Class Handout: Chapter 4 Lyapunov Stability

2006 Fall

• Lyapunov stability (stability in the sense of Lyapunov): Stability of an equilibrium, Stability of a trajectory (limit cycle)
• Input-output stability

I. Autonomous Systems

$$\dot{x} = f(x)$$

where $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, and $f(0) = 0$.

In most cases, we consider an equilibrium as the origin. (Why?)

Definition 4.1

• We say the origin is *stable* if, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|x(0)\| \le \delta \qquad \Rightarrow \qquad \|\phi(t, x(0))\| \le \epsilon, \quad t \ge 0.$$

($\phi(t, x)$ is the solution starting at x when $t = 0$.)

• We say the origin is *globally attractive* if, for each $x(0)$,

$$\|\phi(t, x(0))\| \to 0, \qquad t \to \infty.$$

• We say the origin is *(locally) attractive* if there exists $\delta > 0$ such that

$$\|x(0)\| < \delta \qquad \Rightarrow \qquad \lim_{t\to\infty} \phi(t, x(0)) = 0.$$

• We say the origin is *globally asymptotically stable (GAS)* if it is stable and globally attractive.

• We say the origin is *(locally) asymptotically stable (LAS/AS)* if it is stable and (locally) attractive.

* exponential stability / uniform asymptotic stability

* Issue of global existence of solution in the definition.

* strong / weak stability

Example. Consider a model of pendulum

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - b x_2, \qquad b \ge 0$$

Read Figure 2.2 for the system with $b = 0$.

Read Figure 2.16 for the system with $b > 0$.

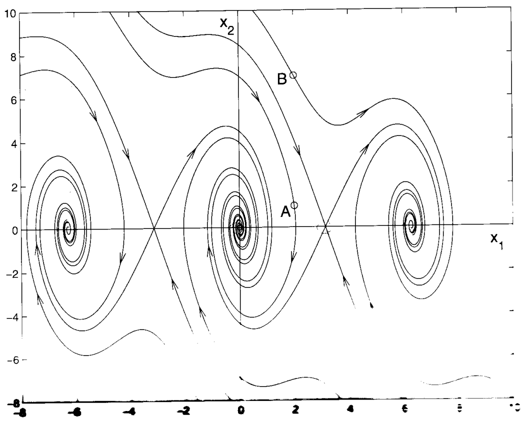


Fig. 1. Figure 2.16

Theorem 4.1

If $\exists$ C^1 *positive definite* function $V : D \to \mathbb{R}$ s.t.

$$\dot{V}(x) \leq 0 \text{ in } D$$

then, the origin is *stable*.

If, in addition,

$$\dot{V}(x) < 0 \text{ in } D - \{0\}$$

then, the origin is *asymptotically stable*.

- The function V is called *Lyapunov function.* (cf. "Lyapunov function candidate")
- $\dot{V}(x) = \frac{\partial V}{\partial x}(x)f(x) =: L_f V(x)$: Directional derivative of $V(x)$ along the direction of $f(x)$ = Lie derivative of V along f.
- *Level set, Lyapunov surface*: $\{x : V(x) = c\}$, $c > 0$.
- *Sublevel set*: $\{x : V(x) \leq c\}$, $c > 0$.
- Meaning of $\dot{V}(x) \leq 0$ on the level set:
- Example of a positive definite function: $V(x) = x^T Px$ where $P > 0$ (positive definite matrix).
- Summary of positive (semi)definite matrix P: (symmetry is assumed)
  - for all nonzero $x \in \mathbb{R}^n$, $x^T Px > (\geq)0$.
  - all eigenvalues of P are positive(nonnegative) real.

– all the leading principal minors of P are positive (all principal minors of P are nonnegative).

– there is a nonsingular matrix N s.t. $P = N^T N$ (there is $N \in \mathbb{R}^{m\times n}$ s.t. $P = N^T N$).

- Example 4.1:

$$P = \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix}$$

which is positive definite for $a > \sqrt{5}$, and negative definite for $a < -\sqrt{5}$.

- Lyapunov function approach is a generalization of decreasing energy concept.
- Lyapunov function is *not* unique.
- Theorem 4.1 is only *sufficient*. (cf. converse theorem of Section 4.7.)

Example 4.2 For $\dot{x} = -g(x)$ where $g(0) = 0$, $xg(x) > 0$ for $x \neq 0$. Try with $V(x) = \int_0^x g(y)dy$. Consider also $V(x) = x^2$ which is simpler. In fact, it is known that, for a scalar system, $V(x) = x^2$ always becomes a Lyapunov function.

Example 4.3 Consider the pendulum without friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1$$

Try $V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2$. Note that, $V(0) = 0$ and is positive definite only over $-\pi < x_1 < \pi$. What is your conclusion about the stability?

Example 4.4 Consider the pendulum with friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - bx_2, \qquad b > 0$$

Try $V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2$. What is your conclusion about the stability?

Now try again with

$$\begin{aligned} V(x) &= \frac{1}{2}x^T P x + a(1 - \cos x_1) \\ &= \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a(1 - \cos x_1) \end{aligned}$$

For P to be positive definite, we must have

$$p_{11} > 0, \qquad p_{11}p_{22} - p_{12}^2 > 0.$$

Also,

$$\dot{V}(x) = (p_{11}x_1 + p_{12}x_2 + a\sin x_1)x_2 + (p_{12}x_1 + p_{22}x_2)(-a\sin x_1 - bx_2)$$
$$= a(1 - p_{22})x_2 \sin x_1 - ap_{12}x_1 \sin x_1 + (p_{11} - p_{12}b)x_1x_2 + (p_{12} - p_{22}b)x_2^2$$

To cancel the indefinite terms, we take $p_{22} = 1$ and $p_{11} = bp_{12}$. Also, let $0 < p_{12} < b$ for $V(x)$ to be positive definite. Let $p_{12} = b/2$. Then, $\dot{V}(x)$ is negative definite on $\{x \in \mathbb{R}^2 : |x_1| < \pi\}$.

Useful facts:

$$\frac{\partial}{\partial x} y^T x = \frac{\partial}{\partial x} x^T y = y$$
$$\frac{\partial}{\partial x} y^T A^T x = \frac{\partial}{\partial x} x^T Ay = Ay$$
$$\frac{\partial}{\partial x} x^T Ax = Ax + A^T x$$

Proof. Roughly stated,

$$V(x(t)) \le V(x(0)), \qquad \forall t \ge 0$$

because $\dot{V}(x(t)) \le 0$, and this proves the stability. To be precise, the above argument need to be converted with the norm $\|x\|$ to fit in the stability definition.

Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ s.t.

$$B_r = \{x : \|x\| < r\} \subset D.$$

Let $\alpha = \min_{\|x\|=r} V(x) > 0$. Take $\beta \in (0, \alpha)$ and let

$$\Omega_\beta = \{x \in B_r : V(x) \le \beta\} \subset B_r.$$

Any trajectory started in Ω_β remains in it. (Why?) Thus, the trajectory (solution) exists for all $t \ge 0$. (Why?)

Since $V(x)$ is continuous and $V(0) = 0$, $\exists \delta > 0$ s.t.

$$\|x\| < \delta \qquad \Rightarrow \qquad V(x) < \beta.$$

So, $B_\delta \subset \Omega_\beta \subset B_r$ and

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < r \le \epsilon, \qquad \forall t \ge 0 \qquad \text{(Stability)}.$$

We now show that $x(t) \to 0$; that is, for each $a > 0$, $\exists T > 0$ s.t. $\|x(t)\| < a$ for all $t > T$. Since for each $a > 0$, we can choose b s.t. $\Omega_b \subset B_a$, it is enough to show that $V(x(t)) \to 0$. (Why?)

Since $V(x(t))$ is nonincreasing and *bounded from below* by zero,

$$V(x(t)) \to c \ge 0 \qquad \text{as } t \to \infty.$$

Suppose $c > 0$. Let $d > 0$ s.t. $B_d \subset \Omega_c$. Let

$$-\gamma = \max_{d \le \|x\| \le r} \dot{V}(x) < 0.$$

Then,

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(s))ds \le V(x(0)) - \gamma t$$

which implies that $V(x(t))$ eventually goes below c, which is a contradiction. (Attractivity).

Region of Attraction (Basin of attraction, Domain of attraction)

If the origin is asymptotically stable, we can consider its region of attraction (ROA) $= \{x : \lim_{t\to\infty} \phi(t,x) = 0\}$.

Estimating ROA

One (conservative) way to find a subset of ROA is to use the level set of a Lyapunov function, that is, if $\Omega_c = \{x : V(x) \le c\}$ is bounded and contained in D, then every trajectory starting in Ω_c remains in Ω_c and approaches the origin as $t \to \infty$.

Theorem 4.2 (Barbashin-Krasovskii theorem)

If $\exists$ C^1 positive definite *radially unbounded* function $V : \mathbb{R}^n \to \mathbb{R}$ s.t.

$$\dot{V}(x) < 0 \qquad \forall x \neq 0$$

then, the origin is *globally asymptotically stable*.

