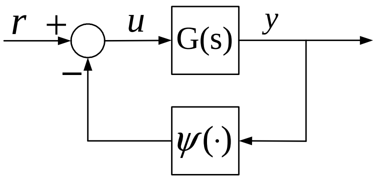
Class Handout: Chapter 7 Frequency Domain Analysis of Feedback Systems

2006 Fall

- Frequency domain analysis of a dynamic system is very useful because it provides much physical insight, has graphical interpretation, and its complexity does not grow much as the system order grows.
- Frequency response functions cannot be defined for nonlinear systems.



Let $r = 0$. We say the system is *absolutely stable* if the closed-loop system is UGAS (ψ may be time-varying) for all ψ satisfying the *given* sector condition.

I. Absolute Stability

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ u &= -\psi(t, y) \end{aligned} \tag{1}$$

where (A, B) controllable, (A, C) observable, $u, y \in \mathbb{R}^p$, and $\psi(\cdot)$ is locally Lipschitz in y.

Suppose that the map $u = -\psi(t, Cx + Du)$ has the unique solution u for all (t, x). (If $D = 0$, then this holds always.) Suppose also that the map ψ satisfies a sector condition on $\mathbb{R}^p$ or a connected subset $Y \subset \mathbb{R}^p$ $(0 \in Y)$. (If the underlying set is not $\mathbb{R}^p$ but Y, then it is absolute stability on a finite domain.)

Problem: Stability under ψ of a given sector (not a particular ψ but all ψ with the sector); **Lure's problem**

Aizerman's conjecture

Let a SISO nonlinearity $\psi \in [k_1, k_2]$ and $G(s) = C(sI - A)^{-1}B + D$. Aizerman made a conjecture: if the matrix $(A - kBC)$ is Hurwitz for all $k_1 \le k \le k_2$, then the closed-loop is absolutely stable. However, it turned out that this conjecture is false, which would be very simple test for the feedback of static nonlinearity.

Circle Criterion

Theorem 7.1 (Multivariable Circle Criterion) The system (1) is absolutely stable if

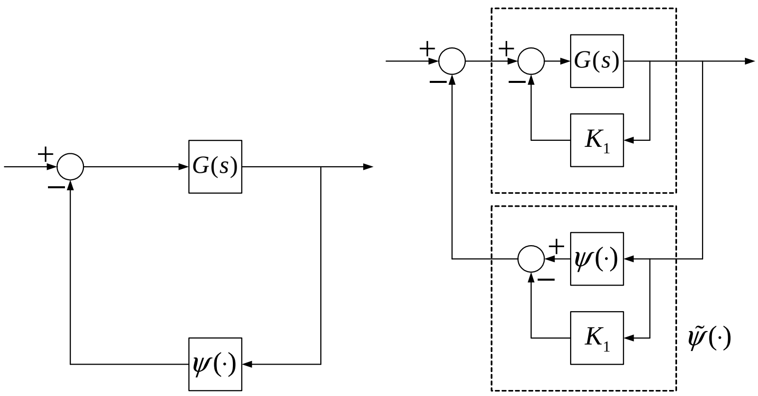
- $\psi \in [K_1, \infty]$ and $G(s)[I + K_1 G(s)]^{-1}$ is SPR, or
- $\psi \in [K_1, K_2]$ with $K := K_2 - K_1 = K^T > 0$, and $[I + K_2 G(s)][I + K_1 G(s)]^{-1}$ is SPR.

Proof. Suppose that $G(s)$ is SPR and $\psi \in [0, \infty]$. Then, $\exists\ P > 0$ s.t. $PA + A^T P = -L^T L - \epsilon P$, $PB = C^T - L^T W$, $W^T W = D + D^T$. Then,

$$
\begin{aligned}
V &= \frac{1}{2} x^T P x \\
\dot{V} &= \frac{1}{2} x^T (PA + A^T P) x + x^T P B u \\
&= -\frac{1}{2} x^T L^T L x - \frac{1}{2} \epsilon x^T P x + x^T (C^T - L^T W) u + u^T D u - u^T D u \\
&= -\frac{1}{2} x^T L^T L x - \frac{1}{2} \epsilon x^T P x + (Cx + Du)^T u - \frac{1}{2} u^T (D + D^T) u - x^T L^T W u \\
&= -\frac{1}{2} \epsilon x^T P x - \frac{1}{2} (Lx + Wu)^T (Lx + Wu) + (Cx + Du)^T u \\
&= -\frac{1}{2} \epsilon x^T P x - \frac{1}{2} (Lx + Wu)^T (Lx + Wu) - y^T \psi(t, y) \\
&\leq -\frac{1}{2} \epsilon x^T P x
\end{aligned}
$$

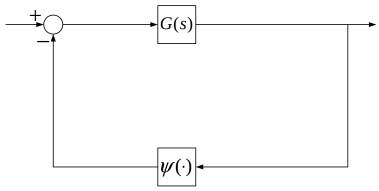
which shows GES.

