노트 제목 2008-05-08

Indep rvs $F_{XY}(x,y) = F_X(x)\,F_Y(y)$

$\forall B, C \; \{X \in B\} \; \{Y \in C\}$

are indep.

$(-\infty, x]$ $(-\infty, y]$

$\iff p_{XY}(x,y) = p_X(x)\,p_Y(y)$

$f_{XY} = f_X \; f_Y =$

$\iff p_{X|Y}(x|y) = p_X(x)$

$f_{X|Y} \qquad f_X.$

X, Y indep $\Rightarrow$ $g(X), h(Y)$ indep

$\Rightarrow$ $\underline{EXY = EX\,EY}$ uncorrel

$\Rightarrow$ $Eg(X)h(Y) = Eg(X)\,Eh(Y)$

$\Rightarrow$ $\underline{Var(X+Y) = \sigma_X^2 + \sigma_Y^2}$

$\not\Rightarrow$ $E(X|Y) = EX$.

jG.

$$\underline{\underline{f_{XY}(x,y)}} = \quad \mu_X \quad \mu_Y \quad \begin{bmatrix} \sigma_X^2 & cov(X,Y) \\ cov(X,Y) & \sigma_Y^2 \end{bmatrix}$$

Cov mtx

$$\Rightarrow f_X(x) = \quad \mu_X \quad \sigma_X^2$$

$$\Rightarrow \underbrace{f_{X|Y}(x|y)} = \frac{1}{\sqrt{2\pi}\,\tilde{\sigma}_1} \exp\left(-\frac{(x-\tilde{\mu}_1)^2}{2\tilde{\sigma}_1^2}\right)$$

$$\tilde{\mu}_1 = \mu_X + \rho \frac{\sigma_X}{\sigma_Y}(y-\mu_Y)$$

$$\tilde{\sigma}_1^2 = \underline{\sigma_X^2(1-\rho^2)} \underset{=}{\leq} \underset{=}{\sigma_X^2}$$

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

- Indep. $\Longleftrightarrow$ uncorrel.

$$\overset{\Longrightarrow}{\not\Longleftarrow}$$

- $\mu_X \ \mu_Y \quad \begin{bmatrix} \sigma_X^2 & \overset{\sigma_X\sigma_Y\rho}{\overset{\shortparallel}{\mathrm{Cov}(X,Y)}} \\ \mathrm{Cov}(X,Y) & \sigma_Y^2 \end{bmatrix} = K$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\,|\det K|^{1/2}} \exp\left(-\frac{1}{2}(\underline{x}-\underline{\mu})^t K^{-1}(\underline{x}-\underline{\mu})\right)$$

$$\underline{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mu = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$$

$$\begin{bmatrix} & A & \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix}$$

