Corr'd coeff $\rho_{XY} = \dfrac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$

$-1 \le \rho_{XY} \le 1$ $=$ iff $X = aY$

Schwarz ineq. $|EXY|^2 \le EX^2 EY^2$

$|\mathrm{Cov}(X,Y)|^2 \le \sigma_X^2 \sigma_Y^2$

Condit jpmf $p_{XY|A}(x,y) = P(X=x, Y=y \,|\, A)$

" jpdf $f_{XY|A}(x,y) =$

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{P(x < X \le x+\Delta x,\ y < Y \le y+\Delta y \mid A)}{\Delta x \Delta y}$$

$$E(g(X,Y) \mid A) = \begin{cases} \sum_x \sum_y g(x,y)\, p_{XY|A}(x,y) \\ \iint g(x,y)\, f_{XY|A}(x,y)\, dx\, dy \end{cases}$$

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)} \qquad \{Y=y\}$$

$$\uparrow$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$E(X \mid Y=y) = \begin{cases} \sum_x x \, p_{X|Y}(x|y) \\ \int x \, f_{X|Y}(x|y)\, dx. \end{cases}$$

$g(x)$

$$= g(y)$$

$$\boxed{E(X \mid Y) = g(Y)}$$

(a) $E(X|Y)$ is a fn of Y.

$$E(X \mid Y+a) = E(X \mid Y) \leftarrow$$

$$E(X+a \mid Y) = E(X \mid Y) + a \Leftarrow$$

$$\neq E(X \mid \cancel{X}) + E(a \mid Y)$$

(b) $E(a_1X_1 + a_2X_2 | Y) = a_1E(X_1|Y) + a_2E(X_2|Y)$

(c) $E(a|Y) = a$

(d) $E(X|a) = EX$

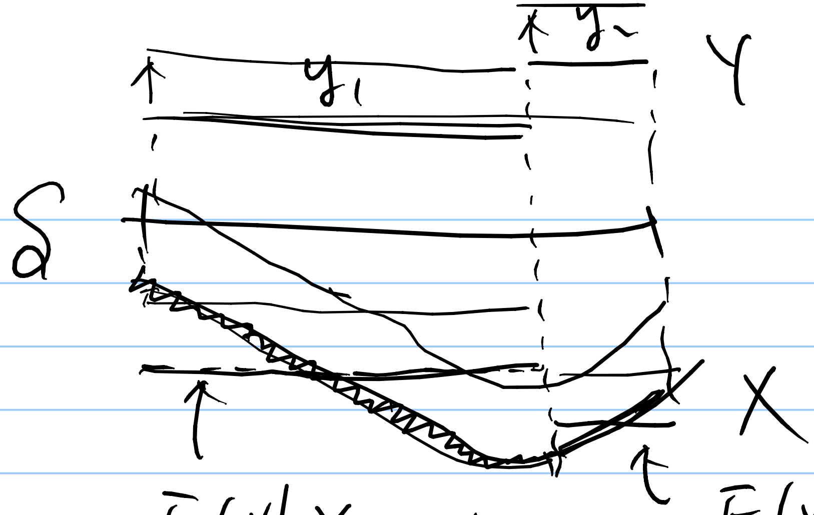
(e) $X \geq 0, \Rightarrow E(X|Y) \geq 0$

(f) $|E(X|Y)| \leq E(|X| \,|\, Y)$

(g) $E(g(\underline{Y})X|Y) = g(Y)E(X|Y)$

(h) $\underline{\underline{E}}\,E(X|Y) = EX$

(i) X, Y $\underline{\underline{\text{indep}}} \Rightarrow E(X|Y) = EX$



$E(X \mid Y = y_1)$

$E(X \mid Y = y_2)$

$E(X \mid Y = \underline{y_1})$ $\quad$ $E(X \mid Y = \underline{y_2})$

Indep rvs X, Y

$$\underline{F_{XY}(x,y) = F_X(x) F_Y(y)} \quad \forall x, y$$

$$\Leftrightarrow \quad p_{XY}(x,y) = p_X(x) p_Y(y) \quad \forall x, y$$

$$\underline{f_{XY}(x,y) = f_X(x) f_Y(y)} \quad \forall x, y \quad \leftarrow$$

$$\searrow \quad f_{X|Y}(x|y)\, f_Y(y)$$

$$\Longleftrightarrow \quad p_{X|Y}(x|y) = p_X(x) \qquad p_{Y|X} = p_Y$$

$$f_{X|Y}(x|y) = f_X(x) \qquad f_{Y|X} = f_Y$$

Condnl dist exists

$$\Longleftrightarrow \quad \forall B, C \quad \{X \in B\} \quad \{Y \in C\} \leftarrow$$

are indep.