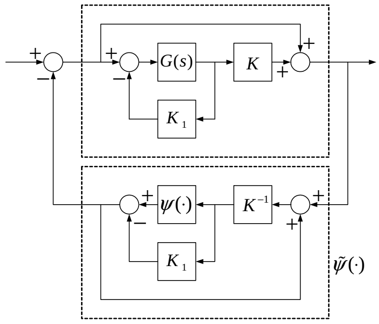
If $\psi \in [K_1, \infty]$, we perform the following loop transformation:



Then, $\tilde{\psi} \in [0, \infty]$ and $\tilde{G}(s) = G(s)[I + K_1 G(s)]^{-1}$. With the above argument, the system is absolutely stable.

If $\psi \in [K_1, K_2]$, we perform the loop transformation of the next page:
Then, $\tilde{\psi} \in [0, \infty]$ and $\tilde{G}(s) = [I + K_2 G(s)][I + K_1 G(s)]^{-1}$. With the above argument, the system is absolutely stable.





Example 7.1 Let $G(s)$ be Hurwitz and strictly proper (i.e., $D = 0$). Let

$$\gamma_1 := \sup_\omega \sigma_{\max}[G(j\omega)] = \sup_\omega \|G(j\omega)\|_2.$$

Suppose that

$$\|\psi(t,y)\|_2 \le \gamma_2 \|y\|_2,$$

that is, $\psi \in [K_1, K_2]$ where $K_1 = -\gamma_2 I$, $K_2 = \gamma_2 I$ (because $(\psi - K_2 y)^T(\psi - K_1 y) \le 0$, so $\|\psi\|^2 - \gamma_2\|y\|^2 \le 0$.)

Now if we show that

$$Z(s) := [I + \gamma_2 G(s)][I - \gamma_2 G(s)]^{-1}$$

is SPR, then the CL is absolutely stable.

Recall Lemma 6.1. First, we know that

$$\det[Z(s) + Z^T(-s)] \not\equiv 0$$

since $Z(\infty) = I$. We now show that

(a) $Z(s)$ is Hurwitz:

If $[I - \gamma_2 G(s)]^{-1}$ is Hurwitz, then $Z(s)$ is Hurwitz (because it is just a cascade of two Hurwitz transfer function). To see this, note that, recalling the Nyquist plot,

$$\sigma_{\min}[I - \gamma_2 G(j\omega)] \ge 1 - \gamma_1\gamma_2.$$

Thus, *if* $\gamma_1\gamma_2 < 1$, then $Z(s)$ is Hurwitz.

(b) $Z(j\omega) + Z^T(-j\omega) > 0$ for all ω:

$$\begin{aligned} Z(j\omega) + Z^T(-j\omega) &= [I + \gamma_2 G][I - \gamma_2 G]^{-1} + [I - \gamma_2 G^*]^{-1}[I + \gamma_2 G^*] \\ &= (I - \gamma_2 G^*)^{-1}(I - \gamma_2 G^*)[I + \gamma_2 G][I - \gamma_2 G]^{-1} \\ &\qquad + [I - \gamma_2 G^*]^{-1}[I + \gamma_2 G^*](I - \gamma_2 G)(I - \gamma_2 G)^{-1} \\ &= 2(I - \gamma_2 G^*)^{-1}[I - \gamma_2^2 G^* G](I - \gamma_2 G)^{-1} \end{aligned}$$

Note that

$\det G \ne 0 \Leftrightarrow \sigma_{\min}[G] > 0$

$\sigma_{\max}[G^{-1}] = 1/\sigma_{\min}[G]$ if $\sigma_{\min}[G] > 0$

$\sigma_{\min}[I + G] \ge 1 - \sigma_{\max}[G]$

$\sigma_{\max}[G_1 G_2] \le \sigma_{\max}[G_1]\sigma_{\max}[G_2]$

So, $Z(j\omega) + Z^T(-j\omega)$ is PD for all ω if and only if

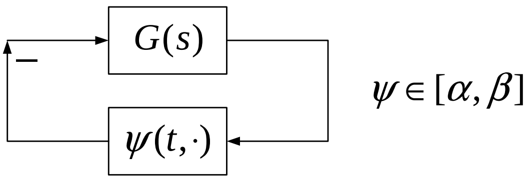
$$\sigma_{\min}[I - \gamma_2^2 G^T(-j\omega)G(j\omega)] > 0, \qquad \forall\omega.$$

Thus, *if* $\gamma_1\gamma_2 < 1$, then this condition holds.
(c) $Z(\infty) + Z^T(\infty) > 0$: Already done.

Hence, the condition $\gamma_1\gamma_2 < 1$ implies that the system is absolutely stable. Compare this with the small gain result of Example 5.13. (This also implies that a Hurwitz transfer function is robust to small feedback perturbation.)

Circle Criterion (SISO case)

We investigate the absolute stability of the system: where $r = 0$.



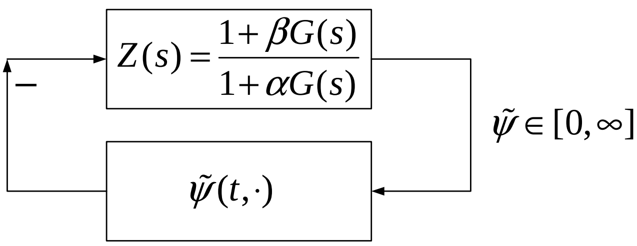
The approach to this question is the following.