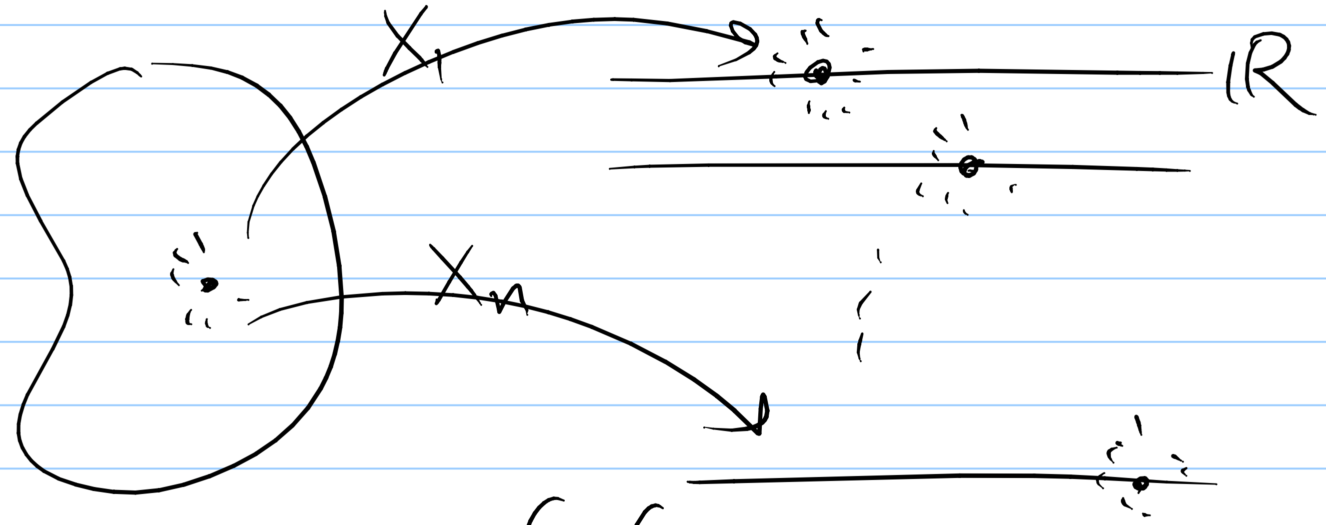
jG jG

~~A~~ n-di rvec $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$

$$F_{\underline{X}}(\underline{x}) = P(X_1 \le x_1 \; \dots \; X_n \le x_n)$$

$$p_{\underline{X}}(\underline{x}) = P(\underline{X} = \underline{x})$$

$$f_{\underline{X}}(\underline{x}) = \frac{\partial^n F_{\underline{X}}(\underline{x})}{\partial x_1 \cdots \partial x_n}$$



$$f_{X_3X_4}(x_3, x_4) = \iiint \cdots \, dx_1\, dx_2\, dx_5 \cdots$$

$$p_{X_1X_2|X_3X_4}(x_1, x_2 | x_3 x_4) = \frac{p_{X_1X_2X_3X_4}(\quad)}{p_{X_3X_4}(\quad)}$$

$$p_{\underline{X_1 X_2 X_3 X_4}}(\quad) = p_{\underline{X_3 X_4}}(\quad)\, p_{\underline{X_1 X_2} | \underline{X_3 X_4}}(\quad | \quad)$$

$$f_X(\underline{x}) = f_{\underline{X_1}}(x_1)\, f_{X_2|\underline{\underline{X_1}}}(x_2|x_1)\, f_{X_3|\underline{X_1, X_2}}(x_3|x_1, x_2) \cdots$$

$$f_{X_n|\underline{X_1 \cdots X_{n-1}}}(x_n|x_1 \cdots x_{n-1})$$

Indep. $F_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} F_{X_i}(x_i)$ $\quad x_3, \cdots, x_n \to \infty$

$$F_{X_1 X_2}(x_1, x_2) = F_{X_1}(x_1)\, F_{X_2}(x_2)$$

$$f_{\underline{X}}(\underline{x}), \quad p_{\underline{X}}(\underline{x})$$

Group indep $(X_1, X_2, X_3, Y_1, Y_2)$

$$F_{X_1 \cdots X_k Y_1 \cdots Y_m}(\quad)$$

$$= F_{X_1 \cdots X_k}(\quad)\, F_{Y_1 \cdots Y_m}(\quad)$$

iid (indep & identically dist) rvs

$$X_1, X_2, \ldots\ldots X_n \qquad f_X(x)$$

$$f_{X_1 X_2}(x_1, x_2) = f(x_1)\, f(x_2)$$

$$F_{\underline{X}}(\underline{x}) = F(x_1)\, F(x_2) \cdots F(x_n)$$

$$p_{\underline{X}}(\underline{x}) = \quad ''$$

Functions of rvecs

$$W = g(\underline{X}) = g(X_1, \ldots, X_n)$$

$$F_W(w) = P(W \leq w)$$

$$= \underset{g(\underline{x}) \leq w}{\int \cdots \int} \underline{\underline{f_{\underline{X}}(\underline{x})}}\, d\underline{x}$$

$$p_W(w) = P(W = w) = \sum_{\underline{x} : g(\underline{x}) = w} \cdots \sum p_{\underline{X}}(\underline{x})$$

$$f_W(w) = \frac{d F_W(w)}{dw}$$

Example: $Y = \max(X_1, \dots, X_n)$

$$F_Y(y) = P(\max(\quad) \le y) = P(X_1 \le y, \dots, X_n \le y)$$

If indep $\quad = P(X_1 \le y) \cdots P(X_n \le y) = \underline{F_{X_1}}(y) \cdots \underline{F_{X_n}}(y)$

