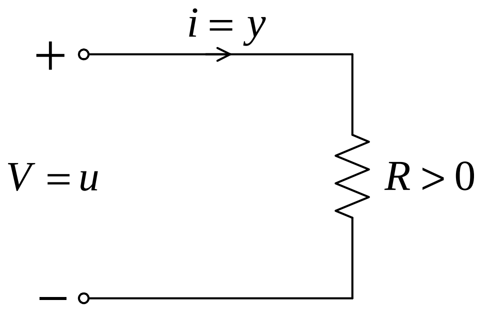
# Class Handout: Chapter 6 Passivity

2006 Fall

Passivity is an important tool for system analysis and control synthesis. It also links between Lyapunov stability and $\mathcal{L}_2$ stability.

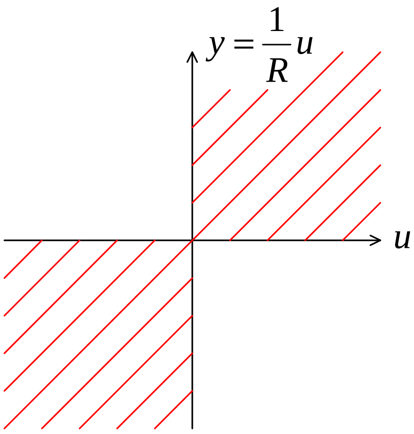
## I. Passivity of a static (memoryless) functions

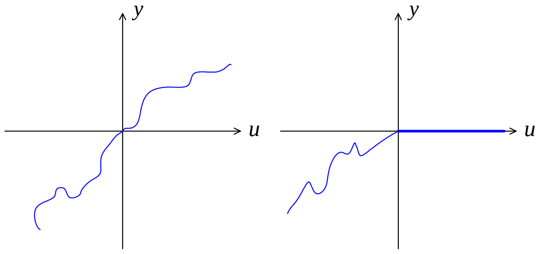



$$V = iR$$
$$u = yR$$
$$\boxed{y = \frac{1}{R}u}$$

$$\text{(power injected into the system =)} \quad u^T y = \frac{1}{R}u^2 \geq 0, \qquad {}^\forall u$$

- *Passive* mapping $y = h(t, u)$ if $u^T y \geq 0$, $\forall t$, $\forall u$.




In these figures, $h$ *belongs to the sector* $[0, \infty]$.

- *Lossless* mapping $y = h(t, u)$ if $u^T y = 0$, $\forall t$, $\forall u$.
- MIMO map must have the same number of inputs and outputs. Easier case would be the

*decentralized*, or *decoupled*, map, i.e.,

$$h(t,u) = \begin{bmatrix} h_1(t,u_1) \\ h_2(t,u_2) \\ \vdots \\ h_p(t,u_p) \end{bmatrix}.$$

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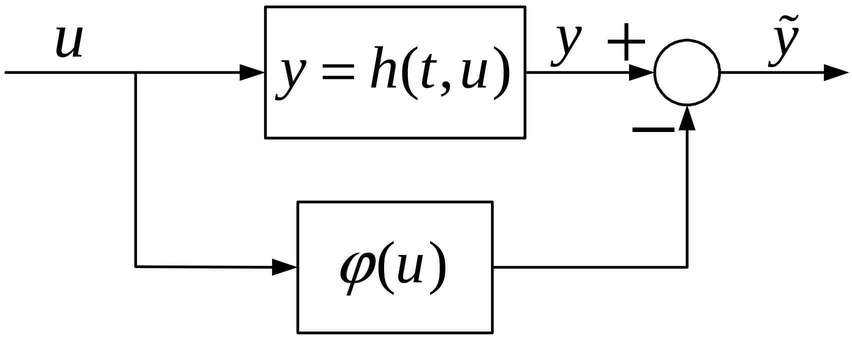
**Definition 6.1** $y = h(t,u)$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^p$

- passive: $u^T y \geq 0$
- lossless: $u^T y = 0$
- input-feedforward passive: $u^T y \geq u^T \psi(u)$, $\psi(u)$: some function
- input strictly passive: $u^T y \geq u^T \psi(u)$ and $u^T \psi(u) > 0$, $^\forall u \neq 0$
- output-feedback passive: $u^T y \geq y^T \rho(y)$, $\rho(y)$: some function
- output strictly passive: $u^T y \geq y^T \rho(y)$ and $y^T \rho(y) > 0$, $^\forall y \neq 0$

Note:

1. For the input-feedforward passive case, if $0 < u^T \rho(u)$, $u \neq 0$, then $u^T \rho(u)$ represents the 'excess of passivity.' If $0 > u^T \rho(u)$ for some $u$, then the map $h$ is still input-feedforward passive but it may not even be passive. In this case, $u^T \rho(u)$ is somehow a shortage of passivity.

2. Removal of excess or shortage of passivity by input-feedforward operation (when $u^T y \geq u^T \psi(u)$): Since $\tilde{y} = y - \psi(u)$, we have

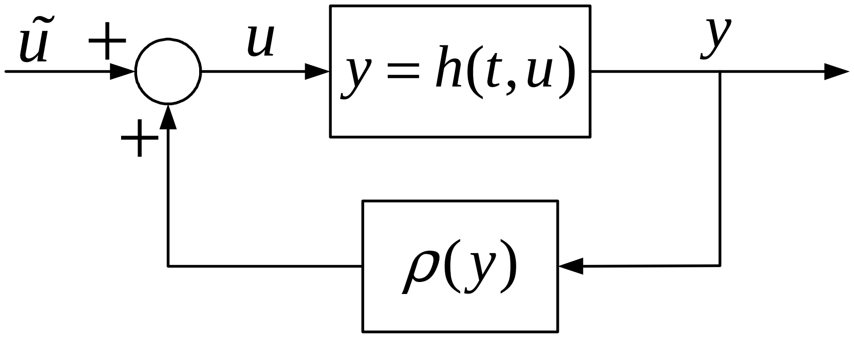



$$u^T \tilde{y} = u^T (y - \psi(u)) = u^T y - u^T \psi(u) \geq 0.$$

That is, $\tilde{y}$ is in the sector $[0, \infty]$.

3. Removal of excess or shortage of passivity by output-feedback operation (when $u^T y \geq y^T \rho(y)$): Since $\tilde{u} = u - \rho(y)$, we have

$$\tilde{u}^T y = u^T y - y^T \rho(y) \geq 0.$$

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**Definition 6.2** $h : [0,\infty) \times \mathbb{R}^p \to \mathbb{R}^p$ belongs to the *sector*

- $[0, \infty]$ if $u^T h(t,u) \geq 0$ (passive)
- $[K_1, \infty]$ if $u^T(h - K_1 u) \geq 0$ (input feedforward passive)
- $[0, K_2]$ with $K_2 = K_2^T$ if $h^T(h - K_2 u) \leq 0$ (output strictly passive if $K_2 = (1/\delta)I$, $\delta > 0$)
- $[K_1, K_2]$ with $K := K_2 - K_1 = K^T > 0$ if $(h - K_1 u)^T(h - K_2 u) \leq 0$.

Note:

- For the sector, the symbol '(' or '[' is used according to $<$ or $\leq$.
- SISO map $h$ is in the sector $[\alpha, \beta]$, that is,

$$\alpha u^2 \leq u h(t,u) \leq \beta u^2,$$

if and only if

$$[h(t,u) - \alpha u][h(t,u) - \beta u] \leq 0.$$

This is clear from a plot for the SISO case. For the decentralized MIMO case, we have

$$[h(t,u) - K_1 u]^T [h(t,u) - K_2 u] \leq 0,$$

where

$$K_1 = \mathrm{diag}(\alpha_1, \alpha_2, \cdots, \alpha_p), K_2 = \mathrm{diag}(\beta_1, \cdots, \beta_p).$$

(Note that $K = K_2 - K_1 = K^T > 0$.)