The system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$u = -\psi(t, y), \qquad \psi \in [\alpha, \beta], \quad \alpha < \beta \quad \text{and} \quad 0 < \beta$$

where (A, B) controllable, (A, C) observable, is absolutely stable.

$\Updownarrow$ (by loop transformation)



is absolutely stable.

$\Uparrow$ (by Theorem 7.1)

$Z(s)$ is SPR.

$\Updownarrow$ (by Lemma 6.1)

(A) $Z(s)$ is Hurwitz.
(B) Re $\left[\frac{1+\beta G(j\omega)}{1+\alpha G(j\omega)}\right] > 0$.

$\Uparrow$ (by **Theorem 7.2 (Circle Criterion)**)

Disk $D(\alpha, \beta)$:

- Case $0 < \alpha < \beta$: the Nyquist plot of $G(j\omega)$ does not enter the disk $D(\alpha, \beta)$ and encircles it m times in the counterclockwise direction, where m is the number of poles of $G(s)$ with positive real parts.
- Case $0 = \alpha < \beta$: $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ lies to the right of the vertical line $\text{Re}[s] = -1/\beta$.
- Case $\alpha < 0 < \beta$: $G(s)$ is Hurwitz and the Nyquist plot of $G(j\omega)$ likes in the interior of the disk $D(\alpha, \beta)$.

Proof of Theorem 7.2:

Two condition (A) $Z(s)$ is Hurwitz

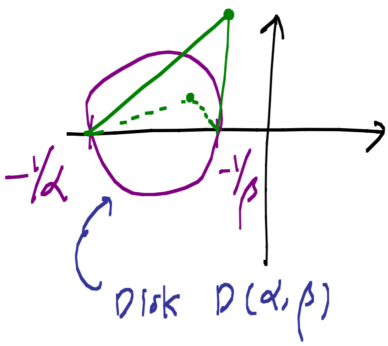
(B) $\mathrm{Re}\left[\dfrac{1+\beta G(j\omega)}{1+\alpha G(j\omega)}\right] > 0 \quad \forall \omega$

Case $0 < \alpha < \beta$:

(B) $\mathrm{Re}\left[\dfrac{1/\beta + G(j\omega)}{1/\alpha + G(j\omega)}\right] > 0 \quad \forall \omega$

$\Leftrightarrow \quad \mathrm{Re}\left[\dfrac{G(j\omega) - (-1/\beta)}{G(j\omega) - (-1/\alpha)}\right] > 0 \quad \forall \omega$

$= M(\omega) e^{j\theta(\omega)}$, then



$\Leftrightarrow \quad -\frac{\pi}{2} < \theta(\omega) < \frac{\pi}{2} \quad \forall \omega$

$\Leftrightarrow$ $G(j\omega)$ resides outside the disk $D(\alpha, \beta)$.

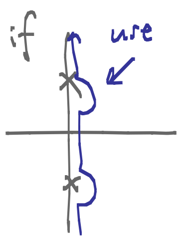
(A) Note: If $G(s) = \dfrac{N(s)}{D(s)}$ ← polynomial,

$$\frac{G}{1+\alpha G} = \frac{\frac{N}{D}}{1+\alpha\frac{N}{D}} = \frac{N}{D+\alpha N}$$

$$Z = \frac{1+\beta G}{1+\alpha G} = \frac{1+\beta N/D}{1+\alpha N/D} = \frac{D+\beta N}{D+\alpha N}$$

$\therefore$ Z is Hurwitz $\Leftrightarrow$ $\dfrac{G}{1+\alpha G}$ is Hurwitz

$\Updownarrow$ by Nyquist THM



$G(j\omega)$ does not intersect $(-1/\alpha) + j\cdot 0$ and encircles it $\underline{m}$ times counter clockwise

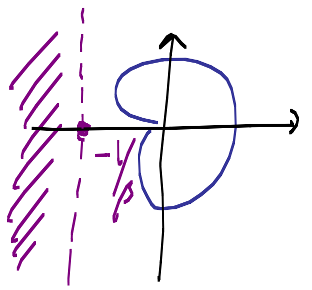
$\parallel$

# of RHP poles of $G(s)$

Case $\alpha = 0 < \beta$: $\quad Z(s) = 1 + \beta G(s)$

(A) $G(s)$ is Hurwitz

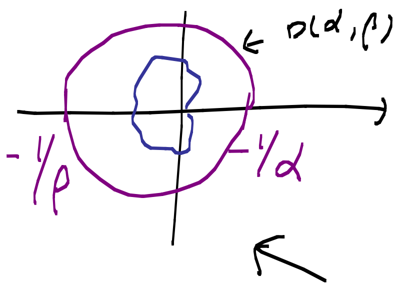
(B) $\mathrm{Re}\,[1 + \beta G(j\omega)] > 0 \iff \mathrm{Re}\,[G(j\omega)] > -1/\beta$



Case $\alpha < 0 < \beta$:

(B) $\mathrm{Re}\left[\dfrac{1/\beta + G(j\omega)}{1/\alpha + G(j\omega)}\right] < 0 \quad \forall \omega$

(∵ $\alpha < 0$)



(A) Note: Nyquist plot cannot encircle $-1/\alpha$ point.