- A positive definite function $V : D \to \mathbb{R}$ is *proper on a set* D if, for each $c > 0$, the sublevel set $\Omega_c := \{x : V(x) \le c\}$ is compact and contained in D. This is equivalent to

$$V(x) \to \infty \qquad \text{as} \qquad x \to \partial D.$$

- radially unbounded = proper on $\mathbb{R}^n$, that is,

$$V(x) \to \infty \qquad \text{as} \qquad \|x\| \to \infty.$$

- Radially unboundedness is needed in the theorem. See Exercise 4.8 for a counterexample. The problem is that for large c, the set Ω_c is not necessarily bounded. See Figure 4.4 for

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2.$$

For small c, Ω_c is closed and bounded (because $V(x)$ is continuous and positive definite). But, for large c, it is unbounded. For Ω_c to be in the interior of a ball B_r, c must satisfy $c < \inf_{\|x\|\ge r} V(x)$. If

$$l = \lim_{r\to\infty} \inf_{\|x\|\ge r} V(x) < \infty$$

then Ω_c will be bounded if $c < l$.

- If an equilibrium is GAS, it means there is no other equilibrium.

Proof. First, prove that radially unboundedness is equivalent to being proper on $\mathbb{R}^n$; that is, prove that

$$V(x) \to \infty \qquad \text{as} \qquad \|x\| \to \infty$$

is equivalent to that, for each $c > 0$, the set Ω_c is compact. (See the textbook.)

Now, for the given initial condition x_0, let $c = V(x_0)$. Then, Ω_c is compact and, since $x(t)$ remains in Ω_c, the previous proof can be employed.

Theorem 4.3 (Chetaev's Theorem) Instability theorem

If $\exists$ C^1 function $V : D \to \mathbb{R}$ s.t. $V(0) = 0$ and $V(x) > 0$ for some x arbitrarily close to the origin, and

$$\dot{V}(x) > 0 \qquad \text{on} \quad U := \{x \in B_r : V(x) > 0\}, \quad r > 0,$$

then, the origin is unstable.

- U is non-empty.
- The boundary of U is the surface $V(x) = 0$ and $\|x\| = r$. The origin is on the boundary. (See Figure 4.5).

Proof.

We show that the trajectory $x(t)$, from $x(0) = x_0$ where x_0 is in the *interior* of U, must leave U.

Let $a = V(x_0) > 0$. Then, since $\dot{V}(x) > 0$, it follows that $V(x(t)) \geq a$ for all $t \geq 0$. This means that $x(t)$ cannot cross the boundary ($V(x) = 0$) of U. We thus show that $x(t)$ will cross the boundary ($\|x\| = r$) of U.

Let $\gamma := \min\{\dot{V}(x) : x \in U \text{ and } V(x) \geq a\} > 0$ which exists since it is a minimization of a continuous function over a compact set. Then,

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))ds \geq a + \int_0^t \gamma ds = a + \gamma t.$$

Hence, $x(t)$ cannot stay in U forever because $V(x)$ is bounded on U, which means $x(t)$ crosses the curve $\|x\| = r$. Because this happens for any x_0 arbitrarily close to the origin, the origin is unstable.

Example 4.7 Consider

$$\dot{x}_1 = x_1 + g_1(x)$$
$$\dot{x}_2 = -x_2 + g_2(x)$$

where $|g_i(x)| \leq k\|x\|_2^2$ in the neighborhood of the origin.

Let $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$. Then,

$$\dot{V}(x) = x_1^2 + x_2^2 + x_1 g_1(x) - x_2 g_2(x) \geq \|x\|_2^2 - 2k\|x\|_2^3 = \|x\|_2^2(1 - 2k\|x\|_2)$$

So, with B_r, $r < 1/(2k)$, we conclude the origin is unstable.

II. The Invariance Principle

Recall Example 4.4 (Pendulum with friction):

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\sin x_1 - bx_2, \qquad b > 0$$

With $V(x) = a(1 - \cos x_1) + \frac{1}{2}x_2^2$,

$$\dot{V}(x) = -bx_2^2.$$

For the system to maintain $\dot{V}(x) = 0$, the trajectory should be confined to line $x_2 = 0$. But, this is impossible unless $x_1 = 0$. This makes it possible to claim $V(x(t)) \to 0$ even though $\dot{V}(x) \leq 0$.

- *Positive limit point* p of the solution $x(t)$: $\exists$ a seq. $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, s.t., $x(t_n) \to p$ as $n \to \infty$.

(Ex. AS equilibrium / any point in AS limit cycle)

- *Positive limit set* of $x(t)$: set of all positive limit points of $x(t)$.

(Ex. AS limit cycle)

- *Invariant set* M (of the system): a set M s.t.

$$x(0) \in M \qquad \Rightarrow \qquad x(t) \in M, \qquad \forall t \in (-\infty, \infty).$$

(Ex. limit cycle, equilibrium, ...)

- *Positive(negative) invariant set* M: in the above, replace $t \in (-\infty, \infty)$ with $t \in [0, \infty)$ ($t \in (-\infty, 0]$).

(Ex. Ω_c with $\dot{V}(x) \leq 0$.)

- Distance of a point x from a set M: $\text{dist}(x, M) = \inf_{y \in M} \|x - y\| = \|x\|_M$

Lemma 4.1

If a sol. $x(t) \in D$ is bounded for all $t \geq 0$,

then, its positive limit set L^+ is nonempty, compact, and invariant.

Moreover, $x(t) \to L^+$ as $t \to \infty$.

Proof. By Bolzano-Weierstrass theorem, L^+ is nonempty because $x(t)$ is bounded.

L^+ is bounded because, for any $y \in L^+$, there is a seq. $\{t_i\}$ s.t. $x(t_i) \to y$. Since $x(t)$ is bounded, y is bounded, too.

L^+ is closed. Let $\{y_i\} \in L^+$ be a seq. s.t. $y_i \to y$. We will prove that $y \in L^+$. For each i, $\exists$ a seq. $\{t_{ij}\}$ s.t.

$$t_{ij} \to \infty, \qquad x(t_{ij}) \to y_i, \qquad \text{as } j \to \infty.$$

Among the sequence elements $\{t_{ij}\}$, we will pick some of them to construct another seq. $\{\tau_i\}$ as follows: choose τ_2 among $\{t_{2j}\}$ s.t. $\tau_2 > t_{12}$ and $\|x(\tau_2) - y_2\| < 1/2$; choose τ_3 among $\{t_{3j}\}$ s.t. $\tau_3 > t_{13}$ and $\|x(\tau_3) - y_3\| < 1/3$; and so on. As a result, $\tau_i \to \infty$ and $\|x(\tau_i) - y_i\| < 1/i$. Now, given $\epsilon > 0$, $\exists N_1, N_2 > 0$ s.t.

$$\|x(\tau_i) - y_i\| < \frac{\epsilon}{2}, \qquad \forall i > N_1 \qquad \text{and} \qquad \|y_i - y\| < \frac{\epsilon}{2}, \qquad \forall i > N_2.$$

From the above, we have

$$\|x(\tau_i) - y\| < \epsilon, \qquad \forall i > N = \max\{N_1, N_2\},$$

which implies that y is also a limit point (so, $y \in L^+$).

L^+ is invariant. Let $y \in L^+$ and $\phi(t; y)$ be the sol. that passes through y at $t = 0$. We show that $\phi(t; y) \in L^+, \forall t \in (-\infty, \infty)$. There is a seq. $\{t_i\}$ s.t. $t_i \to \infty$ and $x(t_i) \to y$. Write $x(t_i) = \phi(t_i; x_0)$ where $x_0 = x(0)$. By uniqueness of the sol.,

$$\phi(t + t_i; x_0) = \phi(t; \phi(t_i; x_0)) = \phi(t; x(t_i)).$$

Then, for any $t \in (-\infty, \infty)$, (by continuity)

$$\lim_{i\to\infty} \phi(t + t_i; x_0) = \lim_{i\to\infty} \phi(t; x(t_i)) = \phi(t; y)$$

which shows $\phi(t; y) \in L^+$.

We now show that $x(t) \to L^+$ as $t \to \infty$. Suppose this is not the case. Then, $\exists \epsilon > 0$ and $\{t_i\}$ with $t_i \to \infty$ s.t. $\|x(t_i)\|_{L^+} > \epsilon$. Since $\{x(t_i)\}$ is bounded, there is a subsequence of it $\{x(t_i')\}$ s.t. $x(t_i') \to x^*$ with some x^*. Then, x^* must be in L^+ because it is a limit point. This contradicts that $\|x(t_i)\|_{L^+} > \epsilon$.

Theorem 4.4 (LaSalle's Invariance Theorem)

$\Omega \subset D$: a positively invariant compact set.

$V : D \to \mathbb{R}$: C^1 function s.t. $\dot{V}(x) \leq 0$ in Ω.

$E \subset \Omega$: the set of points s.t. $\dot{V}(x) = 0$ in Ω.

$M \subset E$: the largest invariant set in E.

Then, every solution starting in Ω approaches M as $t \to \infty$.

Proof. First, since $\dot{V}(x(t)) \leq 0$ and $V(x)$ is bounded below, $\exists a$ s.t. $V(x(t)) \to a$ as $t \to \infty$.

On the other hand, since Ω is compact and positively invariant, $\exists$ a positive limit set L^+ of $x(t)$ *in* Ω. We will show that

$$L^+ \subset M \subset E \subset \Omega,$$

which proves the claim since $x(t)$ is bounded, so $x(t) \to L^+$.