$$B = (-\infty, x)$$

$$C = (-\infty, y)$$

$$\Longrightarrow \quad g(X), h(Y) \text{ are indep } \forall g, h.$$

$$\{g(X)\in B\} \perp\!\!\!\perp \{h(Y)\in C\}$$

$$\Vert \qquad\qquad \Vert$$

$$\{X\in B'\} \perp\!\!\!\perp \{Y\in C'\}$$

Uncorrelated $X, Y \iff EXY = EXEY$

$\iff Cov(X,Y)=0$

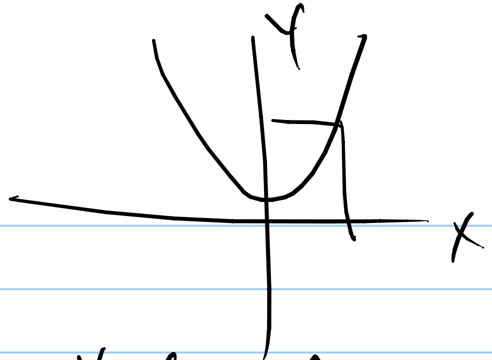
$$EXY = \iint xy\,\underbrace{f_{XY}(x,y)}_{f_X f_Y}\,dx\,dy = \int x f_X(x)\,dx \int y f_Y(y)\,dy$$

$$= EXEY$$

Example: $Y = \underline{\underline{X}}^2$

$$EXY = \int_{-\infty}^{\infty} \underline{\underline{x\cdot x^2}}\,\underbrace{f_X(x)}_{\text{even}}\,dx \qquad EX=0$$

$$= 0 = EX\,\underline{\underline{EY}}$$



Uncorrelate : linear independence

$\rho_{XY} = 0$.

- X, Y indep $\Rightarrow$ $E\,g(X)\,h(Y) = E\,g(X)\,E\,h(Y)$
- $Var(X+Y) = Var(X) + Var(Y) + 2\,\underbrace{Cov(X,Y)}_{=0}$

  if indep or uncorrel.

- $E(X|Y) = EX$ $\quad \left(E(X|Y=y) = EX \ \forall y \right)$

Jointly Gaussian rvs

Bivariate

(Multivariate)

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot$$

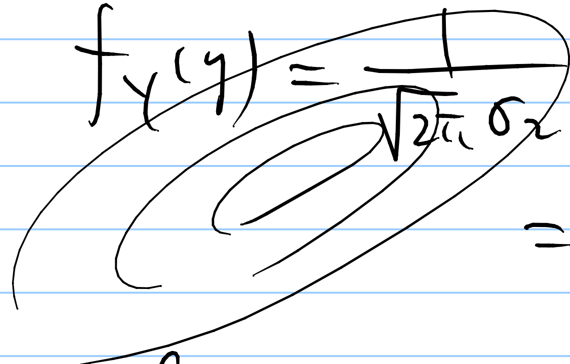
corr coeff X,Y   EX   EY.

$$\exp\left(-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)\right)$$

$\sqrt{Var(X)}$ $\sqrt{Var(Y)}$

$$\Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1}\exp\left(-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right)$$

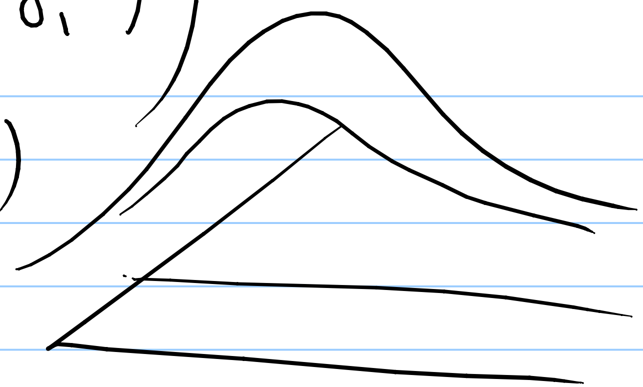
$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_2}\exp\left(-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)$$



$$= \int f_{XY}(x,y)\,dx$$

$$\Rightarrow f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \quad //$$

$$= \frac{1}{\sqrt{2\pi}\,\tilde{\sigma}_1} \exp\left(-\frac{1}{2}\left(\frac{x-\tilde{\mu}_1}{\tilde{\sigma}_1}\right)^2 \right)$$



$$\underline{\tilde{\mu}_1} = \mu_1 + \rho \frac{\sigma_1}{\sigma_2}(y-\mu_2)$$

$$\underline{\underline{\tilde{\sigma}_1}} = \sigma_1 \sqrt{1-\rho^2} \underline{\underline{\leq}} \sigma_1$$

In general Indep $\Longrightarrow$ Uncorrel

$\not\Longleftarrow$

(a) $\boxed{\text{1st \& 2nd moments}}$ $\Longleftarrow$ Gaussian

(b) $\mu_1, \mu_2,$ $\boxed{\begin{matrix}\mathrm{Cov}(X,Y) \\ \text{or} \\ \rho \\ \text{or } r_{XY}\end{matrix}}$ $\Longrightarrow f_{XY}(x,y)$

$\boxed{\sigma_1^2 \quad \sigma_2^2}$

2nd moment

EX^2

$E\underline{XY}$ EY^2

⑨ $aX+bY$, $cX+dY$.

