노트 제목 2008-04-29

$$W = g(X, Y)$$

J. disc $\quad p_W(w) = \sum_{(x,y)\,:\,g(x,y)=w} p_{XY}(x,y)$

$$P(W = g(X,Y) = w)$$

J cts $\quad F_W(w) = \iint_{\boxed{g(x,y)\le w}} f_{XY}(x,y)\,dx\,dy$

$$P(W = g(X,Y) \le w)$$

X+Y

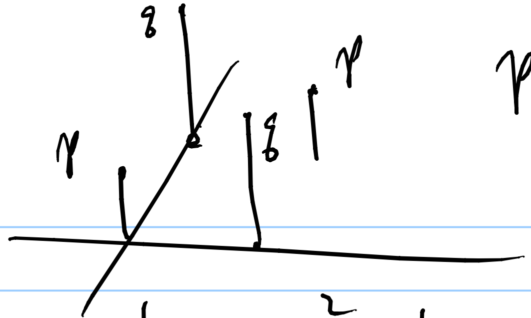
$$f_W(w) = \frac{dF_W(w)}{dw}$$

$$EW = \begin{cases} \sum_w w\, p_W(w) & W: \text{disc} \\ \int w\, f_W(w)\, dw & W: \text{cts} \end{cases}$$

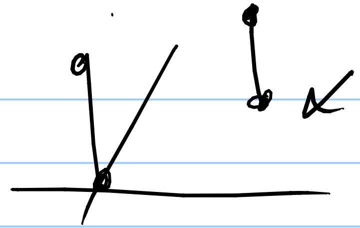
$$= \begin{cases} \sum\sum g(x,y)\, p_{XY}(x,y) \\ \iint g(x,y)\, f_{XY}(x,y)\, dx\, dy \end{cases}$$

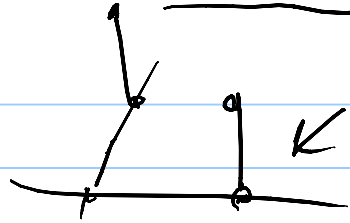
$$\mathrm{Corr}(X,Y) \triangleq r_{XY} = EXY$$

$$\mathrm{Cov}(X,Y) = E(X-\mu_X)(Y-\mu_Y) = r_{XY} - \mu_X \mu_Y$$



$p + q = \frac{1}{2}$ $\quad$ $\mathrm{Cov}(X,Y) = \underline{p - 1/4}$





$EX = \frac{1}{2}$ $\quad$ $EX^2 = \frac{1}{2}$ $\qquad$ $\underline{X = Y}$ $\qquad$ $X = 1 - Y$

$\mathrm{Var}(X) = \frac{1}{4}$

$\mathrm{Cov}(X,Y) = 0 \Rightarrow r_{XY} = \mu_X \mu_Y$ uncorrelated

$EXY = EX\,EY$

$r_{XY} = 0$ $\quad$ X, Y orthogonal.

Corrl coefficient $\rho_{XY} \triangleq \dfrac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$

$$|\rho_{XY}| \le 1$$

Schwarz ineq $\quad |\langle x, y\rangle| \le \|x\| \, \|y\|$

inner prod $\qquad$ norm

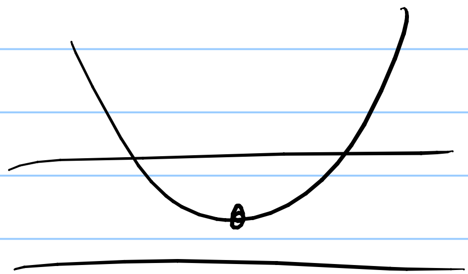
$$|EXY| \le \sqrt{EX^2}\sqrt{EY^2}$$

$$(EXY)^2 \le EX^2 EY^2$$

Pf: $\quad 0 \le E(X - aY)^2 = a^2 EY^2 - 2aEXY + EX^2$

const

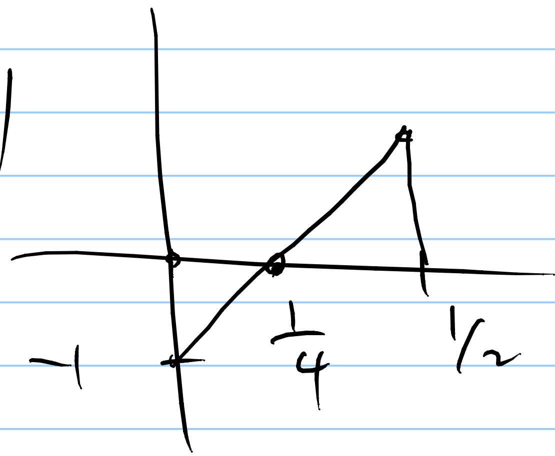
$$= EY^2\left(a - \frac{r_{XY}}{EY^2}\right)^2 + \underbrace{EX^2 - \frac{r_{XY}^2}{EY^2}}_{\ge 0}$$