Pick any $p \in L^+$, then there is a seq. $\{t_n\}$ with $t_n \to \infty$ and $x(t_n) \to p$. Then,

$$V(p) = \lim_{n\to\infty} V(x(t_n)) = a,$$

which means $V(x) = a$ on L^+. Since L^+ is invariant, $\dot{V}(x) = 0$ on L^+, so $L^+ \subset M$.

- Only applicable to autonomous (time-invariant) system.
- $V(x)$ need not be positive definite.
- Ω can be found by a sublevel set of $V(x)$, or by other ways.
- If M consists only of the origin, then it is claimed that $x(t) \to 0$. This is done by showing that no solution can stay identically in E, other than the trivial solution $x(t) \equiv 0$.

Corollary 4.1

$V : D \to \mathbb{R}$: C^1 positive definite function s.t. $\dot{V}(x) \le 0$.

$S = \{x \in D : \dot{V}(x) = 0\}$.

If no sol. can stay identically in S other than $x(t) = 0$, then the origin is AS.

Corollary 4.2

In addition, if $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded, then the origin is GAS.

Example. Show that the system

$$\dot{x}_1 = x_2^3$$
$$\dot{x}_2 = -x_1 - x_2$$

is globally asymptotically stable (using $V(x) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4$).

Example 4.10 Consider

$$\dot{y} = ay + u$$

with an adaptive control law

$$u = -ky, \qquad \dot{k} = \gamma y^2, \qquad \gamma > 0.$$

Taking $x_1 = y$, $x_2 = k$, the closed-loop system becomes

$$\dot{x}_1 = -(x_2 - a)x_1$$
$$\dot{x}_2 = \gamma x_1^2$$

The line $x_1 = 0$ is an equilibrium set. (Meaning?)

Consider

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2, \qquad b > a$$

Then,

$$\dot{V}(x) = -x_1^2(x_2 - a) + x_1^2(x_2 - b) = -x_1^2(b - a) \le 0.$$

Then, $V(x)$ is radially unbounded, Ω_c is compact for any $c > 0$. The set $E = M = \{x : x_1 = 0\}$. So, we conclude that $y(t) \to 0$.

Since we do not know a, we cannot determine the value b. But, the whole argument still holds.

III. Linear Systems and Linearization

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n$$

Theorem 4.5 The origin is stable if and only if

- all eigenvalues of A satisfy $\text{Re}\,\lambda_i \le 0$
- $\text{rank}(A - \lambda_i I) = n - q_i$ for every eigenvalue with $\text{Re}\,\lambda_i = 0$ and algebraic multiplicity $q_i \ge 2$.

The origin is (globally) asymptotically stable if and only if all eigenvalues of A satisfy $\text{Re}\,\lambda_i < 0$ (*Hurwitz* or *stable* matrix).

Proof.

$$T^{-1}AT = J = \text{blockdiag}[J_1, J_2, \cdots, J_r]$$

where J_i is a Jordan block of order m_i corresponding to the eigenvalue λ_i. Then,

$$\exp(At) = T\exp(Jt)T^{-1} = \sum_{i=1}^{r}\sum_{k=1}^{m_i} t^{k-1}\exp(\lambda_i t)R_{ik}$$

E.g., $\exp(Jt) = \begin{bmatrix} e^{-\lambda_1 t} & te^{-\lambda_1 t} \\ 0 & e^{-\lambda_1 t} \end{bmatrix}$

where R_{ik} is an appropriate $n \times n$ matrix. Note that

$$x(t) = \exp(At)x(0).$$

With all the above, the claim can be argued.

Example 4.12 Series and parallel connections of the identical model

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

The resulting system has the system matrix of

$$A_p = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \qquad A_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}.$$

Both A_p and A_s has the same e.v. $\pm j$ with the algebraic multiplicity $q_i = 2$. Since rank $(A_p - jI) = 2$, A_p is stable, and since rank $(A_s - jI) = 3$, A_s is unstable.

* "resonance" for the series connection.

Consider a *quadratic* Lyapunov function $V(x) = x^T P x$ where $P > 0$ for the system $\dot{x} = Ax$. Then,

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x$$

where Q is symmetric given by

$$PA + A^T P = -Q.$$

We know that, if $Q > 0$, then the system is (globally) asymptotically stable. (Why?)

Theorem 4.6 The following are equivalent:

1. A matrix A is Hurwitz.
2. For any $Q > 0$, $\exists$ a $P > 0$ that satisfies

$$PA + A^T P = -Q, \qquad \text{(Lyapunov equation)}.$$

Moreover, if A is Hurwitz, the P is unique for each Q in the above.

* $Q = C^T C$, where (A, C) is observable, gives the same result. (Exercise 4.22) Home Work!

* Sylvester equation: $PA + BP + C = 0$.

Proof. (2)→(1): DIY.

(1)→(2): Since A is Hurwitz, define

$$P := \int_0^\infty \exp(A^T t) Q \exp(At)\, dt,$$

which is well-defined. (Why?)

From the definition, P is symmetric. In addition, P is positive definite. Indeed, supposing it is not so, $\exists\, x \neq 0$ s.t. $x^T P x = 0$. Then,

$$\int_0^\infty x^T \exp(A^T t) Q \exp(At) x \, dt = 0$$
$$\Rightarrow \qquad \exp(At)x \equiv 0 \qquad \Rightarrow \qquad x = 0$$

which is a contradiction.

Therefore,

$$\begin{aligned} PA + A^T P &= \int_0^\infty \exp(A^T t) Q \exp(At) A dt + \int_0^\infty A^T \exp(A^T t) Q \exp(At) dt \\ &= \int_0^\infty \frac{d}{dt} \exp(A^T t) Q \exp(At) dt = \exp(A^T t) Q \exp(At)|_0^\infty = -Q \end{aligned}$$

which means that P is actually a solution of the Lyapunov equation.

Finally, P is unique because, if not, with another solution $\tilde{P} \neq P$, we have

$$(P - \tilde{P})A + A^T (P - \tilde{P}) = 0.$$

Premultiplying by $\exp(A^T t)$ and postmultiplying by $\exp(At)$, we obtain

$$0 = \exp(A^T t)[(P - \tilde{P})A + A^T(P - \tilde{P})]\exp(At) = \frac{d}{dt}\left\{\exp(A^T t)(P - \tilde{P})\exp(At)\right\}.$$

Hence,

$$\exp(A^T t)(P - \tilde{P})\exp(At) \equiv \text{a constant}, \quad \forall t.$$

Then,

$$(P - \tilde{P}) = \exp(A^T t)(P - \tilde{P})\exp(At) \to 0 \quad \text{as} \quad t \to \infty,$$

which proves that $P = \tilde{P}$.

Pitfall: There's no mean value theorem for multi-variable functions. That is, for a C^1 function $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(0) = 0$, it is not true that there exists q for each x such that $f(x) = \frac{\partial f}{\partial x}(q)x$. For example, try with $f(x_1, x_2) = [x_1^2, \exp(x_1) - 1]^T$.

$$\dot{x} = f(x)$$

where f is C^1 and $f(0) = 0$.

By the mean value theorem, for each x, $\exists z_i$ s.t.

$$f_i(x) = f_i(0) + \frac{\partial f_i}{\partial x}(z_i)x = \frac{\partial f_i}{\partial x}(0)x + \left[\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right]x.$$

Hence

$$\dot{x} = f(x) = Ax + g(x)$$

where

$$|g_i(x)| \leq \left\|\frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0)\right\| \|x\|$$

which implies that

$$\frac{\|g(x)\|}{\|x\|} \to 0 \quad \text{as} \quad \|x\| \to 0.$$

$\Rightarrow$ Small o notation: $g(x) = o(\|x\|)$.

Theorem 4.7. (Lyapunov indirect method) Let $A = \frac{\partial f}{\partial x}(0)$.

1. The origin is locally asymptotically stable if A is Hurwitz.
2. The origin is unstable if $\text{Re}\lambda_i > 0$ for one or more of the eigenvalues of A.

- A is called the 'first order approximation of $f(x)$ at the origin' or 'Jacobian of $f(x)$ at the origin'.
- The theorem does not say anything for the case $\text{Re}\lambda_i \leq 0$ for all i.

Proof. (1) Pick any $Q > 0$ and solve P s.t. $PA + A^T P = -Q$. Let $V(x) = x^T P x$. Then,

$$\begin{aligned}
\dot{V}(x) &= x^T P f(x) + f^T(x) P x \\
&= x^T P[Ax + g(x)] + [x^T A^T + g^T(x)]Px \\
&= x^T(PA + A^T P)x + 2x^T P g(x) \\
&= -x^T Q x + 2x^T P g(x)
\end{aligned}$$

Since $g(x) = o(\|x\|)$, for any $\gamma > 0$, $\exists r > 0$ s.t.

$$\|g(x)\| < \gamma\|x\|, \qquad \forall \|x\| < r, x \neq 0.$$

Hence,

$$\begin{aligned} \dot{V}(x) &< -x^T Q x + 2\gamma\|P\|\|x\|^2, \qquad \forall \|x\| < r, x \neq 0, \\ &\leq -[\lambda_{\min}(Q) - 2\gamma\|P\|]\|x\|^2 \end{aligned}$$

which proves (1).

(2) First suppose that A is hyperbolic. Then, $\exists$ a nonsingular T s.t.

