclude that

$$v(t) \leq u(t)$$

where $u(t)$ is the solution

$$\dot{u}(t) = f(t, u(t)).$$

Lemma 3.4 (Comparison Lemma)

Consider a *scalar* differential equation

$$\dot{u} = f(t, u), \qquad u(t_0) = u_0$$

where $f(t, u)$ is continuous in t, locally Lipschitz in u, on $(t, u) \in [0, \infty) \times J$.
Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of $u(t)$, and suppose $u(t) \in J$ for all $t \in [t_0, T)$.

Let $v(t)$ be continuous and satisfy

$$D^+ v(t) \leq f(t, v(t)), \qquad v(t_0) \leq u_0$$

with $v(t) \in J$ for all $t \in [t_0, T)$.

Then, $v(t) \leq u(t)$ for all $t \in [t_0, T)$.

Dini's Derivative

$$
\begin{aligned}
D^+v(t) &= \limsup_{h\to 0^+} \frac{v(t+h)-v(t)}{h}, \qquad \text{Dini's upper right derivative,}\\
D_+v(t) &= \liminf_{h\to 0^+} \frac{v(t+h)-v(t)}{h},\\
D^-v(t) &= \limsup_{h\to 0^-} \frac{v(t+h)-v(t)}{h},\\
D_-v(t) &= \liminf_{h\to 0^-} \frac{v(t+h)-v(t)}{h}
\end{aligned}
$$

- "Non-smooth analysis"
- $y = \limsup_{n\to\infty} x_n$ means that

1. for every $\epsilon > 0$, $\exists$ an integer N s.t. $n > N$ implies $x_n < y + \epsilon$
   (ultimately all terms of the seq. are less than $y + \epsilon$),
2. given $\epsilon > 0$ and $m > 0$, $\exists$ an integer $n > m$ s.t. $x_n > y - \epsilon$
   (infinitely many terms are greater than $y - \epsilon$).

- If $\lim_{n\to\infty} x_n$ exists, then $\lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.
- Therefore, if $\dot{v}(t)$ exists, then $\dot{v}(t) = D^+v(t)$.

Proof. Consider

$$\dot{z} = f(t,z) + \lambda, \qquad z(t_0) = u_0, \quad \lambda > 0$$

On any compact interval $[t_0, t_1] \subset [t_0, T)$, from Theorem 3.5 (continuous dependence on parameter), for any $\epsilon > 0$, $\exists \delta > 0$ s.t., if $\lambda < \delta$, there is a unique solution $z(t,\lambda)$ on $[t_0, t_1]$ and

$$|z(t,\lambda) - u(t)| < \epsilon, \qquad \forall t \in [t_0, t_1]. \tag{2}$$

Claim 1: $v(t) \le z(t,\lambda)$ on $[t_0, t_1]$.

If not, $\exists a, b \in (t_0, t_1]$ s.t. $v(a) = z(a,\lambda)$ and $v(t) > z(t,\lambda)$ for $a < t \le b$. So,

$$v(t) - v(a) > z(t,\lambda) - z(a,\lambda), \qquad \forall t \in (a, b]$$

and thus,

$$D^+v(t) \ge \dot{z}(a,\lambda) = f(a, z(a,\lambda)) + \lambda > f(a, v(a))$$

which is a contradiction.

Claim 2: $v(t) \le u(t)$ on $[t_0, t_1]$.

If not, $\exists a \in (t_0, t_1]$ s.t. $v(a) > u(a)$. Let $\epsilon = [v(a) - u(a)]/2 > 0$. Then, by (2),

$$v(a) - z(a,\lambda) = v(a) - u(a) + u(a) - z(a,\lambda) \ge \epsilon$$

which contradicts Claim 1.

Since the above argument holds for any compact interval in $[t_0, T)$, the proof is completed.

Example 3.8

$$\dot{x} = -(1 + x^2)x, \qquad x(0) = a$$

We know it has a unique sol. on $[0, t_1)$ for some $t_1 > 0$. (Why?)

Let $v(t) = x^2(t)$. Then,

$$\dot{v}(t) = -2x^2(t) - 2x^4(t) \leq -2v(t), \qquad v(0) = a^2$$

Hence,

$$|x(t)| = \sqrt{v(t)} \leq e^{-t}|a|, \qquad \forall t \geq 0$$

Example 3.9

$$\dot{x} = -(1 + x^2)x + e^t, \qquad x(0) = a$$

We know it has a unique sol. on $[0, t_1)$ for some $t_1 > 0$. (Why?)

Let $v(t) = x^2(t)$. Then,

$$\dot{v}(t) = -2x^2(t) - 2x^4(t) + 2x(t)e^t \leq -2v(t) + 2\sqrt{v(t)}e^t$$

(still difficult to solve).

Let $v(t) = |x(t)|$. Then,

for $x(t) \neq 0$,

$$\begin{aligned}\dot{v}(t) &= \frac{d}{dt}\sqrt{x^2(t)} = \frac{x(t)\dot{x}(t)}{|x(t)|} = -|x(t)|(1 + x^2(t)) + \frac{x(t)}{|x(t)|}e^t \\ &\leq -v(t) + e^t\end{aligned}$$

for $x(t) = 0$, we have

$$\begin{aligned}\frac{|v(t+h) - v(t)|}{h} &= \left|\frac{x(t+h)}{h}\right| = \frac{1}{h}\left|\int_t^{t+h} f(\tau, x(\tau))d\tau\right| \\ &= \left|f(t,0) + \frac{1}{h}\int_t^{t+h}[f(\tau, x(\tau)) - f(t, x(t))]d\tau\right| \\ &\leq |f(t,0)| + \frac{1}{h}\int_t^{t+h}|f(\tau, x(\tau)) - f(t, x(t))|d\tau\end{aligned}$$

Thus,

$$D^+v(t) \leq |f(t,0)| = e^t$$

because the second term is zero by L'Hospital's rule and the continuity of $f(t, x(t))$ in t.

Hence, we conclude that

$$D^+v(t) \le -v(t) + e^t, \qquad v(0) = |a|$$

Then, we have

$$|x(t)| \le e^{-t}|a| + \frac{1}{2}|e^t - e^{-t}|, \qquad \forall t \ge 0$$

(Why $t_1 = \infty$?)