- The condition

$$\|h(t,u) - Lu\|_2 \leq \gamma \|u\|_2$$

where $L$ is a matrix can also be written as a sector condition. Let $K_1 = L - \gamma I$ and $K_2 = L + \gamma I$. Then,

$$\begin{aligned}[h - K_1 u]^T [h - K_2 u] &= [h - Lu + \gamma u]^T [h - Lu - \gamma u] \\ &= \|h - Lu\|_2^2 - \gamma^2 \|u\|_2^2 \leq 0\end{aligned}$$

where $K = K_2 - K_1 = 2\gamma I$, which is positive definite.

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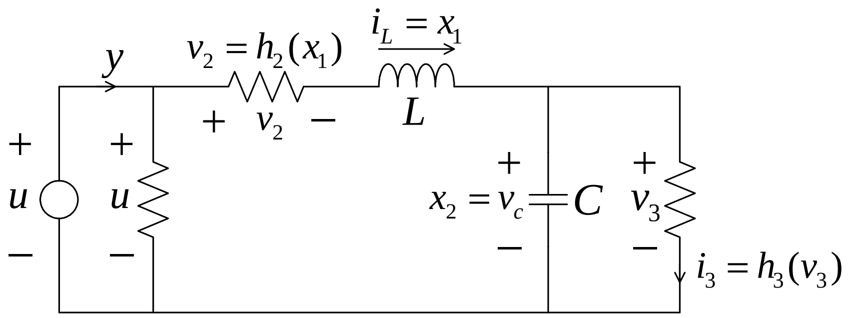
**Exercise 6.1** Verify that a function in the sector $[K_1, K_2]$ can be transformed into a function in the sector $[0, \infty]$ by input feedforward followed by output feedback.

## II. State model

**Example 6.1**




$$L\dot{x}_1 = u - h_2(x_1) - x_2$$
$$C\dot{x}_2 = x_1 - h_3(x_2)$$
$$y = x_1 + h_1(u)$$

The system is said to be *passive* if, for all $t \geq t_0$,

$$\text{(energy supplied)} \quad \int_{t_0}^{t} u(s)y(s)ds \quad \geq \quad V(x(t)) - V(x(t_0)) \quad \text{(energy stored)}$$

where $V(x) = \frac{1}{2}Lx_1^2 + \frac{1}{2}Cx_2^2$ (energy function). Here, the difference in the above inequality is *dissipated* energy.

This is equivalent to

$$u(t)y(t) \geq \dot{V}(x(t)).$$

(Derive it.)

In our example,

$$\dot{V} = uy - uh_1(u) - x_1h_2(x_1) - x_2h_3(x_2).$$

That is,

$$uy = \dot{V} + uh_1(u) + x_1h_2(x_1) + x_2h_3(x_2).$$

For this,

- If $h_1 = h_2 = h_3 \equiv 0$, the system is lossless.
- If $h_1, h_2 \in$ the sector $[0, \infty]$, the system is input feedforward passive.
- If $h_1 \equiv 0$, $h_3 \in [0, \infty]$, then output feedback passive.
- If $h_1 \in [0, \infty]$, $h_2, h_3 \in (0, \infty)$, then state-strict-passive (or, just 'strict passive').

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**Definition 6.3**

$$\dot{x} = f(x, u), \qquad u \in \mathbb{R}^p, y \in \mathbb{R}^p,$$
$$y = h(x, u)$$

is *passive* if $\exists$ $C^1$ positive *semidefinite* function (storage function) $V(x)$ s.t.

$$u^T y \geq \dot{V} = \frac{\partial V}{\partial x}(x) f(x, u), \qquad {}^\forall(x, u) \in \mathbb{R}^n \times \mathbb{R}^p.$$

It is

- lossless if $u^T y = \dot{V}$,
- input feedforward passive if $u^T y \geq \dot{V} + u^T \psi(u)$,
- input strictly passive if $u^T y \geq \dot{V} + u^T \psi(u)$ and $u^T \psi(u) > 0$ for all $u \neq 0$,
- output feedback passive if $u^T y \geq \dot{V} + y^T \rho(y)$,
- output strictly passive if $u^T y \geq \dot{V} + y^T \rho(y)$ and $y^T \rho(y) > 0$ for all $y \neq 0$,
- strictly passive if $u^T y \geq \dot{V} + \psi(x)$ where $\psi(x)$ is positive definite

for all $x$ and $u$.

Note:

- In some cases: $V \equiv 0$ is enough.
- If a system is strictly passive with positive definite storage function $V(x)$, then the system is asymptotic stable with $u = 0$.

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**Example 6.2** Consider an integrator

$$\dot{x} = u, \qquad y = x.$$

This is a lossless system.(Why?) Try using $V(x) = \frac{1}{2}x^2$.

If this system is paralleled with $h(u)$, i.e.,

$$\dot{x} = u, \qquad y = x + h(u),$$

the system becomes input-feedforward passive (because the parallel path $h(u)$ can be canceled by feedforward from the input).

If the integrator is feedbacked by $-h(y)$, i.e.,

$$\dot{x} = -h(x) + u, \qquad y = x,$$

then the system becomes output-feedback passive (since the feedback path can be canceled by a feedback from the output).

## III. Positive Real Transfer Functions

**Definition 6.4** A $p \times p$ proper rational transfer function matrix $G(s)$ is called *positive real* if

(A) poles of all elements of $G(s)$ are in $\mathrm{Re}[s] \le 0$ (CLHP:Closed-Left-Half Plane),

(B) for all real $\omega$ for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite,

(C) any pure imaginary pole $j\omega$ of any element of $G(s)$ is simple and the residual matrix $\lim_{s\to j\omega}(s - j\omega)G(s)$ is positive semidefinite Hermitian.

The transfer function $G(s)$ is *strictly positive real* if $G(s - \epsilon)$ is positive real for some $\epsilon > 0$.

For the first reading, you may want to assume a SISO system with no direct feedthrough term (i.e., $G(\infty) = D = 0$).

**Lemma 6.1** Suppose that

$$\det[G(s) + G^T(-s)] \not\equiv 0.$$

$G(s)$ is SPR if and only if

(a) $G(s)$ is Hurwitz,

(b) $G(j\omega) + G^T(-j\omega)$ is positive definite for all $\omega$,

(c) either $G(\infty) + G^T(\infty) > 0$, or it is $\ge 0$ and

$$\lim_{\omega\to\infty} \omega^2 M^T[G(j\omega) + G^T(-j\omega)]M > 0$$

for any full rank $p \times (p - q)$ matrix $M$, ($q = \mathrm{rank}[G(\infty) + G^T(\infty)]$), such that

$$M^T[G(\infty) + G^T(\infty)]M = 0.$$

Here, $G(j\omega) + G^T(-j\omega)$ is Hermitian and may be complex in skew-symmetric part. But 'positive definite' means its symmetric part is real and positive definite in the usual sense.

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**Example 6.4** Check positive realness of the following transfer functions.

1. $G(s) = \frac{1}{s}$:

- (A) No poles in ORHP.
- (B) $\mathrm{Re}[G(j\omega)] = 0$.
- (C) Simple pole. Residue = 1.