$G(s)$ is Hurwitz $\iff \dfrac{G}{1+\alpha G}$ is Hurwitz

※ If the sector $[\alpha, \beta]$ includes 0, $G(s)$ must be Hurwitz.

※ **Read THM 7.2**.

Example 7.2 Let

$$G(s) = \frac{4}{(s+1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}.$$

For this system, find a sector condition as large as possible under which the closed-loop is GAS.

Example 7.3 Let

$$G(s) = \frac{4}{(s-1)(\frac{1}{2}s+1)(\frac{1}{3}s+1)}.$$

For this system, find a sector condition as large as possible under which the closed-loop is GAS.

Example 7.4 (When the sector condition does not hold globally.) Let

$$G(s) = \frac{s+2}{(s+1)(s-1)}, \qquad \psi(y) = \mathrm{sat}(y).$$

Note that GAS is impossible for this system. (Why?)

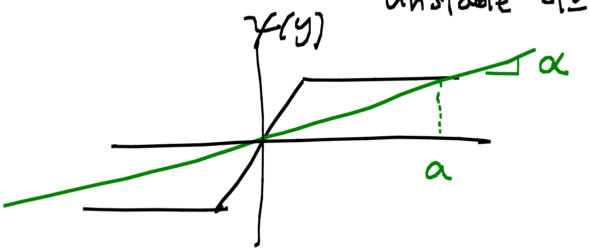
For this system, investigate the stability.

Ex 7.4 Sector condition이 global로 만족되지 않는 경우

※ GAS 불가능.

$$G(s) = \frac{s+2}{(s+1)(s-1)}, \quad \psi(y) = \mathrm{sat}(y)$$

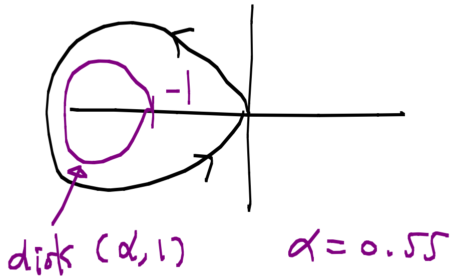
↑ unstable 하므로 $\alpha > 0$ 여야...



interval $[-a, a]$ 에서

$\mathrm{sat} \in [\alpha, 1]$, $\alpha = 1/a$

G(s)의 Nyquist plot



$\mathrm{sat} \in [0.55, 1]$

$\Rightarrow$ interval $[-1.818, 1.818]$ 에서

($|y| < 1.818$ 인 영역의 level set Ω_c가 RoA)

$\therefore$ local result. #

Popov Criterion

Consider the same feedback structure as before, but now we consider

Comparison to Circle Criterion:
- Popov is for TI.
- In Popov, ψ is decoupled.
- Both are sufficient conditions.

$$\dot{x} = Ax + Bu, \qquad u, y \in \mathbb{R}^p,$$
$$y = Cx,$$
$$u_i = -\psi_i(y_i), \qquad 1 \le i \le p,$$

where (A, B) is controllable, (A, C) is observable, and the static nonlinearity ψ is decentralized (decoupled) and TI. Assume that each $\psi_i \in [0, k_i]$ where $0 < k_i \le \infty$.

Theorem 7.3 (Popov Criterion) The CL is absolutely stable if $\exists\ \gamma_i \ge 0$ $(i = 1, \cdots, p)$ s.t.

$$M + (I + s\Gamma)G(s) = \begin{bmatrix} \frac{1}{k_1} & & \\ & \ddots & \\ & & \frac{1}{k_p} \end{bmatrix} + \left(I + s \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_p \end{bmatrix} \right) G(s) \quad \text{is SPR}$$

and $(1 + \lambda_k \gamma_i) \ne 0$ for every e.v. λ_k $(k = 1, \cdots, n)$ of A.

If the sector condition on ψ holds only locally, then the results is also local.

Remarks.

- Here, $G(s)$ should be Hurwitz, which is a necessary condition for absolute stability. (Why?)
- If $G(s)$ is SPR itself, then you can take $\gamma_i = 0$ with which the assumptions hold.
- For SISO case, the Slotine & Li book says: $\exists\ \alpha > 0$ s.t.

$$^\forall\omega \ge 0, \qquad \mathrm{Re}[(1 + j\alpha\omega)G(j\omega)] + \frac{1}{k} \ge \epsilon$$

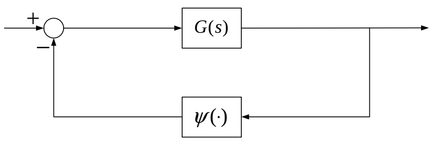
with an $\epsilon > 0$. This is a somewhat restricted condition. Compare and discuss.

- For SISO case, the condition $(1 + \lambda_k \gamma_i) \ne 0$ prevents the pole-zero cancellation between $(1 + s\gamma_i)$ and $G(s)$.