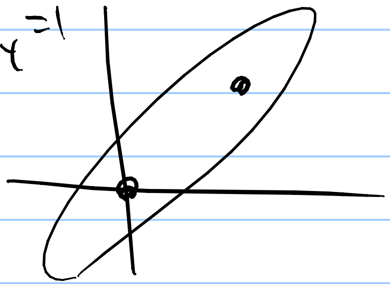
$(EXY)^2 = EX^2EY^2$ iff $X = aY$ for some a

$$\Rightarrow |Cov(X,Y)| = |E(X-\mu_X)(Y-\mu_Y)|$$
$$\le \sqrt{E(X-\mu_X)^2}\sqrt{E(Y-\mu_Y)^2}$$
$$= \sigma_X\sigma_Y$$

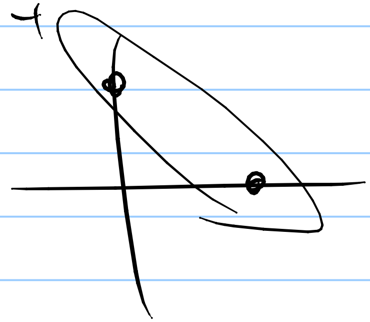
$$\rho_{XY} = \frac{p - 1/4}{\sqrt{1/4}\sqrt{1/4}} = 4p-1$$

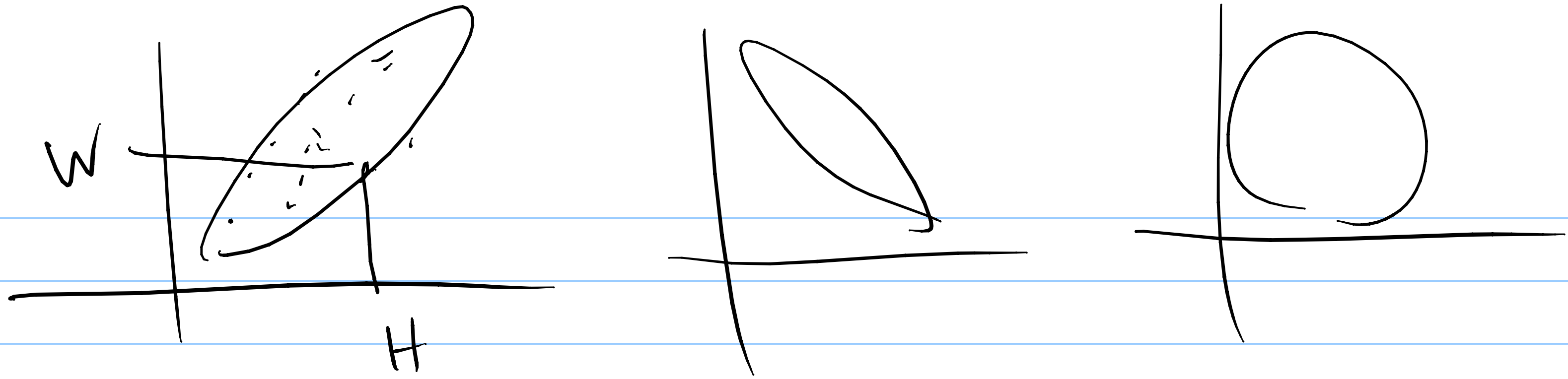


$\rho_{XY} = 1$



$\rho_{XY} = -1$





Conditioning by an event $\underline{\underline{A}}$

$p_{XY}(x,y)$

$p_{XY|A}(x,y) = P(X=x, Y=y \mid A)$

If $A = \{(X,Y) \in B\}$

$$p_{X,Y|A}(x,y) = \begin{cases} \dfrac{p_{XY}(x,y)}{P(A)}, & (x,y) \in B \\ 0, & \text{else} \end{cases}$$

$$f_{XY|A}(x,y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{P\left(x < X \le x+\Delta x,\ y < Y \le y+\Delta x \,\middle|\, A\right)}{\Delta x \Delta y}$$

if $A = \{(X,Y) \in B\}$

$$f_{XY|A}(x,y) = \begin{cases} \dfrac{f_{XY}(x,y)}{P(A)}, & (x,y) \in B \\ 0, & \text{else} \end{cases}$$