If iid $\quad = (F_X(y))^n$

$$f_Y(y) = n(\underline{F_X(y)})^{n-1} f_X(y)$$

$W = \min(X_1, \dots, X_n)$

$$F_W(w) = P(X_1 \le w \text{ or } X_2 \le w \text{ or } \cdots \text{ or } X_n \le w)$$

$$= 1 - P(X_1 > w, \dots, X_n > w)$$

Indep $\quad = 1 - P(X_1 > w) \cdots P(X_n > w)$

$$= 1 - (1 - F_{X_1}(w)) \cdots (1 - F_{X_n}(w))$$

iid $\quad = 1 - (1 - F_X(w))^n$

$$f_W(w) = n(1 - F_X(w))^{n-1} f_X(w)$$

Expectation. $E g(\underline{X}) = \begin{cases} \sum_{\underline{x}} g(\underline{x})\, p_{\underline{X}}(\underline{x}) \\ \int g(\underline{x})\, f_{\underline{X}}(\underline{x})\, d\underline{x}. \end{cases}$

$$E\, g_1(X_1) g_2(X_2) \cdots g_n(X_n)$$

$$= E g_1(X_1) \cdots E g_n(X_n)$$

$$\underline{Y} = \underline{g}(\underline{X}) \Leftarrow \begin{cases} Y_1 = g_1(X_1, X_2) \\ Y_2 = g_2(X_1, X_2) \end{cases}$$

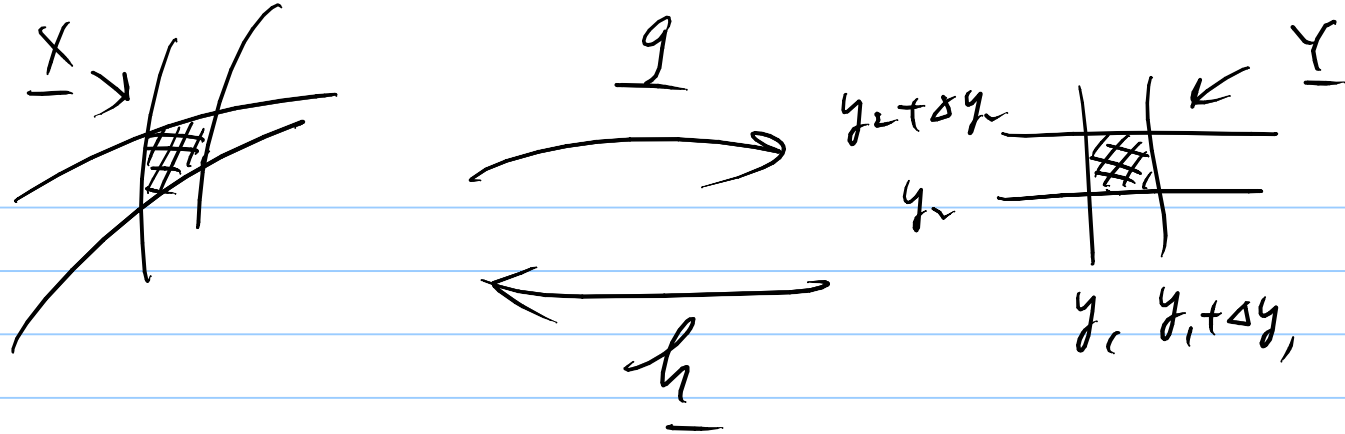
$$\underline{\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}} = \underline{g}\underline{\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}} \quad n \text{ equs. } \quad X_1, \dots, X_n$$

g: cts, ctsly diff'bl, one-to-one,

$$g^{-1} \stackrel{d}{=} h. \qquad \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = h\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \to \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$f_{\underline{Y}}(\underline{y}) = \lim_{\substack{\Delta y_1 \to 0 \\ \Delta y_2 \to 0}} \frac{P(y_1 < Y_1 \le y_1 + \Delta y_1,\ y_2 < Y_2 \le y_2 + \Delta y_2)}{\Delta y_1 \Delta y_2}$$



$$P(\underline{X} \in N_{\underline{x}}) = P(\underline{Y} \in N_{\underline{y}})$$

$$f_{\underline{X}}(\underline{x}) \, \text{area}(N_{\underline{x}}) = f_{\underline{Y}}(\underline{y}) \, \text{area}(N_{\underline{y}})$$

$$\left(P(x < X \leq x+\Delta x) \cong f_X(x)\,\Delta x \right)$$

$$f_{\underline{Y}}(\underline{y}) = f_{\underline{X}}(\underline{x}) \frac{\text{area}(N_{\underline{x}})}{\text{area}(N_{\underline{y}})}$$