'Diffeomorphism' See Exercise 4.26.

$$TAT^{-1} = \begin{bmatrix} -A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

where A_i's are Hurwitz. Let $z = Tx = [z_1; z_2]$. Then, in z-coordinates, the system becomes

$$\begin{aligned} \dot{z}_1 &= -A_1 z_1 + g_1(z) \\ \dot{z}_2 &= A_2 z_2 + g_2(z) \end{aligned}$$

Pick any $Q_1 > 0$ and $Q_2 > 0$, and solve $P_i A_i + A_i^T P_i = -Q_i$. Let

$$V(z) = z_1^T P_1 z_1 - z_2^T P_2 z_2.$$

In the subspace $z_2 = 0$, $V(z) > 0$ at points arbitrarily close to the origin. Let

$$U = \{z \in \mathbb{R}^n : \|z\| \leq r,\ V(z) > 0\}.$$

In U,

$$\begin{aligned} \dot{V}(z) &= -z_1^T(P_1 A_1 + A_1^T P_1)z_1 + 2z_1^T P_1 g_1(z) \\ &\quad - z_2^T(P_2 A_2 + A_2^T P_2)z_2 - 2z_2^T P_2 g_2(z) \\ &= z_1^T Q_1 z_1 + z_2^T Q_2 z_2 + 2z^T[P_1 g_1(z); -P_2 g_2(z)] \\ &\geq \lambda_{\min}(Q_1)\|z_1\|^2 + \lambda_{\min}(Q_2)\|z_2\|^2 - 2\|z\|\sqrt{\|P_1\|^2\|g_1(z)\|^2 + \|P_2\|^2\|g_2(z)\|^2} \\ &> (\alpha - 2\sqrt{2}\beta\gamma)\|z\|^2 \qquad \text{with some } \alpha > 0 \text{ and } \beta > 0, \end{aligned}$$

which leads to the conclusion. Note that the analysis also can be done in x-coordinates with

$$P = T^T \begin{bmatrix} P_1 & 0 \\ 0 & -P_2 \end{bmatrix} T; \qquad Q = T^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} T.$$

If A has some eigenvalues on the imaginary axis (as well as some eigenvalues in the open

The Lyapunov equation $PA + A^T P = -Q$ has, in fact, a unique solution P if and only if $\lambda_i + \lambda_j \neq 0$ for all i and j, where λ_i is an eigenvalue of A.

right-half plane), then consider a matrix $[A - (\delta/2)I]$ that is hyperbolic. With it, we find $P = P^T$ and $Q > 0$ s.t.

$$P\left[A - \frac{\delta}{2}I\right] + \left[A - \frac{\delta}{2}I\right]^T P = Q.$$

Again, $V(x) = x^T P x$ is positive for points arbitrarily close to the origin. With it, we have

$$\begin{aligned}\dot{V}(x) &= x^T(PA + A^T P)x + 2x^T P g(x)\\ &= x^T\left[P\left(A - \frac{\delta}{2}I\right) + \left(A - \frac{\delta}{2}I\right)^T P\right]x + \delta x^T P x + 2x^T P g(x)\\ &= x^T Q x + \delta V(x) + 2x^T P g(x)\end{aligned}$$

In the set

$$\{x \in \mathbb{R}^n : \|x\| \le r,\ V(x) > 0\}$$

it follows that

$$\dot{V}(x) \ge \lambda_{\min}(Q)\|x\|^2 - 2\|P\|\|x\|\|g(x)\|,$$

from which the proof is done.

Example 4.14 Consider $\dot{x} = ax^3$.

Example 4.15 The pendulum system:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a\sin x_1 - bx_2\end{aligned}$$

Inspect stability at $(x_1, x_2) = (0, 0)$ and $(\pi, 0)$.

$$\frac{\partial f}{\partial x}(x) = \begin{bmatrix} 0 & 1\\ -a\cos x_1 & -b\end{bmatrix}$$

Consider two cases: $a, b > 0$ and $a > 0, b = 0$ for both equilibria.

IV. Comparison Functions

Definition 4.2

A continuous function $\alpha : [0, a) \to [0, \infty)$ belongs to class-$\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

If $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$, then it is called class-$\mathcal{K}_\infty$.

Definition 4.3

A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ belongs to class-$\mathcal{KL}$ if, for each fixed s, $\beta(r, s)$ is of class-$\mathcal{K}$ with respect to r, and, for each fixed r, it is decreasing w.r.t. s and $\beta(r, s) \to 0$ as $s \to \infty$.

Lemma 4.2 Let α_1, α_2 be class-$\mathcal{K}$ functions on $[0, a)$, α_3, α_4 be class-$\mathcal{K}_\infty$ functions, β be a class-$\mathcal{KL}$ function.

- α_1^{-1} is of class-$\mathcal{K}$ on $[0, \alpha_1(a))$.
- α_3^{-1} is of class-$\mathcal{K}_\infty$.
- $\alpha_1 \circ \alpha_2$: class-$\mathcal{K}$.
- $\alpha_3 \circ \alpha_4$: class-$\mathcal{K}_\infty$.
- $\alpha_1(\beta(\alpha_2(r), s))$: class-$\mathcal{KL}$.

Lemma 4.3

$V : D \to \mathbb{R}$ a continuous positive definite function where $0 \in D \subset \mathbb{R}^n$.

$B_r \subset D$ with some $r > 0$.

Then, $\exists$ class-$\mathcal{K}$ functions α_1 and α_2 defined on $[0, r]$ s.t.

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

for all $x \in B_r$.

In addition, if $V(x)$ is radially unbounded, such α_1 and α_2 can be of class-$\mathcal{K}_\infty$.

Example. If $V(x) = x^T P x$ with $P > 0$, then $\alpha_1(s) = \lambda_{\min}(P)s^2$ and $\alpha_2(s) = \lambda_{\max}(P)s^2$.

Proof.

$$\psi(s) := \inf_{s \leq \|x\| \leq r} V(x) \quad \text{for } 0 \leq s \leq r$$

$$\phi(s) := \sup_{\|x\| \leq s} V(x) \quad \text{for } 0 \leq s \leq r$$

Then, ψ and ϕ are nondecreasing. So, take class-$\mathcal{K}$ α_1 and α_2 s.t.

$$\alpha_1(s) < \psi(s), \qquad \phi(s) < \alpha_2(s).$$

Then the claim follows.

The case for $V(x)$ that is radially unbounded is the same with $r = \infty$.

Lemma 4.4

Scalar system

$$\dot{y} = -\alpha(y), \qquad y(t_0) = y_0 \geq 0$$

where α is locally Lipschitz and of class-$\mathcal{K}$. There exists a unique solution $y(t)$ defined for all $t \geq t_0$ s.t.

$$y(t) = \sigma(y_0, t - t_0)$$

where σ is a class-$\mathcal{KL}$ function.

Examples.

- For $\dot{y} = -ky, k > 0$, $y(t) = y_0 \exp[-k(t - t_0)]$.
- For $\dot{y} = -ky^2, k > 0$, $y(t) = y_0/(ky_0(t - t_0) + 1)$.

Proof.

$$\frac{dy}{dt} = -\alpha(y) \qquad \Rightarrow \qquad -\int_{y_0}^{y} \frac{dx}{\alpha(x)} = \int_{t_0}^{t} dt.$$

Define

$$\eta(y) := -\int_{y_0}^{y} \frac{dx}{\alpha(x)}.$$

Then, it is strictly decreasing and $\lim_{y \to 0} \eta(y) = \infty$ because, from the system equation, $y(t) \to 0$ as $t \to \infty$.

Let $c = -\lim_{y \to \infty} \eta(y) \quad (\in \mathbb{R} \cup \{\infty\})$. Then, the range of $\eta(y)$ is $(-c, \infty)$. So, η^{-1} is defined on $(-c, \infty)$.

Then,

$$\eta(y(t)) - \eta(y_0) = t - t_0$$
$$y(t) = \eta^{-1}(\eta(y_0) + t - t_0)$$

Now, let

$$\sigma(r, s) = \begin{cases} \eta^{-1}(\eta(r) + s), & r > 0 \\ 0, & r = 0 \end{cases},$$

which is our class-$\mathcal{KL}$ function because

- it is continuous because $\lim_{x \to \infty} \eta^{-1}(x) = 0$,
- it is strictly increasing in r because

$$\frac{\partial}{\partial r}\sigma(r, s) = \frac{\alpha(\sigma(r, s))}{\alpha(r)} > 0,$$

- it is strictly decreasing in s because

$$\frac{\partial}{\partial s}\sigma(r, s) = -\alpha(\sigma(r, s)) < 0,$$

- $\sigma(r, s) \to 0$ as $s \to \infty$.

Note: Since $\eta^{-1}(\eta(x)) = x$,

$$D_x\eta^{-1}(\eta(x)) \cdot D\eta(x) = I.$$

Thus,

$$\begin{aligned} \frac{\partial}{\partial r}\sigma &= D\eta^{-1}(\eta(r) + s)D\eta(r) \\ &= \frac{\alpha(\sigma(r, s))}{\alpha(r)} \\ \frac{\partial}{\partial s}\sigma &= D\eta^{-1}(\eta(r) + s) \\ &= -\alpha(\sigma(r, s)) \end{aligned}$$

An Example: Usefulness in Lyapunov Stability Analysis (Proof of Theorem 4.1)

We have chosen β and δ s.t. $B_\delta \subset \Omega_\beta \subset B_r$. This can also be done as follows: With α_1 and α_2 s.t.