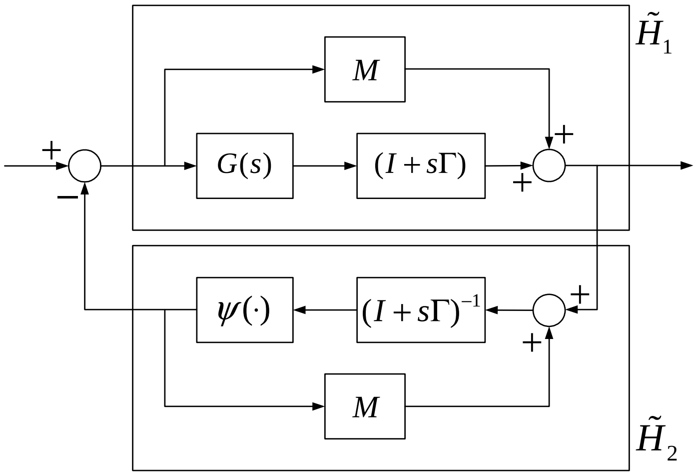
Proof. Proof is done in the state-space. To do so, we need a minimal realization of $\tilde{H}_1$ in the Figure 7.12 (loop transformation).

That is,

$$\begin{aligned} \tilde{H}_1 : \quad M + (I + s\Gamma)G(s) &= M + (I + s\Gamma)C(sI - A)^{-1}B \\ &= M + C(sI - A)^{-1}B + \Gamma C s(sI - A)^{-1}B \\ &= M + C(sI - A)^{-1}B + \Gamma C(sI - A + A)(sI - A)^{-1}B \\ &= M + [C + \Gamma CA](sI - A)^{-1}B + \Gamma CB \end{aligned}$$



(a)



(b)

Figure 7.12

with which, a realization of $\tilde{H}_1$ will be

$$(A, B, C + \Gamma CA, M + \Gamma CB).$$

This realization is minimal because of the following: (A, B) is already assumed to be controllable, and to see the observability, let λ_k $(1 \leq k \leq n)$ be the e.v. of A and v_k be the eigenvector of A. Note that

$$(C + \Gamma CA)v_k = Cv_k + \lambda_k \Gamma Cv_k = (I + \lambda_k \Gamma)Cv_k \neq 0. \qquad \text{(Why?)}$$

Therefore, the pair $(A, C + \Gamma CA)$ is observable.

Assumption. $\Rightarrow$ $\tilde{H}_1$ is SPR. $\Rightarrow \exists P = P^T > 0, L, W, \epsilon > 0$ s.t.

$$PA + A^T P = -L^T L - \epsilon P$$
$$PB = (C + \Gamma CA)^T - L^T W$$
$$W^T W = 2M + \Gamma CB + B^T C^T \Gamma.$$

We now can show that $\tilde{H}_1$ is strict passive with the storage function $\tilde{V}_1 = \frac{1}{2}x^T Px$.

Now, we show that $\tilde{H}_2$ is passive. $\tilde{H}_2$ is written as a parallel connection of

$$\gamma_i \dot{z}_i = -z_i + \frac{1}{k_i}\psi_i(z_i) + u_i, \qquad y_i = \psi_i(z_i).$$

Take $\tilde{V}_2 = \sum_{i=1}^p v_i = \sum_{i=1}^p \gamma_i \int_0^{z_i} \psi_i(\sigma)d\sigma$. Then, the function $v_i = \gamma_i \int_0^{z_i} \psi_i(\sigma)d\sigma$ has the derivative

$$\dot{v}_i = \psi_i(z_i)(-z_i + \frac{1}{k_i}\psi_i + u_i) = \frac{1}{k_i}(\psi_i - k_i z_i)\psi_i + y_i u_i \leq y_i u_i,$$

which shows the passivity of $\tilde{H}_2$.

Up to now, we have a SPR $\tilde{H}_1$ and a passive $\tilde{H}_2$. However, the above argument is done with the virtual additional dynamics. In order to show the absolute stability for the original system, we just try the storage function with a slight modification:

$$V = \frac{1}{2}x^T Px + \sum_{i=1}^{p} \gamma_i \int_0^{y_i} \psi_i(\sigma)d\sigma.$$

Then,

$$\begin{aligned}\dot{V} &= \frac{1}{2}x^T(PA + A^T P)x + x^T PBu + \psi^T(y)\Gamma\dot{y} \qquad (\text{with } \dot{y} = C(Ax + Bu)) \\ &= -\frac{1}{2}x^T L^T Lx - \frac{1}{2}\epsilon x^T Px + x^T(C^T + A^T C^T\Gamma - L^T W)u + \psi^T(y)\Gamma CAx + \psi^T(y)\Gamma CBu\end{aligned}$$

With the fact that $u = -\psi$, we continue the equality to the following (which is a little tedious, so we omit it):

$$\dot{V} = -\frac{1}{2}\epsilon x^T Px - \frac{1}{2}(Lx + Wu)^T(Lx + Wu) - \psi^T(y)[y - M\psi(y)] \le -\frac{1}{2}\epsilon x^T Px.$$

Therefore, the CL is GAS for all ψ in the sector (absolute stability).