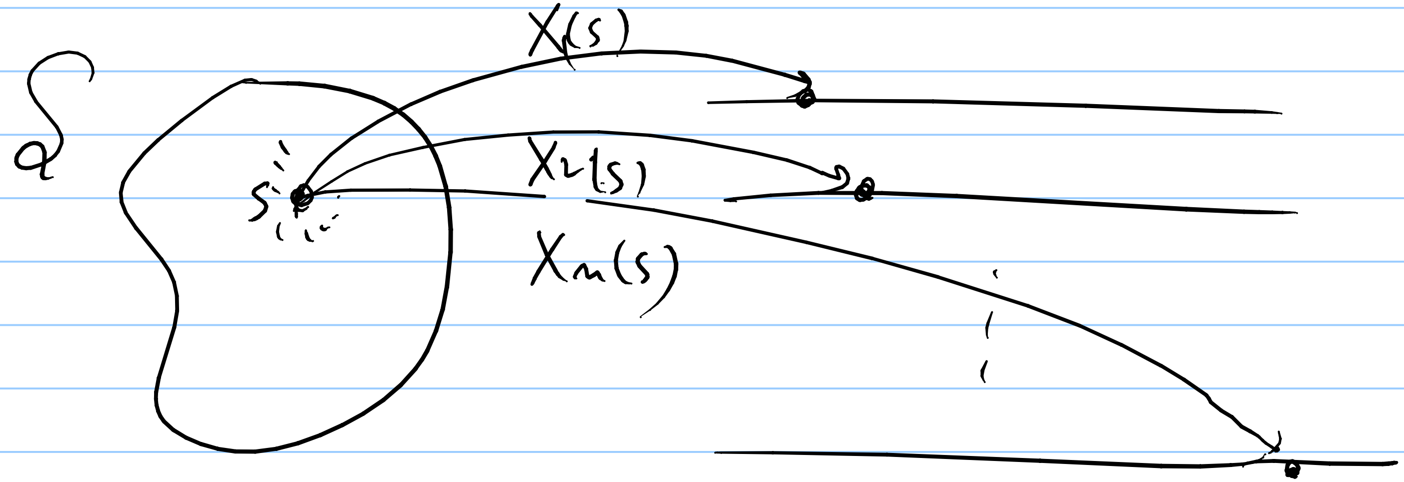
U V

j Gaussian

$$\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

linear tfm.

M. rvs. $X_1\ X_2\ \cdots\ X_n$.



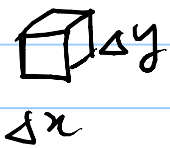
$$\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$$ random vector

jcdf $F_{X_1 \cdots X_n}(x_1, \dots, x_n) \stackrel{d}{=} F_{\underline{X}}(\underline{x})$

$$= P(X_1 \le x_1, \dots, X_n \le x_n)$$

jpmf $p_{\underline{X}}(\underline{x}) = P(X_1 = x_1, \dots, X_n = x_n)$

jpdf $f_{\underline{X}}(\underline{x}) = \dfrac{\partial^n F_{\underline{X}}(\underline{x})}{\partial x_1 \cdots \partial x_n}$



$$= \lim_{|\Delta \underline{x}| \to 0} \frac{P(\underline{X} \in \Delta \underline{x})}{|\Delta \underline{x}|}$$

Volume.

j disc. $P(\underline{X} \in B) = \sum_{\underline{x} \in B} p_{\underline{X}}(\underline{x})$ $\quad \Sigma \cdots \Sigma$

j cts.

$$= \int f_{\underline{X}}(\underline{x})\, d\underline{x}$$

$$\iint \cdots \int \quad dx_1\, dx_2 \cdots dx_n$$

$$\underline{p_{X_2X_3}(x_2, x_3)} = \sum_{x_1} \sum_{x_4} \sum_{x_5} \cdots \sum_{x_n} p_{\underline{X}}(\underline{x})$$

$$f \qquad \int \quad \int \quad \int \qquad \int \qquad d\underline{x}$$

$$p_{X_i}(x_i) = \sum_{x_1} \cdots \sum_{x_{i-1}} \sum_{x_{i+1}} \cdots \sum_{x_n} p_{\underline{X}}(x)$$

$$\int \qquad \int \quad \int \quad \int \qquad \int \qquad d\underline{x}$$

Condit pmf & pdf

$$p_{X_1 \underline{X_2} | X_3 \underline{\underline{X_4}} X_5}(x_1 \underline{x_2} | x_3\, \underline{\underline{x_4}}\, x_5)$$

$$f$$

$$= \frac{p^{f}_{X_1 \underline{X_2} X_3 \underline{\underline{X_4}} X_5}\left(x_1, \underline{\ }, \ldots, \underline{\underline{\ }}, x_5\right)}{p^{f}_{X_3 \underline{\underline{X_4}} X_5}\left(x_3, \underline{\underline{x_4}}, x_5\right)}$$