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$

choose $\beta \leq \alpha_1(r)$ and $\delta \leq \alpha_2^{-1}(\beta)$, because

$$V(x) \leq \beta \quad \Rightarrow \quad \alpha_1(\|x\|) \leq \alpha_1(r) \Leftrightarrow \|x\| \leq r$$
$$\|x\| \leq \delta \quad \Rightarrow \quad V(x) \leq \alpha_2(\delta) \leq \beta.$$

Now we show (again) that $\dot{V}(x)$ is negative definite, $x(t)$ tends to zero. There exists a class-$\mathcal{K}$ function α_3 s.t. $\dot{V}(x) \leq -\alpha_3(\|x\|)$. Hence,

$$\dot{V} \leq -\alpha_3(\alpha_2^{-1}(V)).$$

Since the differential equation

$$\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \qquad y(0) = V(x(0))$$

has a solution $y(t) = \beta(y(0), t)$ where β is a class-$\mathcal{KL}$ function, we know that

$$V(x(t)) \leq \beta(V(x(0)), t).$$

Is the function $\alpha_3(\alpha_2^{-1}(s))$ locally Lipschitz? Yes, without loss of generality by modifying α_2 if necessary.

This is nice because we can go beyond the proof of Theorem 4.1. Now we have

$$\alpha_1(\|x(t)\|) \leq V(x(t)) \leq V(x(0)) \leq \alpha_2(\|x(0)\|),$$

which leads to $\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x(0)\|))$. Also,

$$\alpha_1(\|x(t)\|) \leq V(x(t)) \leq \beta(V(x(0)), t) \leq \beta(\alpha_2(\|x(0)\|), t),$$

which leads to $\|x(t)\| \leq \alpha_1^{-1}(\beta(\alpha_2(\|x(0)\|), t))$.

V. Nonautonomous Systems

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x, and

$$f(t, 0) = 0, \qquad \forall t \geq 0.$$

An equilibrium at the origin could be a translation of a nonzero equilibrium point or, more generally, a translation of *a nonzero solution* of the system. Read p. 147 of the textbook to see what this means.

Example 4.17

$$
\begin{aligned}
\dot{x} &= (6t\sin t - 2t)x \\
x(t) &= x(t_0)\exp\left[\int_{t_0}^{t}(6\tau\sin\tau - 2\tau)d\tau\right] \\
&= x(t_0)\exp\left[6\sin t - 6t\cos t - t^2 - 6\sin t_0 + 6t_0\cos t_0 + t_0^2\right]
\end{aligned}
$$

For any t_0, $\exists$ a constant $c(t_0)$ s.t.

$$|x(t)| < |x(t_0)|c(t_0), \qquad \forall t \geq t_0.$$

For any $\epsilon > 0$, take $\delta = \epsilon / c(t_0)$, which shows that the origin is (not uniformly) stable.

Now consider $t_0 = 2n\pi$ for $n = 0, 1, 2, \cdots$, and let $t = t_0 + \pi$. Then,

$$x(t_0 + \pi) = x(t_0)\exp[(4n+1)(6-\pi)\pi],$$

which shows that it is impossible to take δ independently of t_0.

Example 4.18

$$
\begin{aligned}
\dot{x} &= -\frac{x}{1+t} \\
x(t) &= x(t_0)\exp\left(\int_{t_0}^{t}\frac{-1}{1+\tau}d\tau\right) = x(t_0)\frac{1+t_0}{1+t}
\end{aligned}
$$

In this case, uniformly stable, but not uniformly convergent; that is, given any $\epsilon > 0$, $\exists T = T(\epsilon, t_0) > 0$ s.t. $|x(t)| < \epsilon$ for $t \geq t_0 + T$, but T depends on t_0.

Definition 4.4 The origin is

- stable if, for each $\epsilon > 0$, $\exists \delta(\epsilon, t_0) > 0$ s.t.

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \qquad \forall t \geq t_0 \geq 0$$

- uniformly stable if the origin is stable but the δ is independent of t_0
- unstable if it is not stable
- asymptotically stable if the origin is stable and $\exists c(t_0) > 0$ s.t.

$$x(t) \to 0 \quad \text{as} \quad t \to \infty, \qquad \text{for all } \|x(t_0)\| < c(t_0)$$

- uniformly asymptotically stable if the origin is *uniformly* stable, the c above is independent of t_0, and the convergence is uniform, i.e., for each $\eta > 0$, $\exists T = T(\eta) > 0$ s.t.

$$\|x(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta), \ \forall \|x(t_0)\| < c$$

- globally uniformly asymptotically stable (UGAS) if the origin is uniformly stable, $\delta(\epsilon)$ can be chosen to satisfy $\lim_{\epsilon\to\infty}\delta(\epsilon)=\infty$, and, for each (η,c), $\exists T=T(\eta,c)>0$ s.t.

$$\|x(t)\|<\eta, \quad \forall t\geq t_0+T(\eta,c),\ \forall\|x(t_0)\|<c$$

- exponentially stable if $\exists\ c,k,\lambda>0$ s.t.

$$\|x(t)\|\leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \qquad \forall\|x(t_0)\|<c$$

- globally exponentially stable (GES) if, in addition, $c=\infty$.

Lemma 4.5 The origin is

- uniformly stable if and only if $\exists$ a $\mathcal{K}$-function α and $c>0$ independent of t_0 s.t.

$$\|x(t)\|\leq\alpha(\|x(t_0)\|), \qquad \forall t\geq t_0\geq 0,\ \forall\|x(t_0)\|<c$$

- uniformly asymptotically stable if and only if $\exists$ a $\mathcal{KL}$-function β and $c>0$ independent of t_0 s.t.

$$\|x(t)\|\leq\beta(\|x(t_0)\|,t-t_0), \qquad \forall t\geq t_0\geq 0,\ \forall\|x(t_0)\|<c$$

- UGAS if and only if $\exists$ a $\mathcal{KL}$-function β s.t.

$$\|x(t)\|\leq\beta(\|x(t_0)\|,t-t_0), \qquad \forall t\geq t_0\geq 0,\ \forall x(t_0)$$

Proof. Read Appendix C.6.

Homework: Summarize Appendix C.6.

A function $V(t,x)$ is

- positive definite if $V(t,x)\geq W_1(x)$ where W_1 is a positive definite function
- radially unbounded if $V(t,x)\geq W_1(x)$ where W_1 is radially unbounded
- decrescent if $V(t,x)\leq W_2(x)$ with a function W_2

Theorem 4.8

IF $\exists$ a C^1 $V(t,x)$ s.t.

$$W_1(x)\leq V(t,x)\leq W_2(x)$$
$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x)\leq 0$$

for all $t\geq 0$ and $x\in D$, where $W_1(x)$ and $W_2(x)$ are continuous positive definite functions, THEN $x=0$ is uniformly stable.

Recall $t_0\geq 0$.

Proof. Read the book. The key is

$$\{x\in B_r: W_2(x)\leq c\}\subset\Omega_{t,c}\subset\{x\in B_r: W_1(x)\leq c\}\subset B_r\subset D.$$

Theorem 4.9

- IF, in addition to the assumptions of Theorem 4.8,

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x)$$

for all $t \geq 0$ and $x \in D$, where $W_3(x)$ is a continuous positive definite function, THEN $x = 0$ is uniformly asymptotically stable.

In particular, let r and c be s.t. $B_r \subset D$ and $c < \min_{\|x\|=r} W_1(x)$. Then,

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0,\ x(t_0) \in \{x \in B_r : W_2(x) \leq c\}$$

where β is a $\mathcal{KL}$-function.

- IF $D = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded, THEN $x = 0$ is UGAS.

Note. $\dot{V}(t,x) < 0$ for $x \neq 0$ is not enough. W_3 is necessary. Consider

$$\dot{x} = -\frac{1}{(1+t)^2} x$$

Then, with $V = x^2$, $\dot{V} < 0$ for $x \neq 0$, but

$$x(t) = x(t_0) e^{1/(1+t)-1}.$$

Proof. $\exists$ a $\mathcal{K}$-function $\alpha_3 : [0,r] \to \mathbb{R}$ s.t.

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x) \leq -\alpha_3(\|x\|).$$

Then,

$$\dot{V} \leq -\alpha_3(\alpha_2^{-1}(V)) =: -\bar{\alpha}(V) \leq -\alpha(V)$$

where α is a *locally Lipschitz* class-$\mathcal{K}$ function defined on $[0,r]$.

By the comparison lemma (Lemma 3.4) and Lemma 4.4, $\exists$ a $\mathcal{KL}$-function $\sigma : [0,r] \times [0,\infty)$ s.t.

$$V(t,x(t)) \leq \sigma(V(t_0,x(t_0)), t - t_0), \qquad \forall V(t_0,x(t_0)) \in [0,c].$$

Thus,

$$\|x(t)\| \leq .... \leq \beta(\|x(t_0)\|, t - t_0)$$

for $x(t_0) \in \{x \in B_r : W_2(x) \leq c\}$, where β is a $\mathcal{KL}$-function. (Fill the blank.)

The rest is omitted. See the book.

Theorem 4.10

IF V is a C^1 function s.t.

$$k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^a$$

for all $t \geq 0$ and $x \in D$, where k_i's and a are positive constants,

THEN $x = 0$ is exponentially stable.