SISO case: Popov plot

For SISO case, we test if

$$\tilde{H}_1 : Z(s) = \frac{1}{k} + (1 + s\gamma)G(s)$$

is SPR with some γ s.t. $(1 + \lambda_k\gamma) \neq 0$ for all k.

For this, we apply Lemma 6.1 (since $Z(s) + Z^T(-s) \not\equiv 0$), and the following should hold:

(a) $G(s)$ is Hurwitz so that $Z(s)$ is Hurwitz,

(b) For all ω,

$$\frac{2}{k} + G(j\omega) + G(-j\omega) + j\omega\gamma G(j\omega) - j\omega\gamma G(-j\omega) > 0$$

which is equivalent to

$$\frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega\text{Im}[G(j\omega)] > 0. \tag{2}$$

(c) The above inequality (2) holds with $\omega = \infty$, or

$$\lim_{\omega\to\infty} \omega^2 \left(\frac{1}{k} + \text{Re}[G(j\omega)] - \gamma\omega\text{Im}[G(j\omega)]\right) > 0.$$

Draw the *Popov plot* in the complex plane; that is, draw the graph of

$$(x, y) = (\text{Re}[G(j\omega)], \omega\text{Im}[G(j\omega)])$$

for all $\omega \ge 0$. Note that we just draw the plot for $\omega \ge 0$ because the graph is even with respect to ω. (Why?)

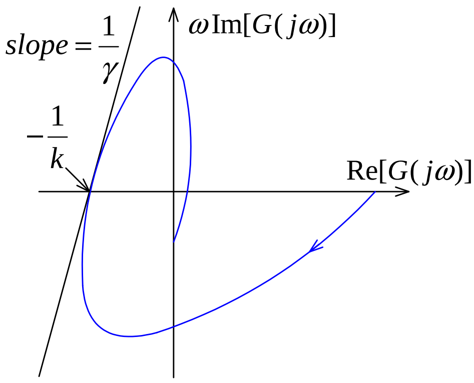


Figure 7.13.

From the figure, we get to know that ($\gamma > 0$)

$$y < \frac{1}{\gamma}\left(x + \frac{1}{k}\right) \qquad \Leftrightarrow \qquad \frac{1}{k} + x - \gamma y > 0$$

which is known as *Popov criterion*.

- See what happens by increasing k.
- Popov criterion gives weaker restriction than the circle criterion for the case of sector $[0, k]$.

(Why?)

Example 7.5 Let $h \in [\alpha, \beta]$ where $0 < \alpha < \beta$ and $G(s)$ be

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + u$$
$$y = x_1,$$

with a feedback connection $u = -h(y)$. For applying the Popov criterion, the system $G(s)$ is not Hurwitz because

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

The idea is to consider $\psi(y) = h(y) - \alpha y$ (to borrow the passivity from $h(y)$ and add more stability into $G(s)$). So, we consider

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\alpha x_1 - x_2 + u$$
$$y = x_1, \qquad u = -\psi(y) = -h(y) + \alpha y$$

where $\psi \in [0, k]$ ($k = \beta - \alpha$) and

$$A = \begin{bmatrix} 0 & 1 \\ -\alpha & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The inequality (2) becomes

$$\frac{1}{k} + \frac{\alpha - \omega^2 + \gamma\omega^2}{(\alpha - \omega)^2 + \omega^2} > 0, \qquad {}^{\forall}\omega \in [0, \infty]$$

if $\gamma > 1$.

Even in the case $k = \infty$, the above inequality holds for $\omega \in [0, \infty)$, and for $\omega = \infty$, we have

$$\lim_{\omega \to \infty} \omega^2 \frac{\alpha - \omega^2 + \gamma\omega^2}{(\alpha - \omega)^2 + \omega^2} = \gamma - 1 > 0.$$

Therefore, the CL is absolutely stable even when $\beta = \infty$.

II. The Describing Function Method

Main Goal: Prediction of limit cycles

This section is adopted mainly from Slotine & Li.

Why are we interested in the existence of limit cycle?

- Existence of unstable limit cycle around the origin prohibits the origin being GAS, or AS with a large domain of attraction.
- Small stable limit cycle leads to poor control accuracy, and also implies wear and tear of control system.

Sometimes, the limit cycle is very useful for our purpose, e.g., designing an oscillator. However, as above, it is an obstacle for the set-point regulation problem. Usually a precise knowledge of the waveform of a limit cycle is not mandatory while the knowledge of their existence with approximate amplitude and frequency is crucial.

For example, consider the Van der Pol equation:

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0, \qquad \alpha > 0$$

which is equivalently described in the following figure.

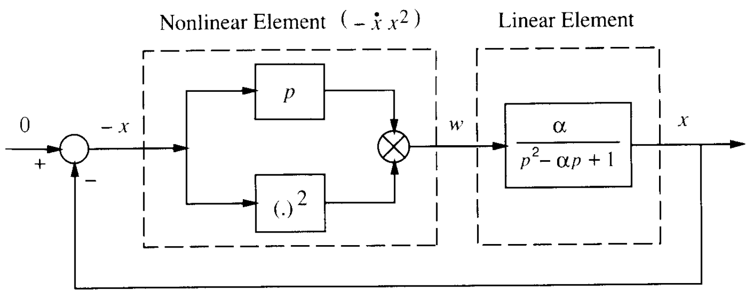


Figure A.