Thus, PR, but not SPR.

2. $G(s) = \frac{1}{s+a}$, $a > 0$:

- (A) Passed.
- (B) $\mathrm{Re}[G(j\omega)] = \mathrm{Re}\frac{1}{j\omega+a} = \mathrm{Re}\frac{a-j\omega}{\omega^2+a^2} = \frac{a}{\omega^2+a^2} > 0$ At the same time, $G(s - \epsilon) = \frac{1}{s+a-\epsilon}$ is also PR with small $\epsilon$
- (C) NC (No Concern)

Thus, PR and SPR.

Also, by Lemma 6.1, (a) and (b) are okay, and for (c),

$$\lim_{\omega\to\infty} \omega^2 \mathrm{Re}[G(j\omega)] = \lim_{\omega\to\infty} \frac{\omega^2 a}{\omega^2 + a^2} = a > 0.$$

Thus, we again have SPR.

3. $G(s) = \frac{1}{s^2+s+1}$: This system has the relative degree 2, which cannot be PR. In fact, its Nyquist plot goes into OLHP. (Why?) Truly,

$$\mathrm{Re}[G(j\omega)] = \frac{1-\omega^2}{(1-\omega^2)^2 + \omega^2} < 0, \qquad {}^\forall|\omega| > 1.$$

4. $G(s) = \frac{1}{s+1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$: For this,

$$\det[G(s) + G^T(-s)] \equiv 0, \qquad {}^\forall s.$$

So, Lemma 6.1 is useless for this case. We then apply Definition 6.1 instead. Let us check the positive realness of $G(s-\epsilon)$. With small $\epsilon$,

- (A) Passed.
- (B) $G(j\omega - \epsilon) + G^T(-j\omega - \epsilon) = \frac{2(1-\epsilon)}{(1-\epsilon)^2+\omega^2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. So, (B) is passed.
- (C) NC

Thus, SPR.

5. $G(s) = \frac{1}{s+1}\begin{bmatrix} s & 1 \\ -1 & 2s+1 \end{bmatrix}$: We apply Lemma 6.1.

- (a) Passed.
- (b) $G(j\omega - \epsilon) + G^T(-j\omega - \epsilon) = \frac{2}{\omega^2+1}\begin{bmatrix} \omega^2 & -j\omega \\ j\omega & 2\omega^2+1 \end{bmatrix} > 0$. So, passed.
- (c) $G(\infty) + G^T(\infty) = 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} > 0$. So, passed.

Thus, SPR.

6. $G(s) = \begin{bmatrix} \frac{s}{s+1} & \frac{1}{s+2} \\ \frac{-1}{s+2} & \frac{2}{s+1} \end{bmatrix}$: We apply Lemma 6.1.

- (a) Passed.
- (b) $G(j\omega - \epsilon) + G^T(-j\omega - \epsilon) = \begin{bmatrix} \frac{2\omega^2}{1+\omega^2} & \frac{-2j\omega}{4+\omega^2} \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix} > 0.$
- (c) $G(\infty) + G^T(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. So, let

$$M = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so that

$$M^T(G(\infty) + G^T(\infty))M = 0.$$

Then,

$$\omega^2 M^T[G(j\omega) + G^T(-j\omega)]M = \frac{4\omega^2}{1+\omega^2} \to 4 > 0.$$

Thus, SPR.

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Let $G(s) = C(sI-A)^{-1}B+D$ be a $p\times p$ transfer function matrix, and $(A, B)$ is controllable, $(A, C)$ is observable.

**Lemma 6.2 (Positive Real Lemma)** $G(s)$ is positive real if and only if

$\exists P = P^T > 0$, $L$, $W$ s.t.

$$\begin{aligned} PA + A^T P &= -L^T L \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

For the proof, see Appendix C.12.

**Lemma 6.3 (Kalman-Yakubovich-Popov Lemma)** $G(s)$ is strictly positive real if and only if

$\exists P = P^T > 0$, $L$, $W$, $\epsilon > 0$ s.t.

$$\begin{aligned} PA + A^T P &= -L^T L - \epsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

*Proof.* (Necessity): If $G(s)$ is SPR, then $\exists\epsilon > 0$ s.t. $G(s - \epsilon/2)$ is PR.

Since $G(s - \epsilon/2) = C(sI - \frac{\epsilon}{2}I - A)^{-1}B + D$, from the PR lemma, $\exists P$ s.t.

$$P(A + \frac{\epsilon}{2}I) + (A + \frac{\epsilon}{2}I)^T P = -L^T L$$

which implies the condition of KYP lemma.

(Sufficiency): Let $\mu := \epsilon/2$. Then, from the assumption,

$$P(A + \mu I) + (A + \mu I)^T P = -L^T L.$$

Since $G(s - \mu) = C(sI - \mu I - A)^{-1}B + D$, and from PR Lemma above, it concludes that $G(s - \mu)$ is PR, i.e., $G(s)$ is SPR.

**Lemma 6.4** (Passivity and PR)

The LTI minimal realization

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

with $G(s) = C(sI - A)^{-1}B + D$ is

- passive if $G(s)$ is PR;
- strictly passive if $G(s)$ is SPR.

*Proof.* We apply Lemmas 6.2 and 6.3.

Let $V(x) = \frac{1}{2}x^T Px$.

$$\begin{aligned} u^T y - \frac{\partial V}{\partial x}(Ax + Bu) &= u^T(Cx + Du) - x^T P(Ax + Bu) \\ &= u^T Cx + \frac{1}{2}u^T(D + D^T)u - \frac{1}{2}x^T(PA + A^T P)x - x^T PBu \\ &= u^T(B^T P + W^T L)x + \frac{1}{2}u^T W^T Wu + \frac{1}{2}x^T L^T Lx + \frac{1}{2}\epsilon x^T Px - x^T PBu \\ &= \frac{1}{2}(Lx + Wu)^T(Lx + Wu) + \frac{1}{2}\epsilon x^T Px \ge \frac{1}{2}\epsilon x^T Px \end{aligned}$$

So, PR implies passivity (with $\epsilon = 0$) while SPR implies strict passivity (with $\epsilon > 0$).

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Note: For $(A, B, C, D)$, if $D = 0$, then the system should have the *relative degree* 1 if it is passive. (Why?)

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*Proof of Lemma 6.1*

(Necessity): Let $G(s)$ be SPR. Then, $\exists \mu > 0$ s.t. $G(s - \mu)$ is PR (no poles in ORHP). Thus, $G(s)$ is Hurwitz. $\Rightarrow$ (a)

For (b), let $(A, B, C, D)$: minimal realization of $G(s)$. By Lemma 6.3, $\exists\ P, L, W$ s.t.

$$A^T P + PA \le -L^T L, \qquad PB = C^T - L^T W, \qquad D + D^T = W^T W.$$

Let $\Phi(s) = (sI - A)^{-1}$. Then,

$$\begin{aligned} G + G^* &= D + D^T + C\Phi B + B^T \Phi^T(-s)C^T \\ &= W^T W + (B^T P + W^T L)\Phi B + B^T \Phi^T(-s)(PB + L^T W) \\ &= W^T W + W^T L\Phi(s)B + B^T \Phi^T(-s)L^T W + B^T P\Phi(s)B + B^T \Phi^T(-s)PB \end{aligned}$$