$\Delta x_1 \; \Delta y_1$, $\Delta x_2 \; \Delta y_2 \to 0$

$$= f_{\underline{X}}(\underline{h}(\underline{y}))\,|\text{Jacobian}|$$

$$\| $$

$$\left| \det \begin{pmatrix} \dfrac{\partial h_1(y_1, y_2)}{\partial y_1} & \dfrac{\partial h_1(y_1, y_2)}{\partial y_2} \\ \dfrac{\partial h_2(y_1, y_2)}{\partial y_1} & \dfrac{\partial h_2(y_1, y_2)}{\partial y_2} \end{pmatrix} \right|$$

$$\|$$

$$\neq \quad \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right|$$

$$\|$$

$$\underline{f_{\underline{Y}}(\underline{y}) = f_{\underline{X}}(\underline{h}(\underline{y}))} \left(\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}\right) \qquad \underline{\left|\left(\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right)^{-1}\right|}$$

$$\underline{Y} = A\underline{X} + \underline{b} \qquad A \text{ invertible.}$$

$$Y_1 = a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n$$

$$\frac{\partial y_1}{\partial x_1} = a_{11} \cdots \qquad \left|\left(\frac{\partial(\underline{x})}{\partial(\underline{y})}\right)\right| = A^{-1}$$

$$f_{\underline{Y}}(\underline{y}) = f_{\underline{X}}\left(A^{-1}(\underline{y}-\underline{b})\right)\left|\det(A^{-1})\right|$$

$$\left|\det(A^{-1})\right| = \frac{1}{\det(A)}$$

Moments. mean vector

$$\mu_{\underline{X}} = E\underline{X} = \begin{bmatrix} EX_1 \\ \vdots \\ EX_n \end{bmatrix}$$

Correl mtx

$$R_{\underline{X}} = \begin{bmatrix} EX_1^2 & EX_1X_2 & \cdots & EX_1X_n \\ EX_2X_1 & EX_2^2 & \cdots & EX_2X_n \\ \vdots & \vdots & & \vdots \\ EX_nX_1 & EX_nX_2 & \cdots & EX_n^2 \end{bmatrix}$$

$$= E\underline{X}\,\underline{X}^t = E\begin{bmatrix} \\ \\ \\ \end{bmatrix}\begin{bmatrix} & & & \end{bmatrix}$$

Cov mtx.

$$C_{\underline{X}} = E(\underline{X} - \underline{\mu}_X)(\underline{X} - \underline{\mu}_X)^t.$$

$$\begin{pmatrix} \sigma_1^2 & cov(X_1,X_2) \\ cov(\ ) & \sigma_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1^2 & cov(X_1,X_2) & \cdots & cov(X_1,X_n) \\ cov(X_2,X_1) & \sigma_2^2 & \cdots & cov(X_2,X_n) \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

$$\begin{pmatrix} \vdots & \vdots & \ddots & \vdots \\ \mathrm{cov}(X_n, X_1) & \mathrm{cov}(X_n, X_2) & \cdots & \sigma_n^2 \end{pmatrix}$$

- $X_1, X_2 \cdots$ uncorrel. $\Rightarrow$ $C_{\underline{X}} = \begin{pmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \ddots \ \sigma_n^2 \end{pmatrix}$

- $C_{\underline{X}}$ symmetric
  - $\to$ real eigenvalues.
  - $\to$ orthonormal eigenvectors.

- $C_{\underline{X}}$ non-negative definite
  (positive semi-definite)

$$\left(\begin{array}{ll} \underline{a}^t M \underline{a} > 0 & \text{pos def} \\ \quad \text{''} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \geq 0 & \text{non-neg def} \end{array} \right)$$

$\rightarrow$ eigenvalues ≥ 0

$\rightarrow$ $\det(C_{\underline{x}}) \geq 0$ $\boxed{> 0}$

$= 0$