Noting that the linear block acts like a low-pass filter, we assume that

$$x(t) = A\sin(\omega t)$$

that is, in the signal $x(t)$ of Figure A, only the fundamental frequency component exists while the higher order harmonics are attenuated. Under this assumption, if suitable A and ω exist in the feedback-loop, this may emphasize the possible existence of a limit cycle. This is the idea of the describing function approach.

With the above $x(t)$,

$$\dot{x} = A\omega\cos(\omega t)$$
$$w = -x^2\dot{x} = -\frac{A^3\omega}{4}(\cos(\omega t) - \cos(3\omega t))$$

The signal of the frequency 3ω will be attenuated through the LTI block, so we ignore it. Then,

$$w \approx -\frac{A^3}{4}\omega\cos(\omega t) = \frac{A^2}{4}\frac{d}{dt}[-A\sin(\omega t)] = \frac{A^2}{4}\frac{d}{dt}\left(-x(t)\right).$$

This equality leads to the Figure B.

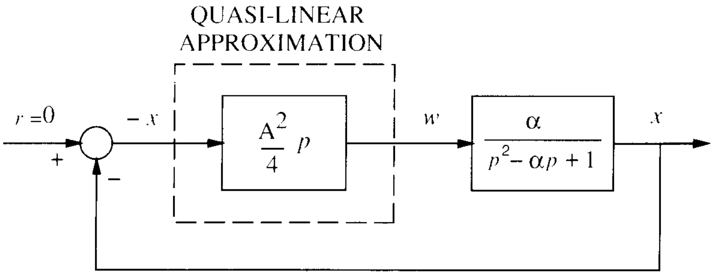


Figure B.

Here, the block from the nonlinearity of the system is called a quasi-linear approximation, which is more realistic if $G(s) = \frac{\alpha}{s^2-\alpha s+1}$ attenuates the higher order sinusoidal signals. We will call

$$N(A,\omega) := \frac{A^2}{4}(j\omega)$$

as a *describing function.* It means that the nonlinear block is approximated by the frequency response function $N(A,\omega)$.

Now, since $x = G(j\omega)w = G(j\omega)N(A,\omega)(-x)$, we have

$$1 + G(j\omega)N(A,\omega) = 0. \qquad (3)$$

From this, we solve

$$1 + \frac{A^2(j\omega)}{4}\frac{\alpha}{(j\omega)^2 - \alpha(j\omega) + 1} = 0$$

and get $A = 2$, $\omega = 1$. This implies that a limit cycle with the amplitude about 2 and the frequency about 1 *may* exist.

In fact, there exists a limit cycle as in Figure C. The limit cycle depends on the value α, and if α is small, it becomes similar to the one expected from the describing function approach.

Definition of Describing Function

Although the nonlinear element is represented by $w = f(x)$ in Figure: Typical Configuration, the block may be a backlash or a hysteresis (which is not a static map, i.e., a single-valued map, but a map with memory). Therefore, we will denote $w(t)$ as the output signal of the nonlinear block.

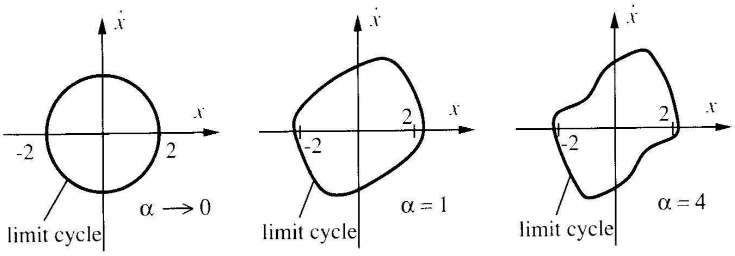


Figure C.

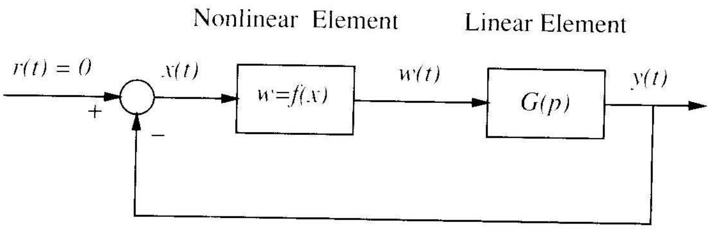


Figure: Typical Configuration

Basic approximation assumption: The oscillation in the loop is a sinusoidal function with only one frequency ω, that is, the $G(s)$ attenuates the higher order harmonics completely.

Assuming $x(t) = A\sin(\omega t)$, the signal $w(t)$ is periodic with the frequency ω. (If the nonlinear block is a single-valued map $w = f(x)$, then $w(t) = f(A\sin(\omega t))$ that is periodic with the frequency ω only.) Then, the Fourier series gives $w(t) = a_0 + a_1\cos(\omega t) + b_1\sin(\omega t) + a_2\cos(2\omega t) + \cdots$ where

Although $w(t)$ is periodic with the frequency ω, its shape determines its higher order harmonics.