Here, note that

$$\begin{aligned} &B^T P\Phi(s)B + B^T \Phi^T(-s)PB \\ &\quad = B^T(-sI - A^T)^{-1}(-sI - A^T)P\Phi(s)B + B^T \Phi^T(-s)P(sI - A)(sI - A)^{-1}B \\ &\quad = B^T \Phi^T(-s)[-sP - A^T P + sP - PA]\Phi(s)B. \end{aligned}$$

With this, we continue that

$$\begin{aligned} G + G^* &= W^T W + W^T L\Phi(s)B + B^T\Phi^T(-s)L^T W + B^T\Phi^T(-s)[-A^T P - PA]\Phi(s)B \\ &= [W^T + B^T\Phi^T(-s)L^T][W + L\Phi(s)B] + \epsilon B^T\Phi^T(-s)P\Phi(s)B. \end{aligned}$$

Note that, since $G(s)$ is PR, $G(j\omega) + G^T(-j\omega) \geq 0$ for all $\omega$ which implies that

$$G(\infty) + G^T(\infty) \quad (= D + D^T) \quad \geq \quad 0. \tag{1}$$

In order to claim that $G(j\omega) + G^T(-j\omega)$ is PD, $^\forall\omega$, we suppose that it it not; $\exists\omega$ s.t. $G(j\omega) + G^T(-j\omega)$ is singular.

$$\begin{aligned} &\Rightarrow \quad \exists x \in \mathbb{C}^p, x \neq 0, \qquad x^*[G(j\omega) + G^T(-j\omega)]x = 0 \\ &\Rightarrow \quad x^* B^T\Phi^T(-j\omega)P\Phi(j\omega)Bx = 0 \qquad \Rightarrow \qquad Bx = 0 \\ &\Rightarrow \quad x^*[W^T + B^T\Phi^T(-j\omega)L^T][W + L\Phi(j\omega)B]x = 0 \qquad \Rightarrow \qquad Wx = 0 \\ &\Rightarrow \quad x^*[G(s) + G^T(-s)]x = 0, \qquad \forall s \qquad \text{(Can you agree with } j\omega \to s\text{?)} \\ &\Rightarrow \quad \det[G(s) + G^T(-s)] \equiv 0 \end{aligned}$$

which contradicts the assumption of the Lemma. This proves (b).

Now if $G(\infty) + G^T(\infty)$ is PD (i.e., (c)), the proof completes. If not, let $M$: any $p \times (p - q)$ full rank matrix s.t.

$$M^T(D + D^T)M = M^T W^T W M = 0.$$

Then,

$$M^T[G(j\omega) + G^T(-j\omega)]M = M^T B^T\Phi^T(-j\omega)(L^T L + \epsilon P)\Phi(j\omega)BM.$$

Here $BM$ is of full column rank because, if not, $\exists x \neq 0$ s.t. $BMx = 0$. Then,

$$x^T M^T[G(j\omega) + G^T(-j\omega)]Mx = 0, \qquad ^\forall\omega$$

which is a contradiction to the fact that $(G(j\omega) + G^T(-j\omega))$ is PD.

Finally,

$$\lim_{\omega\to\infty} \omega^2 M^T[G(j\omega) + G^T(-j\omega)]M = M^T B^T(L^T L + \epsilon P)BM$$

is PD because $BH$ has full column rank. ($\Rightarrow$ (c))

(Sufficiency): We will show that $G(s - \epsilon)$ is PR assuming that (a), (b), and (c) hold.

Since $G(s)$ is Hurwitz, $\exists\mu^* > 0$ s.t. $G(s - \mu)$ is Hurwitz for all $\mu \in [0, \mu^*]$. Thus, (A) holds and we don't have to check (C). It is left to show that

$$G(j\omega - \mu) + G^T(-j\omega - \mu) \geq 0, \qquad ^\forall\omega.$$

Let $(A, B, C, D)$ be the minimal realization of $G(s)$. Then,

$$\begin{aligned} G(s-\mu) &= D + C(sI - \mu I - A)^{-1}B \\ &= D + C(sI - A)^{-1}(\mu I + sI - \mu I - A)(sI - \mu I - A)^{-1}B \\ &= D + C(sI - A)^{-1}B + \mu C(sI - A)^{-1}(sI - \mu I - A)^{-1}B \\ &=: G(s) + \mu N_\mu(s). \end{aligned}$$

Let $k_0$ and $k_1$ be s.t., for all $\omega$ and for all $\mu \in [0, \mu^*]$,

$$\begin{aligned} \sigma_{\max}[N_\mu(j\omega) + N_\mu^T(-j\omega)] &\le k_0 \\ \omega^2 \sigma_{\max}[N_\mu(j\omega) + N_\mu^T(-j\omega)] &\le k_1 \end{aligned}$$

where $k_0$ and $k_1$ are finite. (Why?)

Now we have two cases. First, when $G(\infty) + G^T(\infty) > 0$, $\exists \sigma_0$ s.t.

$$\sigma_{\min}[G(j\omega) + G^T(-j\omega)] \ge \sigma_0, \qquad {}^\forall \omega.$$

Then, for all $\omega$, it holds that

$$\sigma_{\min}[G(j\omega - \mu) + G^T(-j\omega - \mu)] \ge \sigma_0 - \mu k_0$$

which is positive with sufficiently small $\mu$. On the other hand, if $G(\infty) + G^T(\infty)$ is singular, $\exists \sigma_1, \sigma_2 > 0$ and $\omega_2 > 0$ s.t.

$$\begin{aligned} \omega^2 \sigma_{\min}[G(j\omega) + G^T(-j\omega)] &\ge \sigma_1, \qquad {}^\forall |\omega| \ge \omega_2 \\ \sigma_{\min}[G(j\omega) + G^T(-j\omega)] &\ge \sigma_2, \qquad {}^\forall |\omega| \le \omega_2. \end{aligned}$$

Therefore,

$$\begin{aligned} \omega^2 \sigma_{\min}[G(j\omega - \mu) + G^T(-j\omega - \mu)] &\ge \sigma_1 - \mu k_1, \qquad {}^\forall |\omega| \ge \omega_2 \\ \sigma_{\min}[G(j\omega - \mu) + G^T(-j\omega - \mu)] &\ge \sigma_2 - \mu k_0, \qquad {}^\forall |\omega| \le \omega_2 \end{aligned}$$

which are positive with sufficiently small $\mu$. This proves (B).

# IV. $\mathcal{L}_2$ and Lyapunov Stability

$$\dot{x} = f(x,u) \qquad u \in \mathbb{R}^p, y \in \mathbb{R}^p$$
$$y = h(x,u)$$

where $f$: locally Lipschitz, $h$: continuous and $f(0,0) = 0$, $h(0,0) = 0$.