Condit expectatn:

$$W = g(X,Y)$$

$$E(W|A) = \begin{cases} \sum\sum g(x,y)\, p_{XY|A}(x,y) \\ \iint g(x,y)\, f_{XY|A}(x,y)\, dx\, dy \end{cases}$$

$$A = \{ Y = y \}$$

j disc $\quad \underbrace{p_{X|Y}(x|y)} = P\left(X = x \,\middle|\, Y = y\right)$

$$= \frac{P(X = x, Y = y)}{P(Y = y)}$$

$$= \frac{p_{XY}(x,y)}{p_Y(y)}$$

$$p_{XY}(x,y) = p_{X|Y}(x|y)\, p_Y(y) \quad \text{chain rule.}$$

$$\left(P(A \cap B) = P(A|B)P(B) \right)$$

j cts.

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$f_{XY}(x,y) = f_{X|Y}(x|y)\, f_Y(y)$$

$(Y=y)$

$(y < Y \leq y+\Delta y)$

$$f_{X|Y}(x|y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{P(x < X \leq x+\Delta x \mid y < Y \leq y+\Delta y)}{\Delta x \quad P(y < Y \leq y+\Delta y)}$$

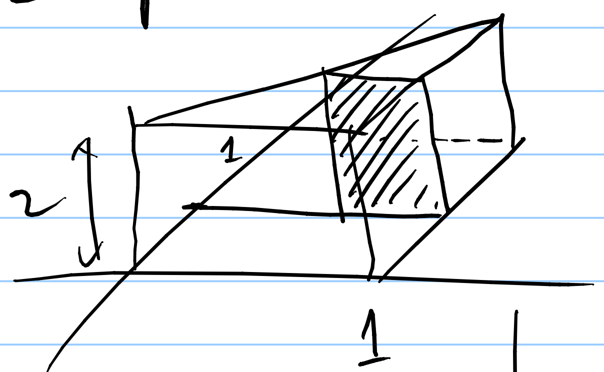
Condit expectatn. $W = g(X,Y)$

$$E(W \mid \underline{\underline{Y=y}}) = \begin{cases} \sum g(x,y)\, p_{X|Y}(\underline{\underline{x}}|y) \\ \int g(x,y)\, f_{X|Y}(\underline{\underline{x}}|y)\, dx \end{cases}$$

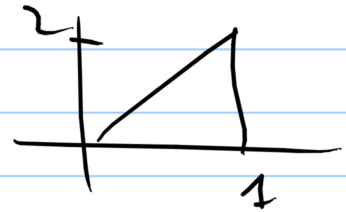
$$E(g(X)\mid Y=y) = \begin{cases} \sum g(x)\, p_{X|Y}(x|y) \\ \int g(x)\, f_{X|Y}(x|y)\, dx \end{cases}$$

$$E(X\mid Y=y) = \begin{cases} \sum x\, p_{X|Y}(x|y) \\ \int x\, f_{X|Y}(x|y)\, dx. \end{cases}$$

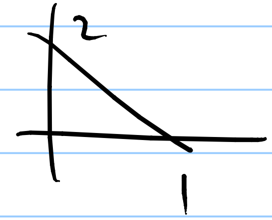
Example



$f_X(x)$

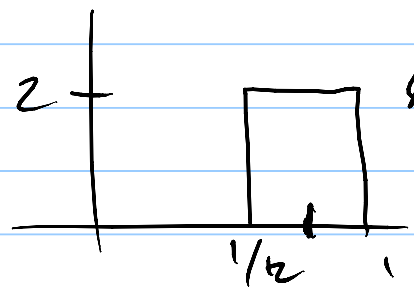


$f_Y(y)$

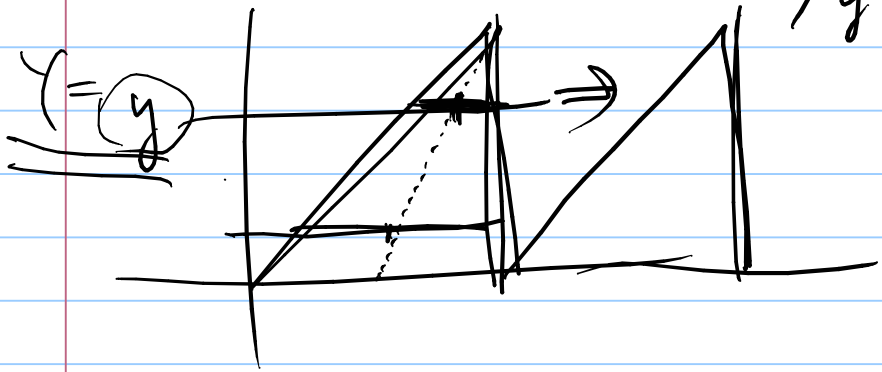


$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{1-y} & y \le x \le 1 \\ 0, & \text{else} \end{cases}$$