If $D = \mathbb{R}^n$, then $x = 0$ is GES.

Proof.

It can be seen that trajectories starting in $\{k_2\|x\|^a \leq c\}$, for sufficiently small c, remain bounded for all $t \geq t_0$, and satisfies

$$\dot{V} \leq -\frac{k_3}{k_2}V.$$

By the comparison lemma,

$$V(t, x(t)) \leq V(t_0, x(t_0))e^{-(k_3/k_2)(t-t_0)}.$$

Hence,

$$\begin{aligned}\|x(t)\| &\leq \left[\frac{V(t, x(t))}{k_1}\right]^{1/a} \leq \left[\frac{V(t_0, x(t_0))e^{-(k_3/k_2)(t-t_0)}}{k_1}\right]^{1/a} \\ &\leq \left[\frac{k_2\|x(t_0)\|^a e^{-(k_3/k_2)(t-t_0)}}{k_1}\right]^{1/a} = \left(\frac{k_2}{k_1}\right)^{1/a}\|x(t_0)\|e^{-(k_3/k_2 a)(t-t_0)}.\end{aligned}$$

If all the assumptions hold globally, GES follows.

Example 4.19

$$\dot{x} = -[1 + g(t)]x^3$$

where continuous $g(t) \geq 0$. Take $V(x) = \frac{1}{2}x^2$.

Result: UGAS.

Example 4.20

$$\begin{aligned}\dot{x}_1 &= -x_1 - g(t)x_2 \\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

where

$$0 \leq g(t) \leq k, \qquad \dot{g}(t) \leq g(t).$$

Take $V(t, x) = x_1^2 + [1 + g(t)]x_2^2$, which satisfies

$$x_1^2 + x_2^2 \leq V(t, x) \leq x_1^2 + (1 + k)x_2^2.$$

Then,

$$\dot{V} = -2x_1^2 + 2x_1x_2 - [2 + 2g(t) - \dot{g}(t)]x_2^2 \leq -2x_1^2 + 2x_1x_2 - 2x_2^2 = -x^TQx$$

where $Q > 0$.

Result: GES

Example 4.21

$$\dot{x} = A(t)x$$

where $A(t)$ is continuous. Suppose that $\exists$ a C^1 bounded $P(t) > 0$; that is,

$$0 < c_1 I \le P(t) \le c_2 I, \qquad \forall t \ge 0$$

which satisfies

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$$

where continuous $Q(t) \ge c_3 I > 0$.

Then, with $V(t,x) = x^T P(t)x$,

$$\dot{V} = x^T \dot{P} x + x^T P \dot{x} + \dot{x}^T P x = x^T[\dot{P} + PA + A^T P]x = -x^T Q x \le -c_3 \|x\|^2.$$

Result: GES

VI. Linear Time-varying Systems and Linearization

$$\dot{x} = A(t)x$$
$$x(t) = \Phi(t, t_0)x(t_0)$$

where $\Phi(t, t_0)$ is the state transition matrix. (Note that the local and the global behaviors are the same in linear systems.)

Theorem 4.11 The origin is UGAS if and only if

$$\|\Phi(t, t_0)\| \leq ke^{-\lambda(t-t_0)}$$

for some $k, \lambda > 0$.

Note. For linear systems, GES = UGAS = ES = UAS.

Proof. (Sufficiency)

$$\|x(t)\| \leq \|\Phi(t, t_0)\|\|x(t_0)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}.$$

(Necessity) From UGAS,

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \qquad \forall t \geq t_0, \forall x(t_0) \in \mathbb{R}^n$$

On the other hand,

$$\|\Phi(t, t_0)\| = \max_{\|x\|=1} \|\Phi(t, t_0)x\| \leq \max_{\|x\|=1} \beta(\|x\|, t - t_0) = \beta(1, t - t_0).$$

Pick T s.t. $\beta(1, T) \leq 1/e$. For any $t \geq t_0$, let N be the smallest positive integer s.t. $t \leq t_0 + NT$. Divide the interval $[t_0, t_0 + (N-1)T]$ into $(N-1)$ equal subintervals. Then,

$$\Phi(t, t_0) = \Phi(t, t_0 + (N-1)T)\Phi(t_0 + (N-1)T, t_0 + (N-2)T) \cdots \Phi(t_0 + T, t_0).$$

Hence,

$$\begin{aligned}
\|\Phi(t, t_0)\| &\leq \|\Phi(t, t_0 + (N-1)T)\|\Pi_{k=1}^{k=N-1}\|\Phi(t_0 + kT, t_0 + (k-1)T)\| \\
&\leq \beta(1, 0)\Pi_{k=1}^{k=N-1}\frac{1}{e} = e\beta(1, 0)e^{-N} \\
&\leq e\beta(1, 0)e^{-(t-t_0)/T} = ke^{-\lambda(t-t_0)}
\end{aligned}$$

where $k = e\beta(1, 0)$ and $\lambda = 1/T$.

Is there easier characterization of the stability for $\dot{x} = A(t)x$?

Wrong conjecture: If $A(t)$ is Hurwitz for every fixed t, then the origin is UGAS.

Example 4.22

$$A(t) = \begin{bmatrix} -1 + 1.5\cos^2 t & 1 - 1.5\sin t\cos t \\ -1 - 1.5\sin t\cos t & -1 + 1.5\sin^2 t \end{bmatrix}$$

For each t, the eigenvalues of $A(t)$ are $-0.25 \pm 0.25\sqrt{7}j$. But,

$$\Phi(t, 0) = \begin{bmatrix} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & -e^{-t}\cos t \end{bmatrix}.$$

Theorem 4.12

Suppose that the origin of $\dot{x} = A(t)x$ where $A(t)$ is *continuous and bounded* is GES. If $Q(t)$ is continuous, bounded, positive definite, then $\exists$ continuously differentiable, bounded, positive definite $P(t)$ satisfying

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t). \tag{1}$$

$P(t)$ is bounded and positive definite (so, symmetric):

$$0 < c_1 I \leq P(t) \leq c_2 I$$

for all t.

Thus, $V(t, x) = x^T P(t)x$ is a suitable Lyapunov function for the system. (See also Example 4.21.)

Proof. Let

$$P(t) := \int_t^\infty \Phi^T(\tau, t)Q(\tau)\Phi(\tau, t)d\tau$$

and $\phi(\tau; t, x)$ be the solution that starts at (t, x) (so that $\phi(\tau; t, x) = \Phi(\tau, t)x$). (We also know $c_1 I \leq Q(t) \leq c_2 I$.)

(a) $P(t)$ is positive definite and bounded.

Note that

$$x^T P(t)x = \int_t^\infty \phi^T(\tau; t, x)Q(\tau)\phi(\tau; t, x)d\tau.$$

Since $\|\Phi(t, t_0)\| \leq ke^{-\lambda(t-t_0)}$, we have

$$\begin{aligned} x^T P(t)x &\leq \int_t^\infty c_4\|\Phi(\tau, t)\|^2\|x\|^2 d\tau \\ &\leq \int_t^\infty k^2 e^{-2\lambda(\tau-t)}d\tau c_4\|x\|^2 = \frac{k^2 c_4}{2\lambda}\|x\|^2 =: c_2\|x\|^2 \end{aligned}$$

On the other hand, since

$$\|A(t)\| \leq L, \qquad \forall t \geq 0$$

by Exercise 3.17, we have

$$\|\phi(\tau; t, x)\|^2 \geq \|x\|^2 e^{-2L(\tau-t)}.$$

Exercise 3.17 was your homework.

Hence,

$$x^T P(t)x \geq \int_t^\infty c_3 \|\phi(\tau; t, x)\|^2 d\tau$$
$$\geq \int_t^\infty e^{-2L(\tau - t)} d\tau c_3 \|x\|^2 = \frac{c_3}{2L}\|x\|^2 =: c_1\|x\|^2$$

Thus,

$$c_1\|x\|^2 \leq x^T P(t)x \leq c_2\|x\|^2.$$

(b) $P(t)$ is symmetric and C^1 by the definition.

(c) $P(t)$ satisfies (1).

Note that

$$\frac{\partial}{\partial t}\Phi(\tau, t) = -\Phi(\tau, t)A(t). \quad (2)$$

In particular,

$$\dot{P}(t) = \int_t^\infty \Phi^T(\tau, t)Q(\tau)\frac{\partial}{\partial t}\Phi(\tau, t)d\tau$$
$$+ \int_t^\infty \left[\frac{\partial}{\partial t}\Phi^T(\tau, t)\right] Q(\tau)\Phi(\tau, t)d\tau - Q(t)$$
$$= -\int_t^\infty \Phi^T(\tau, t)Q(\tau)\Phi(\tau, t)d\tau A(t) - A^T(t)\int_t^\infty \Phi^T(\tau, t)Q(\tau)\Phi(\tau, t)d\tau - Q(t)$$
$$= -P(t)A(t) - A^T(t)P(t) - Q(t)$$

From the property of state transition matrix, we know that

$$\frac{\partial}{\partial t}\Phi(t, \tau) = A(t)\Phi(t, \tau).$$

From this, derive (2).
Ans.: $A^{-1}(t)A(t) = I$, thus,

$$\frac{d}{dt}A^{-1}\cdot A + A^{-1}\cdot\frac{d}{dt}A^{-1} = 0,$$

and so,

$$\frac{dA^{-1}}{dt} = -A^{-1}\frac{dA}{dt}A^{-1}.$$

Therefore,

$$\frac{\partial}{\partial t}\Phi(\tau, t) = \frac{\partial}{\partial t}\Phi^{-1}(t, \tau)$$
$$= -\Phi^{-1}(t, \tau)\frac{\partial}{\partial t}\Phi(t, \tau)\Phi^{-1}(t, \tau).$$

Stability of

$$\dot{x} = f(t, x)$$

through linearization.