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**Relation with $\mathcal{L}_2$ stability**

Assume that $\exists$ PSD $V(x)$ s.t. $u^T y \geq \dot{V} + \delta y^T y$, $\delta > 0$ (i.e., OSP). Then,

$$\dot{V} \leq u^T y - \delta y^T y \leq \frac{1}{2\delta} u^T u + \frac{\delta}{2} y^T y - \delta y^T y = \frac{1}{2\delta}\|u\|^2 - \frac{\delta}{2}\|y\|^2$$

$$\Rightarrow \qquad V(x(\tau)) - V(x(0)) \leq \frac{1}{2\delta}\int_0^\tau \|u\|^2 d\tau - \frac{\delta}{2}\int_0^\tau \|y\|^2 d\tau$$
$$0 \leq V(x(\tau)) \leq ... + V(x(0))$$

$$\Rightarrow \qquad \|y_\tau\|_{\mathcal{L}_2}^2 \leq \frac{1}{\delta^2}\|u_\tau\|_{\mathcal{L}_2}^2 + \frac{2}{\delta} V(x(0))$$
$$\Rightarrow \qquad \|y_\tau\|_{\mathcal{L}_2} \leq \frac{1}{\delta}\|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{2}{\delta} V(x(0))}$$

Therefore, *if the system is OSP with $\rho(y) = \delta y$, then it is finite-gain $\mathcal{L}_2$ stable with the gain $\leq 1/\delta$.* **(Lemma 6.5)**

---

**Relation with Lyapunov stability**

It is easy to see that the origin of $\dot{x} = f(x,0)$ is *stable* if the system $\dot{x} = f(x,u)$ is passive with PD storage function $V(x)$, because

$$\dot{V} \leq 0.$$

Then, what about the *asypmtotic stability*? We may want to use the invariance principle. That is, we may want to follow the path:

$$\dot{V} = 0 \qquad \Rightarrow \qquad y = 0 \qquad \Rightarrow \qquad x = 0.$$

This is, in fact, a kind of observability. For example, consider

$$\dot{x} = Ax, \qquad y = Cx.$$

Then,

$$y(t) = Ce^{At}x(0) \equiv 0 \qquad \Leftrightarrow \qquad x(0) = 0 \qquad \Leftrightarrow \qquad x(t) = 0.$$

**Definition 6.5**

In the textbook, only ZSO with $D = \mathbb{R}^n$ is presented. But, ....

- Zero-state observability (ZSO) on a domain $D$: No sol. of $\dot{x} = f(x, 0)$ can stay *identically* in $S = \{x \in D : h(x, 0) = 0\}$ other than the trivial solution $x(t) = 0$.
- Local ZSO on a domain $D$: No sol. of $\dot{x} = f(x, 0)$ can stay *for some open interval of time* in $S = \{x \in D : h(x, 0) = 0\}$ other than the trivial solution $x(t) = 0$.

**My Lemma 6.7**

The system $\dot{x} = f(x, 0)$ is AS if the system $\dot{x} = f(x, u)$ is

- strictly passive on $D$ with a PD storage function,

or,

- OSP with a PD storage function, and ZSO on $D$.

In addition, if $D = \mathbb{R}^n$ and the storage function is radially unbounded, then the origin is GAS.

*Proof.* The first case is trivial. For the second case, note that OSP implies that

$$\dot{V} \le u^T y - y^T \rho(y), \qquad \text{where} \quad y^T \rho(y) > 0, ^{\forall} y \ne 0.$$

With $u = 0$, and using the LaSalle's invariant principle, we get the conclusion.

**Lemma 6.7**

The system $\dot{x} = f(x, 0)$ is AS if the system $\dot{x} = f(x, u)$ is

- strictly passive on $\mathbb{R}^n$

or,

- OSP and ZSO on $\mathbb{R}^n$.

In addition, if the storage function is radially unbounded, then the origin is GAS.

*Proof.* (First case):

$$\dot{V} \le -\psi(x)$$

where $\psi$ is PD.

We now show that, this also, in fact, implies that $V(x)$ is PD. For any $x \in \mathbb{R}^n$, the system $\dot{x} = f(x, 0)$ has a sol. $\phi(t; x)$ on the *maximal* interval $t \in [0, \delta)$. Then, for all $\tau \in [0, \delta)$,

$$V(\phi(\tau; x)) - V(x) \le -\int_0^\tau \psi(\phi(t; x))dt.$$

This again implies that

$$V(x) \geq \int_0^\tau \psi(\phi(t;x))dt.$$

If $\exists \bar{x} \neq 0$ and $V(\bar{x}) = 0$, then

$$\begin{aligned} 0 = V(\bar{x}) &\geq \int_0^\tau \psi(\phi(t;\bar{x}))dt \\ &\Rightarrow \quad \psi(\phi(t;\bar{x})) \equiv 0 \quad \Rightarrow \quad \phi(t;\bar{x}) \equiv 0 \quad \Rightarrow \quad \bar{x} = 0. \end{aligned}$$

(Second case): OSP implies that, with $u = 0$,

$$\dot{V} \leq -y^T \rho(y) < 0, \qquad y \neq 0.$$

Then, we have

$$V(x) \geq \int_0^\tau h^T(\phi(t;x),0)\rho(h(\phi(t;x),0))dt.$$

Similarly, if $\bar{x} = 0$ and $V(\bar{x}) = 0$, then

$$h(\phi(t,\bar{x}),0) = 0 \qquad \text{on the maximal interval } [0,\delta),$$

which is impossible by the ZSO property on $\mathbb{R}^n$ except the case when $\bar{x} = 0$.

**My Another Lemma 6.7**

The system $\dot{x} = f(x,0)$ is AS if the system $\dot{x} = f(x,u)$ is

- strictly passive on $D$

or,

- OSP and *locally* ZSO on $D$.

*Proof.* DIY.

---

**Example 6.5 (Nonlinear KYP Lemma)**

Kalman-Yakubovich-Popov

$$\begin{aligned} \dot{x} &= f(x) + G(x)u, \qquad u \in \mathbb{R}^p, y \in \mathbb{R}^p, \\ y &= h(x), \qquad\qquad f(0) = 0 : \text{locally Lipschitz}, h(0) = 0 : \text{continuous}. \end{aligned}$$

The system is (strictly) passive if and only if $\exists$ $C^1$ PSD $V(x)$ s.t.

$$\begin{aligned} L_f V(x) &\leq 0 \quad (< 0) \\ L_G V(x) &= h^T(x). \end{aligned}$$

*Proof of this example.*

$$u^T y - \frac{\partial V}{\partial x}(f + Gu) = u^T h - L_f V - h^T u = -L_f V$$

With this, can you show the sufficiency *and* the necessity?

**Example 6.6**

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -ax_1^3 - kx_2 + u\\ y &= x_2, \qquad a > 0, k > 0.\end{aligned}$$

Let $V = \frac{1}{4}ax_1^2 + \frac{1}{2}x_2^2$. Then,

$$\dot{V} = \cdots = -ky^2 + yu.$$

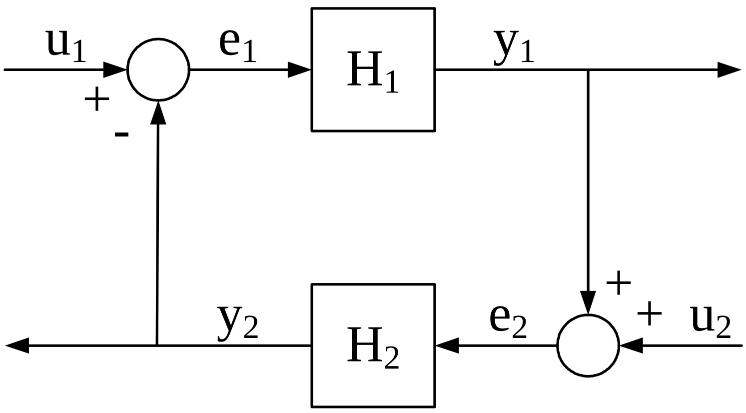
Therefore, it is OSP. (Thus, it is finite-gain $\mathcal{L}_2$ stable with the gain $\leq 1/k$.)

And, when $u = 0$, it is ZSO (with PD storage function) on $\mathbb{R}^2$, because

$$y \equiv 0 \quad \Rightarrow \quad x_2 = 0 \quad \Rightarrow \quad ax_1^3 = 0 \quad \Rightarrow \quad x_1 = 0.$$

So, the system is GAS.