$y = \frac{1}{2}$



$$\underline{E(X|Y=y)} = \int_y^1 x\, f_{X|Y}(x|y)\, dx = \frac{y+1}{2}$$

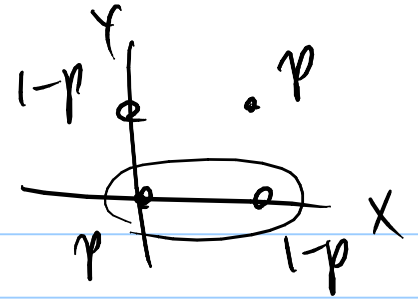


$$\to \frac{y+1}{2}+1$$

$$\underline{E(X|Y)} = \frac{Y+1}{2} \quad \text{New r.v.}$$

$$E(X|Y) = g(Y) \quad \text{fn of } Y$$

$$\text{when } g(y) = E(X|Y=y)$$



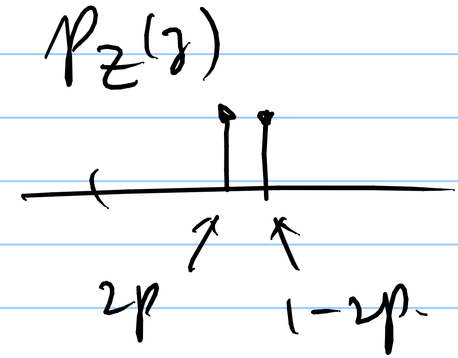
$E(X|Y)$ $\quad E(X|\underline{\underline{Y=0}})$

$$= 0 \cdot \underline{2p} + 1 \cdot 2\left(\tfrac{1}{2}-p\right) = 2\left(\tfrac{1}{2}-p\right)$$
$$= 1-2p.$$

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{\underset{\substack{\shortparallel \\ 1/2}}{p_Y(y)}}$$

$$E(X|Y=1) = 0 \cdot 2\left(\tfrac{1}{2}-p\right) + 1 \cdot 2p = 2p$$

$$Z = E(X|Y) = \begin{cases} \underline{1-2p}, & w/p \ 1/2 \\ \underline{2p}, & w/p \ 1/2 \end{cases}$$



Thm $\quad \underline{E\left[E(X|Y)\right] = EX}$

$$= \int E(X \mid Y = y)\, f_Y(y)\, dy$$

$$= \int \left(\int x\, f_{X|Y}(x|y)\, dx \right) f_Y(y)\, dy$$

$$= \int x \underbrace{\left(\int \underbrace{f_{X|Y}(x|y)\, f_Y(y)}_{f_{XY}(x,y)}\, \underline{\underline{dy}} \right)}_{f_X(x)} dx = EX$$

$$E\, E(g(X) \mid Y) = E g(X)$$

$E(X|Y)$

(a) $E(X|Y)$ is a fn of Y.

(b) $E(a_1 X_1 + a_2 X_2 \,|\, Y) = a_1 E(X_1|Y) + a_2 E(X_2|Y)$

(c) $E(a|Y) = a$

(d) $E(X|a) = EX$

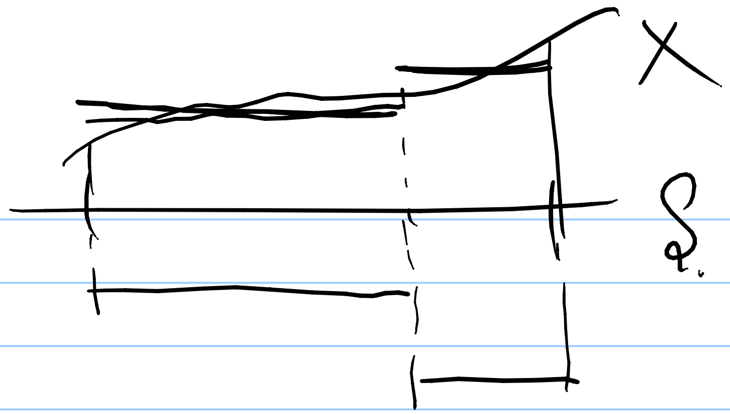
(e) $X \geq 0 \Rightarrow E(X|Y) \geq 0$

(f) $|E(X|Y)| \leq E(|X| \,|\, Y)$

(g) $E(\underline{g(Y)}\, X \,|\, Y) = g(Y)\, E(X|Y)$

(h) $E\, E(X|Y) = EX$.

(i) X, Y indep $\Rightarrow E(X|Y) = EX$



$E(X|Y)$