Theorem 4.13 Let

- the Jacobian $\frac{\partial f}{\partial x}(t, x)$ be Lipschitz on D uniformly in t, that is,

$$\left\|\frac{\partial f}{\partial x}(t, x_1) - \frac{\partial f}{\partial x}(t, x_2)\right\| \leq L\|x_1 - x_2\|, \qquad \forall x_1, x_2 \in D, \forall t$$

- $A(t) := \frac{\partial f}{\partial x}(t, x)\big|_{x=0}$ be bounded for all t.

If the origin of

$$\dot{x} = A(t)x$$

is exponentially stable, then the origin of nonlinear system is also exponentially stable.

Proof. By Theorem 4.12, we have $V(t, x) = x^T P(t)x$. Since $f(t, x) = A(t)x + g(t, x)$ where

$$g_i(t, x) = \left[\frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0)\right] x,$$

with which, we have

$$\|g(t,x)\| \le L\|x\|^2.$$

So,

$$\begin{aligned}\dot{V}(t,x) &= x^T(PA + A^TP + \dot{P})x + 2x^TPg(t,x)\\ &= -x^TQ(t)x + 2x^TP(t)g(t,x)\\ &\le -c_3\|x\|^2 + 2c_2L\|x\|^3\\ &\le -(c_3 - 2c_2L\rho)\|x\|^2\end{aligned}$$

for all $\|x\| < \rho$.

VII. Converse Theorems

What is converse theorem?

Theorem 4.14 Consider

$$\dot{x} = f(t,x)$$

where f is C^1 on $[0,\infty)\times D$, $D = B_r$, and $\frac{\partial f}{\partial x}(t,x)$ is bounded on D uniformly in t.

IF

$$\|x(t)\| \le k\|x(t_0)\|e^{-\lambda(t-t_0)}, \qquad \forall x(t_0)\in D_0, \forall t \ge t_0 \ge 0,$$

where $D_0 = B_{r_0}$ with $r_0 < r/k$,

THEN $\exists$ a C^1 function $V : [0,\infty)\times D_0 \to \mathbb{R}$ s.t.

$$\begin{gathered}c_1\|x\|^2 \le V(t,x) \le c_2\|x\|^2\\ \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x}f(t,x) \le -c_3\|x\|^2\\ \left\|\frac{\partial V}{\partial x}\right\| \le c_4\|x\|\end{gathered}$$

Moreover, if the origin is GES, then global Lyapunov function $V(t,x)$ exists, and if the system is autonomous, then V can be independent of t.

Proof. DIY while comparing the proof of Theorem 4.12. A Lyapunov function will be given by

$$V(t,x) = \int_t^{t+\delta} \phi^T(\tau;t,x)\phi(\tau;t,x)d\tau.$$

Theorem 4.15 Consider

$$\dot{x} = f(t,x)$$

where f is C^1 on $[0,\infty) \times D$, $D = B_r$, and $\frac{\partial f}{\partial x}(t,x)$ is bounded on D uniformly in t.

Suppose also that $\frac{\partial f}{\partial x}(t,x)$ is Lipschitz on D uniformly in t.

The origin is ES if and only if the origin of

$$\dot{x} = \frac{\partial f}{\partial x}(t,x)\big|_{x=0} x$$

is ES.

Corollary 4.3 Consider

$$\dot{x} = f(x)$$

where $f(x)$ is C^1 and $f(0) = 0$.

The origin is ES if and only if $A = \frac{\partial f}{\partial x}(0)$ is Hurwitz.

* Note that AS (rather than ES) has no such simple relation as above.

Proof. DIY.

Theorem 4.16 Consider

$$\dot{x} = f(t,x)$$

where f is C^1 on $[0,\infty) \times D$, $D = B_r$, and $\frac{\partial f}{\partial x}(t,x)$ is bounded on D uniformly in t.

IF

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \qquad \forall x(t_0) \in D_0, \forall t \geq t_0 \geq 0,$$

where $D_0 = B_{r_0}$ with r_0 s.t. $\beta(r_0, 0) < r$,

THEN $\exists$ a C^1 function $V : [0,\infty) \times D_0 \to \mathbb{R}$ s.t.

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$$
$$\frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x} f(t,x) \leq -\alpha_3(\|x\|)$$
$$\left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_4(\|x\|)$$

If the system is autonomous, then V can be independent of t.

Theorem 4.17 Consider

$$\dot{x} = f(x)$$

where f is locally Lipschitz.

IF the origin is AS with its region of attraction R_A,

THEN $\exists$ a C^∞ positive definite function $V(x)$ (defined on R_A) s.t.

$$V(x) \to \infty \quad \text{as} \quad x \to \partial R_A \quad (V(x) \text{ is proper on } R_A)$$
$$L_f V(x) \leq -W(x)$$

for all $x \in R_A$ where $W(x)$ is a continuous positive definite function defined on R_A.

When $R_A = \mathbb{R}^n$, $V(x)$ is radially unbounded.

Proofs of Theorems 4.16 and 17: Skipped.

VIII. Boundedness and Ultimate Boundedness

$$\dot{x} = f(t, x)$$

Case 1: $\dot{V}(x(t)) \leq 0$

Case 2: $\dot{V}(x(t)) \leq -V(x) + d$ (e.g., $\dot{x} = -x + \delta \sin t$)

Definition 4.6 The solution $x(t)$ is

- *uniformly bounded* if $\exists c > 0$, indep. of t_0, and for every $a \in (0, c)$, $\exists \beta = \beta(a) > 0$, indep. of t_0, s.t.

$$\|x(t_0)\| \leq a \qquad \Rightarrow \qquad \|x(t)\| \leq \beta, \quad \forall t \geq t_0$$

- *globally uniformly bounded* if $c = \infty$ in the above,
- *uniformly ultimately bounded with ultimate bound* b if $\exists b, c > 0$, indep. of t_0, and for every $a \in (0, c)$, $\exists T = T(a, b) \geq 0$, indep. of t_0, s.t.

$$\|x(t_0)\| \leq a \qquad \Rightarrow \qquad \|x(t)\| \leq b, \quad \forall t \geq t_0 + T$$

- *globally uniformly ultimately bounded* if $c = \infty$ in the above.

For time-invariant systems, we may drop the word 'uniformly' in the above.

Theorem 4.18 Let $V : [0, \infty) \times D \to \mathbb{R}$ and take r s.t. $B_r \subset D$.

IF $\exists$ C^1 function $V(t, x)$, class-$\mathcal{K}$ functions α_1 and α_2, continuous positive definite function $W_3(x)$, s.t.

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$
$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x), \qquad \forall \|x\| \geq \mu > 0,$$

where

$$\mu < \alpha_2^{-1} \circ \alpha_1(r)$$

THEN $\exists$ a class-$\mathcal{KL}$ function β, and for each $\|x(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$, there is $T(x(t_0), \mu) \geq 0$ s.t.

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \qquad \forall t_0 \leq t \leq t_0 + T$$
$$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu)), \qquad \forall t \geq t_0 + T$$

IF $D = \mathbb{R}^n$ and α_1 is of class-$\mathcal{K}_\infty$, THEN the conclusion holds globally.

Proof. Two key points follow:

- Let

$$\dot{x} = f(t, x)$$
$$\exists V(x) \quad \text{and} \quad \Lambda := \{x : \epsilon \leq V(x) \leq c\}$$
$$\dot{V} \leq -W_3(x) \quad \text{on } \Lambda$$

where W_3 is continuous positive definite.

Then, both Ω_ϵ and Ω_c are positively invariant. Any trajectory with initial condition in Λ reaches in Ω_ϵ in finite time and remains there (because, $\dot{V} \leq -W_3(x) \leq -k$ and thus, $V(t) \leq V(0) - kt$).

- In many cases, we have

$$\dot{V} \leq -W_3(x), \qquad \forall \mu \leq \|x\| \leq r$$

where the range is given in terms of the norm, not of the sub-level set.

Then, if μ and r are too close, it may happen there's no Λ in $\{x : \mu \leq \|x\| \leq r\}$. Recall that,

– If $c \leq \alpha_1(r)$, then $\Omega_c \subset B_r$. (Why?)

– If $\epsilon \geq \alpha_2(\mu)$, then $B_\mu \subset \Omega_\epsilon$. (Why?)

So, in order to have $\epsilon < c$, it should be

$$\mu < \alpha_2^{-1}(\alpha_1(r)).$$

The ultimate bound b, in this case, is

$$b = \alpha_1^{-1}(\alpha_2(\mu)).$$

For details, see the Appendix C.9.

IX. Input-to-State Stability (ISS)

Example.