## V. Feedback Systems: Passivity Theorems




$$
\begin{aligned}
\dot{x}_i &= f_i(x_i, e_i) \\
y_i &= h_i(x_i, e_i), \qquad \text{or} \qquad y_i = h_i(e_i) \\
\Rightarrow \qquad \dot{x} &= f(x, u) \\
y &= h(x, u)
\end{aligned}
$$

Let both $H_1$ and $H_2$ be passive, i.e.,

$$e_i^T y_i \geq \dot{V}_i(x_i) \qquad (V_i \equiv 0 \text{ for memoryless system})$$

Then,

$$
\begin{aligned}
&\Rightarrow \qquad e_1^T y_1 + e_2^T y_2 = (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2 = u^T y \text{ !!} \\
&\Rightarrow \qquad u^T y \geq \dot{V}_1 + \dot{V}_2 = \dot{V} \quad \text{taking } V(x) = V_1(x_1) + V_2(x_2).
\end{aligned}
$$

Therefore,

**Theorem 6.1** Negative feedback of two passive systems is passive.

**Lemma 6.8** Negative feedback of two output strictly passive systems

$$e_i^T y_i \geq \dot{V}_i + \delta_i y_i^T y_i, \qquad \delta_i > 0,$$

is FG $\mathcal{L}_2$ stable with the gain $\leq 1/\min(\delta_1, \delta_2)$.

*Proof.*

$$
\begin{aligned}
V(x) &= V_1(x_1) + V_2(x_2) \\
u^T y &= e_1^T y_1 + e_2^T y_2 \geq \dot{V}_1 + \delta_1 y_1^T y_1 + \dot{V}_2 + \delta_2 y_2^T y_2 \\
&\geq \dot{V} + \delta(y_1^T y_1 + y_2^T y_2) = \dot{V} + \delta y^T y
\end{aligned}
$$

Then the conclusion follows (see Lemma 6.5).

**Theorem 6.2** Let

$$e_i^T y_i \geq \dot{V}_i(x_i) + \epsilon_i e_i^T e_i + \delta_i y_i^T y_i$$

(where $\epsilon_i$ and $\delta_i$ can have negative values).

IF

$$\epsilon_1 + \delta_2 > 0 \qquad \text{and} \qquad \epsilon_2 + \delta_1 > 0$$

THEN the closed-loop system is FG $\mathcal{L}_2$ stable from $u$ to $y$.

*Proof.*

$$\begin{aligned} e_1^T y_1 + e_2^T y_2 &= u_1^T y_1 + u_2^T y_2 \\ e_1^T e_1 &= u_1^T u_1 - 2u_1^T y_2 + y_2^T y_2 \\ e_2^T e_2 &= u_2^T u_2 + 2u_2^T y_1 + y_1^T y_1 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \dot{V} = \dot{V}_1 + \dot{V}_2 &\leq e_1^T y_1 + e_2^T y_2 - \epsilon_1 e_1^T e_1 - \delta_1 y_1^T y_1 - \epsilon_2 e_2^T e_2 - \delta_2 y_2^T y_2 \\ &= u_1^T y_1 + u_2^T y_2 - \epsilon_1 u_1^T u_1 + 2\epsilon_1 u_1^T y_2 - \epsilon_1 y_2^T y_2 - \delta_1 y_1^T y_1 \\ &\quad - \epsilon_2 u_2^T u_2 - 2\epsilon_2 u_2^T y_1 - \epsilon_2 y_1^T y_1 - \delta_2 y_2^T y_2 \\ &= -y^T \begin{bmatrix} (\epsilon_2 + \delta_1)I & 0 \\ 0 & (\epsilon_1 + \delta_2)I \end{bmatrix} y - u^T \begin{bmatrix} \epsilon_1 I & 0 \\ 0 & \epsilon_2 I \end{bmatrix} u + u^T \begin{bmatrix} I & 2\epsilon_1 \\ -2\epsilon_2 & I \end{bmatrix} y \end{aligned}$$

Then, with $a := \min(\epsilon_2 + \delta_1, \epsilon_1 + \delta_2)$ and some $b \geq 0$ and $c \geq 0$,

$$\begin{aligned} \dot{V} &\leq -a\|y\|^2 + b\|u\|\|y\| + c\|u\|^2 \\ &\leq -a\|y\|^2 + \frac{a}{2}\|y\|^2 + \frac{b^2}{2a}\|u\|^2 + c\|u\|^2 \\ &= -\frac{a}{2}\|y\|^2 + \left(\frac{b^2}{2a} + c\right)\|u\|^2. \end{aligned}$$

**Example 6.7** Let

$$\begin{aligned} H_1: \qquad \dot{x} &= f(x) + G(x)e_1 \\ y_1 &= h(x) \end{aligned}$$

and $H_2 : y_2 = ke_2$ where $k > 0$, $e_i, y_i \in \mathbb{R}^p$. Let $H_1$ be passive (but not in the strict sense). That is, $\exists$ PD $V_1(x)$ s.t. $L_f V_1(x) \leq 0$ and $L_G V_1(x) = h^T(x)$. Note that $H_2$ is passive, especially, it is seen as being OSP because

$$e_2^T y_2 = ke_2^T e_2 = \gamma k e_2^T e_2 + (1-\gamma)ke_2^T e_2 = \gamma k e_2^T e_2 + \frac{1-\gamma}{k} y_2^T y_2$$

with $0 < \gamma < 1$. Then, we have $\epsilon_1 = \delta_1 = 0$, $\epsilon_2 = \gamma k$, and $\delta_2 = (1-\gamma)/k$. By Theorem 6.2, the closed-loop is FG $\mathcal{L}_2$ stable.

**Example 6.8** Let

$$
\begin{aligned}
H_1: \qquad \dot{x}_1 &= x_2 \\
\dot{x}_2 &= -ax_1 - \sigma(x_2) + e_1 \\
y_1 &= x_2
\end{aligned}
$$

where $a > 0$, $\sigma \in [-\alpha, \infty]$ $(\alpha > 0)$, and $H_2 : y_2 = ke_2$ where $k > 0$.

Let $V_1 = \frac{a}{4}x_1^4 + \frac{1}{2}x_2^2$.

$$\dot{V}_1 = ax_1^3x_2 - ax_1^3x_2 - x_2\sigma(x_2) + x_2e_1 \le \alpha x_2^2 + x_2e_1 = \alpha y_1^2 + y_1e_1.$$

Thus, for $H_1$, we have $\epsilon_1 = 0$, $\delta_1 = -\alpha$.

Also, as in Example 6.7, we have $\epsilon_2 = \gamma k$, and $\delta_2 = (1-\gamma)/k$. If $k > \alpha$, then we can choose $\gamma$ s.t. $\gamma k > \alpha$ so that the closed-loop is FG $\mathcal{L}_2$ stable by Theorem 6.2.

---

**Asymptotic stability of the feedback (with $u = 0$)**

**Theorem 6.3** For two time-invariant systems $H_1$ and $H_2$, if

1. both are strictly passive,
2. both are OSP and ZSO,
3. one is strictly passive and the other is OSP and ZSO,

then, with $u = 0$, the origin of the closed-loop is AS. In addition, if $V$ is radially unbounded, it is GAS.

*Proof.* Let

$$V(x) = V_1(x_1) + V_2(x_2).$$

Then, we know from the proof of Lemma 6.7 that both $V_1$ and $V_2$ are *positive definite*, which is implicitly implied by the assumption (strict passivity, or OSP+ZSO).

For the first case, note that

$$\dot{V} \le \cdots = u^T y - \psi_1(x_1) - \psi_2(x_2)$$

which implies AS.

For the second case, note that

$$\dot{V} \le -y_1^T\rho(y_1) - y_2^T\rho(y_2).$$

From the feedback structure and the fact that the systems are ZSO,

$$
\begin{array}{ccccccccc}
(y_2 \equiv 0 & \Rightarrow & e_1 \equiv 0) & + & (y_1 \equiv 0) & \Rightarrow & x_1 \equiv 0 \\
(y_1 \equiv 0 & \Rightarrow & e_2 \equiv 0) & + & (y_2 \equiv 0) & \Rightarrow & x_2 \equiv 0
\end{array}
$$

which leads to the conclusion by the invariant principle.

For the third case, let's skip it.

**Example 6.9** Let $H_1$ be

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -ax_1^3 - kx_2 + e_1$$
$$y_1 = x_2,$$

and $H_2$ be

$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = -bx_3 - x_4^3 + e_2$$
$$y_2 = x_4.$$

Show that both are ZSO. And, with $V_1 = (a/4)x_1^4 + (1/2)x_2^2$ and $V_2 = (b/2)x_3^2 + (1/2)x_4^2$, show that they are OSP. For example,

$$\dot{V}_1 = ax_1^3x_2 + x_2(-ax_1^3 - kx_2 + e_1) = -kx_2^2 + x_2e_1,$$
$$\dot{V}_2 = bx_3x_4 + x_4(-bx_3 - x_4^3 + e_2) = -x_4^4 + x_4e_2,$$

which implies that $H_1$ and $H_2$ are OSP.

Then, the closed-loop of both are GAS with $u = 0$.

**Example 6.10** What if, in Example 6.9, $y_1 = x_2 + e_1$?

Then,

$$\dot{V}_1 = -kx_2^2 + x_2e_1 = -k(y_1 - e_1)^2 - e_1^2 + y_1e_1.$$

With this, $H_1$ is passive (but not strictly passive or OSP yet).

With $V = V_1 + V_2$,

$$\begin{aligned}\dot{V} &= -kx_2^2 + x_2e_1 - x_4^4 + x_4e_2 \\ &= -kx_2^2 - x_2x_4 - x_4^4 + x_4(x_2 - x_4) \\ &= -kx_2^2 - x_4^4 - x_4^2 \le 0\end{aligned}$$
$$\dot{V} = 0 \qquad \Rightarrow \qquad x_2 = x_4 = 0 \qquad \Rightarrow \qquad x_1 = x_3 = 0.$$

Thus, the closed-loop is GAS by the invariance principle.

---

**Incorporating Time-varying Static Nonlinearity**

**Theorem 6.4** IF

- $H_1$: strict passive, TI
- $H_2$: TV, memoryless passive

THEN the origin of the CL ($u = 0$) is UAS.

(IF $V$ is also radially unbounded, THEN UGAS.)

*Proof.* $V_1(\cdot)$ is in fact PD.

$$\dot{V}_1 = \frac{\partial V_1}{\partial x} f_1(x_1, e_1) \leq e_1^T y_1 - \psi_1(x_1) = -e_2^T y_2 - \psi_1(x_1) \leq -\psi_1(x_1)$$

in which, $e_1^T y_1 = -e_2^T y_2$ from the structure, and $-e_2^T y_2 \leq 0$ due to passivity.

**Theorem 6.5** Let

- $H_1$: TI, ZSO and $\exists$ PD $V_1$ s.t.

$$e_1^T y_1 \geq \dot{V}_1 + y_1^T \rho_1(y_1)$$

- $H_2$: TI, memoryless, $e_2^T y_2 \geq e_2^T \psi_2(e_2)$

(Here, $\rho_1$ and $\psi_2$ can have any sign.)

IF $v^T[\rho_1(v) + \psi_2(v)] > 0$, $^\forall v \neq 0$,

THEN the origin of the CL ($u = 0$) is UAS.

(IF $V$ is also radially unbounded, THEN UGAS.)

*Proof.*

$$\begin{aligned}
\dot{V}_1 &= L_{f_1} V_1(x_1, e_1) \leq e_1^T y_1 - y_1^T \rho_1(y_1) \\
&= -e_2^T y_2 - y_1^T \rho_1(y_1) \leq -[y_1^T \psi_2(y_1) + y_1^T \rho_1(y_1)]
\end{aligned}$$

$$\dot{V}_1 \equiv 0 \quad \Rightarrow \quad y_1 \equiv 0 \quad \Rightarrow \quad e_2 \equiv 0 \quad \Rightarrow \quad e_1 \equiv 0$$

Conclusion that $x_1 \equiv 0$ follows from the invariance theorem.

**Example 6.12** $H_1$: SPR, linear system, $H_2$: TV, passive memoryless function. THEN the origin is UGAS.

This also follows from the 'circle criterion'.

**Example 6.13** $H_1$: $\dot{x} = f(x) + G(x)u$, $y = h(x)$, for which, $\exists$ a PD $V_1$ s.t. $L_f V_1 \leq 0$ and $L_G V_1 = h^T$. Also, suppose that $H_1$ is ZSO.

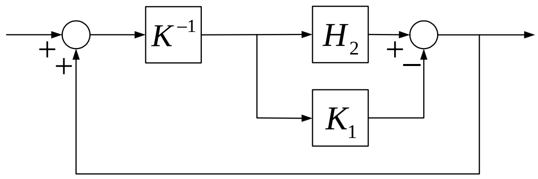
$H_2$: $y_2 = \sigma(e_2)$, $\sigma \in (0, \infty)$.

THEN, by Theorem 6.5, the origin is GAS.

---

**Loop Transformation**

If a static map $H_2 \in [K_1, K_2]$ where $K := K_2 - K_1 = K^T > 0$, then by the following transformation defines a new map $\tilde{H}_2$ and $\tilde{H}_2 \in [0, \infty]$. (Exercise 6.1.)