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ A\text{: Hurwitz}$$

$$\dot{x} = -x + x^2 u, \qquad x \in \mathbb{R}, u \in \mathbb{R}$$

$$\dot{x} = -x^3 + x^2 u, \qquad x \in \mathbb{R}, u \in \mathbb{R}$$

$$\dot{x} = f(t, x, u), \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

Definition 4.7 (ISS) The system is input-to-state stable if $\exists$ a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ s.t.

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) + \gamma \left(\sup_{t_0 \le \tau \le t} \|u(\tau)\| \right).$$

- With $u(t) \equiv 0$, the system is UGAS.
- For any bounded input $u(\cdot)$, the solution $x(t)$ is bounded. (In fact, it is uniformly ultimately bounded with a bound determined by $\sup_{t \ge t_0} \|u(t)\|$).
- If $u(t) \to 0$ as $t \to \infty$, then $x(t)$ also goes to zero. (Exercise 4.58)
- Local version of ISS is also available. See Exercise 4.60.

Theorem 4.19

IF $\exists$ C^1 function $V(t, x)$, class-$\mathcal{K}$ function ρ, class-$\mathcal{K}_\infty$ functions α_1 and α_2, continuous positive definite function $W_3(x)$, s.t.

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x} f(t, x, u) \le -W_3(x), \qquad \forall \|x\| \ge \rho(\|u\|) > 0,$$

THEN the system is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Proof. The proof is done by employing Theorem 4.18 with a bounded input $u(t)$ and $\mu = \sup_{\tau \ge t_0} \|u(\tau)\|$. Then, we have

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) + \gamma \left(\sup_{t_0 \le \tau} \|u(\tau)\| \right).$$

Here, $\sup_{t_0 \le \tau}$ can be substituted by $\sup_{t_0 \le \tau \le t}$ due to causality.

Additional Theorem (from [183])

For the system

$$\dot{x} = f(x, u)$$

the following are equivalent.

- the system is ISS,
- there exists a smooth ISS-Lyapunov function (a function satisfying the assumption of Theorem 4.19),
- $\exists$ a smooth positive definite radially unbounded function V and class-$\mathcal{K}_\infty$ functions ρ_1 and ρ_2 s.t.

$$\frac{\partial V}{\partial x} f(x, u) \le -\rho_1(\|x\|) + \rho_2(\|u\|).$$

Lemma 4.6

IF $f(t,x,u)$ is C^1 and globally Lipschitz in (x,u) uniformly in t, and the origin of $\dot{x} = f(t,x,0)$ is GES,

THEN the system $\dot{x} = f(t,x,u)$ is ISS.

Proof. Apply the converse Lyapunov Theorem 4.14 for the unforced system, obtain a Lyapunov function satisfying

$$\begin{aligned} c_1\|x\|^2 \le V(t,x) \le c_2\|x\|^2 \\ \frac{\partial V}{\partial t}(t,x) + \frac{\partial V}{\partial x} f(t,x,0) \le -c_3\|x\|^2 \\ \left\| \frac{\partial V}{\partial x}(t,x) \right\| \le c_4\|x\| \end{aligned}$$

Then, the derivative of V along $\dot{x} = f(t,x,u)$ is

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,0) + \frac{\partial V}{\partial x}[f(t,x,u) - f(t,x,0)] \\ &\le -c_3\|x\|^2 + c_4\|x\|L\|u\| \end{aligned}$$

Then, with $0 < \theta < 1$,

$$\dot{V} \le -c_3(1-\theta)\|x\|^2 - c_3\theta\|x\|^2 + c_4L\|x\|\|u\|$$

That it,

$$\dot{V} \le -c_3(1-\theta)\|x\|^2, \qquad \forall \|x\| \ge \frac{c_4L\|u\|}{c_3\theta}.$$

Two examples that do not satisfy the assumption of Lemma 4.6.

Ex.1.: $\dot{x} = -3x + (1+x^2)u$. GES with $u = 0$. Not globally Lipschitz. Not ISS.

Ex.2.: $\dot{x} = -\frac{x}{1+x^2} + u$. Globally Lipschitz. Not GES (but, LES) with $u = 0$. Not ISS.

Example 4.25 Not GES, but ISS.

$$\dot{x} = -x^3 + u$$

GAS with $u = 0$. Let $V = x^2/2$.

$$\dot{V} = -x^4 + xu = -(1-\theta)x^4 - \theta x^4 + xu \le -(1-\theta)x^4, \quad \forall |x| \ge \left(\frac{|u|}{\theta}\right)^{1/3}$$

where $0 < \theta < 1$.

Example 4.26 GES with $u = 0$. Not globally Lipschitz, but ISS.

$$\dot{x} = -x - 2x^3 + (1+x^2)u^2$$

Let $V = x^2/2$. Then

$$\dot{V} = -x^2 - 2x^4 + x(1+x^2)u^2 \le -x^4, \qquad \forall |x| \ge u^2.$$

Cascaded system:

$$\begin{aligned}\dot{x}_1 &= f_1(t, x_1, x_2)\\ \dot{x}_2 &= f_2(t, x_2)\end{aligned}$$

Lemma 4.7 (ISS of Cascade)

IF the second system is UGAS and the first system is ISS with x_2 as an input,

THEN the whole system is UGAS.

Proof.

Let t_0 be the initial time. The solution satisfies that

$$\begin{aligned}\|x_1(t)\| &\le \beta_1(\|x_1(s)\|, t-s) + \gamma_1\left(\sup_{s\le\tau\le t}\|x_2(\tau)\|\right)\\ \|x_2(t)\| &\le \beta_2(\|x_2(s)\|, t-s)\end{aligned}$$

where $t \ge s \ge t_0$. Then, we obtain

$$\begin{aligned}\|x_1(t)\| &\le \beta_1\left(\left\|x_1\left(\frac{t+t_0}{2}\right)\right\|, \frac{t-t_0}{2}\right) + \gamma_1\left(\sup_{\frac{t+t_0}{2}\le\tau\le t}\|x_2(\tau)\|\right)\\ \left\|x_1\left(\frac{t+t_0}{2}\right)\right\| &\le \beta_1\left(\|x_1(t_0)\|, \frac{t-t_0}{2}\right) + \gamma_1\left(\sup_{t_0\le\tau\le\frac{t+t_0}{2}}\|x_2(\tau)\|\right)\\ \sup_{t_0\le\tau\le\frac{t+t_0}{2}}\|x_2(\tau)\| &\le \beta_2(\|x_2(t_0)\|, 0)\\ \sup_{\frac{t+t_0}{2}\le\tau\le t}\|x_2(\tau)\| &\le \beta_2\left(\|x_2(t_0)\|, \frac{t-t_0}{2}\right)\end{aligned}$$

Then, since

$$\|x_1(t_0)\| \le \|x(t_0)\|, \quad \|x_2(t_0)\| \le \|x(t_0)\|, \quad \|x(t)\| \le \|x_1(t)\| + \|x_2(t)\|,$$

we have

$$\|x(t)\| \le \beta(\|x(t_0)\|, t-t_0)$$

where

$$\beta(r,s) = \beta_1\left(\beta_1\left(r, \frac{s}{2}\right) + \gamma_1(\beta_2(r,0)), \frac{s}{2}\right) + \gamma_1\left(\beta_2\left(r, \frac{s}{2}\right)\right) + \beta_2(r,s).$$

Chapter Comments.

- A time-varying system can be written as a time-invariant system by augmenting a time state.
- Simple statement of LaSalle's theorem:

Consider $\dot{x} = f(x)$. Let $V(\cdot)$ be positive definite, radially unbounded and such that

$$\dot{V}(x) \leq 0, \qquad \forall x.$$

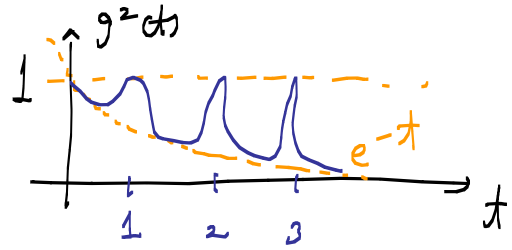
Then, the state $x(t)$ converges to the 'largest invariant set' that is contained in the set $\{x : \dot{V}(x) = 0\}$.

- Intrinsic robustness: $\dot{x} = f(x) + g(x)$.

Consider

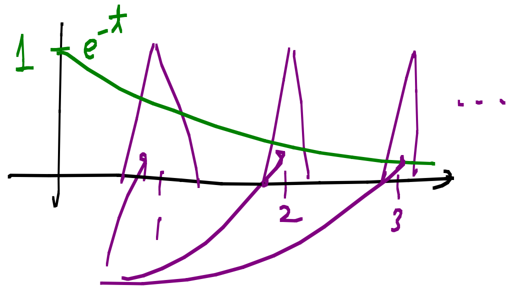
$$\dot{x} = \frac{\dot{g}(t)}{g(t)}\, x$$

where $g(t)$: C^1 function s.t



$$\int_0^\infty g^2(t)\,dt < \int_0^\infty e^{-t}\,dt + \sum_{n=1}^{\infty} \frac{1}{2^n} = 2$$

that is,



Let $V(t,x) = \dfrac{x^2}{g^2(t)}\left[3 - \int_0^t g^2(s)\,ds\right]$

Then, $x^2 < V(t,x)$, but NOT decrescent.

$$\dot{V} = \cdots = -x^2$$

However, $x(t) = \dfrac{g(t)}{g(t_0)}\, x(t_0)$,

and the origin is NOT A.S.