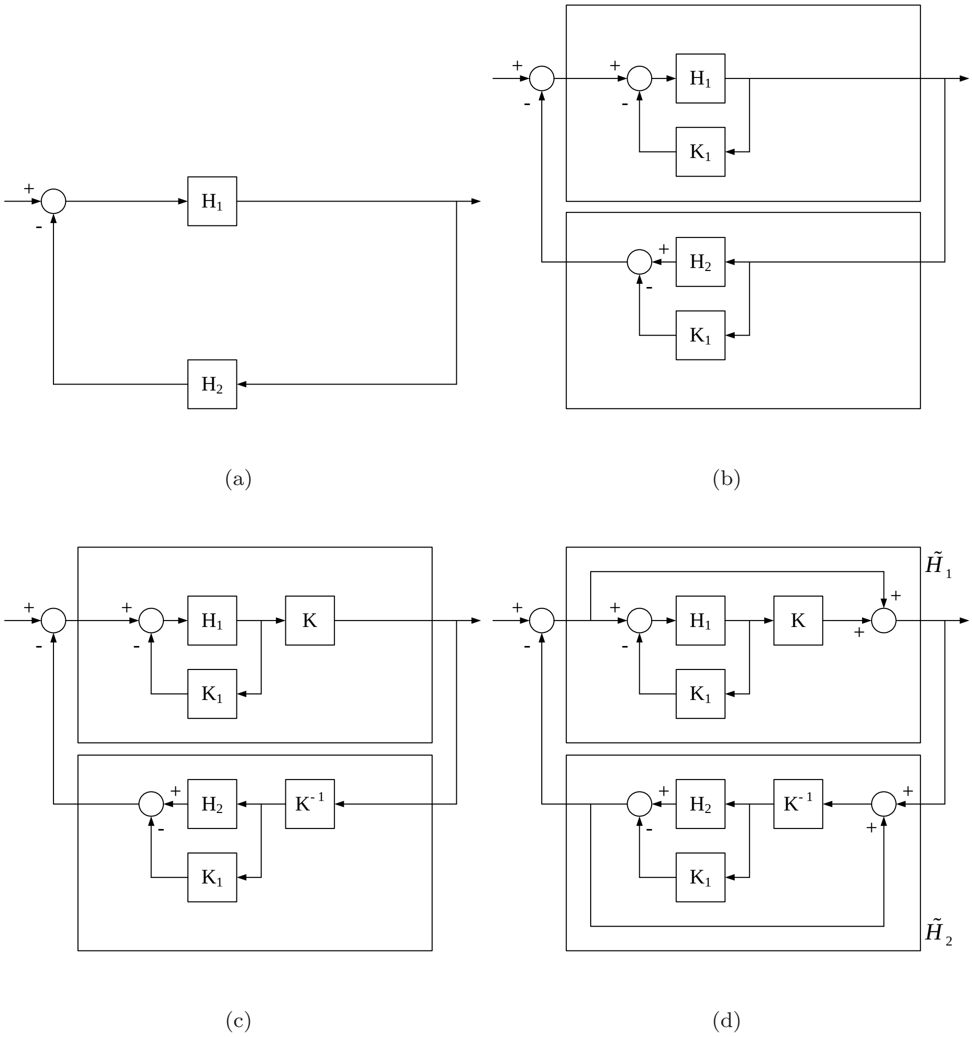




Fig. 1. Figure 6.13

Now, look at Figure 6.13 and discuss the possible utility of the loop transformation.

**Example 6.14** Let $H_1$ be

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -h(x_1) + bx_2 + e_1$$

$$y_1 = x_2$$

where $b > 0$, $h \in [\alpha_1, \infty]$ and $\alpha_1 > 0$, and $H_2$ be

$$y_2 = \sigma(e_2)$$

where $\sigma \in [\alpha, \beta]$ ($K = \beta - \alpha > 0$), which may not even be passive.

By the loop transformation, $H_2$ is converted into

$$\begin{aligned} &\tilde{y}_2 = \sigma\left(K^{-1}(\tilde{e}_2 + \tilde{y}_2)\right) - \alpha(K^{-1}(\tilde{e}_2 + \tilde{y}_2)) \\ \Rightarrow \qquad &\tilde{y}_2 = \tilde{\sigma}(\tilde{e}_2) \end{aligned}$$

which becomes s.t. $\tilde{\sigma} \in [0, \infty]$.

On the other hand, according to Figure 6.13, $H_1$ is converted into

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) + (b - \alpha)x_2 + \tilde{e}_1 \\ \tilde{y}_1 &= Kx_2 + \tilde{e}_1 \end{aligned}$$

which can be shown to be strictly passive by $V_1 = K \int_0^{x_1} h(s)ds + x^T P x$ where $p_{11} = ap_{12}$, $p_{22} = K/2$, and $0 < p_{12} < \min\{2\alpha_1, aK/2\}$ (from Exercise 6.4 which is your homework).

Therefore, the origin of the overall system is GAS (from Theorem 6.4).

---

**Dynamic Weighting(Multiplier)**

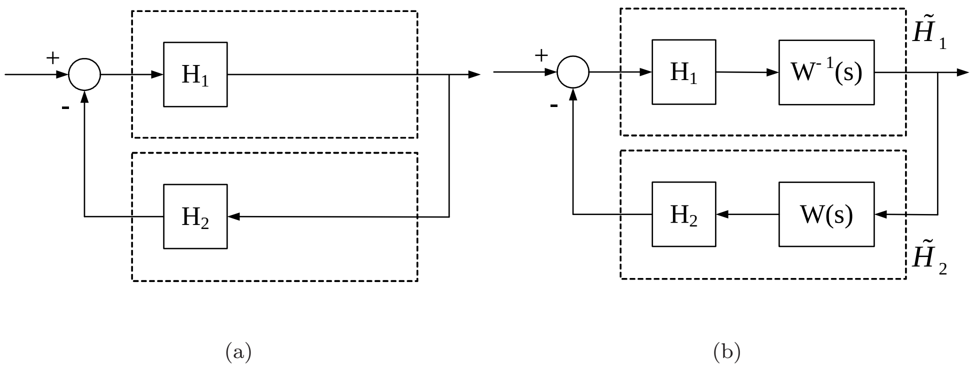



Fig. 2. Figure 6.14

**Example 6.15** Let $H_1$ be

$$\dot{x} = Ax + Be_1, \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$y_1 = Cx$$

that is, $G_1(s) = \frac{1}{s^2+s+1}$. So, let $W^{-1}(s) = as+1$. Then

$$\Rightarrow \quad \frac{as+1}{s^2+s+1} \quad \Rightarrow \quad \mathrm{Re}\left[\frac{1+j\omega a}{1-\omega^2+j\omega}\right] = \frac{1+(a-1)\omega^2}{(1-\omega^2)^2+\omega^2} > 0, \quad {}^\forall\omega.$$

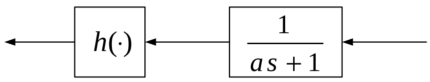
Therefore, it is SPR if $a \geq 1$. Its realization would be

$$\dot{x} = Ax + Be_1$$
$$\tilde{y}_1 = Cx + aCAx =: \tilde{C}x$$

which is strictly passive with $V_1(x) = \frac{1}{2}x^T Px$ where $P > 0$ satisfies that

$$PA + A^T P = -L^T L - \epsilon P, \qquad PB = \tilde{C}^T, \qquad \epsilon > 0.$$

Let $H_2$ be $y_2 = h(e_2)$, $h \in (0, \infty)$. The system is strictly passive with $V_2(x) = a\int_0^{e_2} h(s)ds$.




(Example 6.3).

Now, by Theorem 6.3, the origin is AS with $V(x) = \frac{1}{2}x^T Px + a\int_0^{e_2} h(s)ds$.

However, up to now, we have shown the stability for the third-order system (*with* $W(s)$) while the system of interest was the second-order original system. For the clear presentation, we now try to show the stability of the original system *with* the $V$ so obtained.

$$\begin{aligned}
\dot{V} &= \frac{1}{2}x^T P[Ax - Bh(e_2)] + \frac{1}{2}[...]^T Px + ah(e_2)C[Ax - Bh(e_2)] \\
&= -\frac{1}{2}x^T L^T Lx - \frac{\epsilon}{2}x^T Px - x^T PBh(e_2) + ah(e_2)CAx - ah(e_2)CBh(e_2) \\
&\qquad (\text{where } \tilde{C}^T = PB \text{ and } CB = 0) \\
&= -\frac{1}{2}x^T L^T Lx - \frac{\epsilon}{2}x^T Px - x^T C^T h(e_2) \\
&= -\frac{1}{2}x^T L^T Lx - \frac{\epsilon}{2}x^T Px - e_2^T h(e_2) \leq -\frac{\epsilon}{2}x^T Px.
\end{aligned}$$

This shows that the second-order system is GAS.