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} w(t)d(\omega t) \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\cos(n\omega t)d(\omega t) \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} w(t)\sin(n\omega t)d(\omega t). \end{aligned}$$

Under the basic assumption, we regard that $a_2 = b_2 = a_3 = \cdots = 0$ because they will be rejected after $G(s)$ and thus, they are not necessary in the analysis. (I mean, we just consider $w(t)$ as

$$w(t) = a_0 + a_1\cos(\omega t) + b_1\sin(\omega t)$$

although it is not.)

Assumption: Assume that the nonlinear block is odd.

Even for backlash or hysteresis.

With this assumption, $a_0 = 0$. Then, with the input $x(t) = A\sin(\omega t)$, the output of the nonlinear block is

$$w(t) = a_1\cos(\omega t) + b_1\sin(\omega t) = M\sin(\omega t + \phi)$$

where $M(A,\omega) = \sqrt{a_1^2 + b_1^2}$ and $\phi(A,\omega) = \arctan(a_1/b_1)$. Note that M and ϕ depends on A as well as ω. Then, the nonlinear block is represented by a transfer function

Transfer function for LTI system is independent of A.

$$N(A,\omega) = \frac{1}{A}(b_1(A,\omega) + ja_1(A,\omega))$$

which is defined as the *describing function* of the nonlinear block.

Note: If the nonlinear block is odd and single-valued ($w = f(x) = -f(-x)$), then $a_0 = a_1 = 0$ and $N(A,\omega)$ is real and independent of ω. (Why?)

How to get DF $N(A,\omega)$?

This can be obtained by analytic integration or numerical integration of the Fourier series formula, or by an experimental evaluation. For the practice, refer to Sections 5.2 and 5.3 of the Slotine & Li's book. Also see Examples 7.6, 7.7 and 7.8 in Khalil's book.

From Example 7.8 in Khalil's book, it turns out that if

$$\alpha x^2 \le x f(x) \le \beta x^2,$$

then

$$\alpha \le N(A,\omega) \le \beta.$$

(Check it.)

Existence of Limit Cycle and Graphical Method

The existence of limit cycle is determined by solving A and ω for $1 + G(j\omega)N(A,\omega) = 0$, i.e.,

$$G(j\omega) = -\frac{1}{N(A,\omega)}.$$

If N is independent of ω, then this can be easily solved by a graphical method.

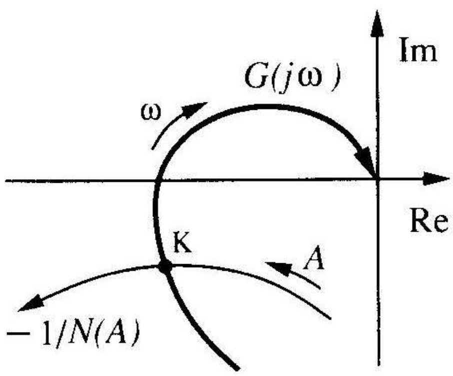


Figure D.

Usually, $N(A)$ is real. Then, the graph of $-1/N(A)$ will be on the real line.

If N depends on ω, then we may plot $N(A,\omega)G(j\omega)$ and find the values of A and ω that leads to the intersection of the plot with the point $(-1,0)$ as in Figure E.

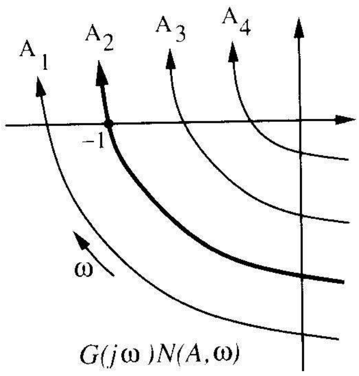


Figure E.

About the stability of a limit cycle, refer to Figure F. Assuming that $G(s)$ does not have an open loop unstable pole, the point L_1 represents the unstable limit cycle because, if the magnitude A is increased, the plot of $G(j\omega)$ encircles the point $-1/N(A)$, which leads to unstability by the Nyquist theory so that the magnitude is ever increasing. Likewise, the point L_2 represents the stable limit cycle.

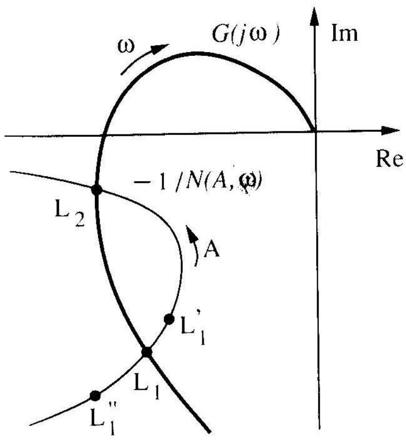


Figure F.

How much can we trust the describing function approach? Rule of thumb says that, if the frequency response of $G(s)$ does not have the resonant peaks, $|G(nj\omega)| << 1$, $n = 2, \cdots$, for the expected frequency ω, and the plot of $G(j\omega)$ and $-1/N(A)$ meets almost perpendicular (see Figure G), then the prediction is quite good.

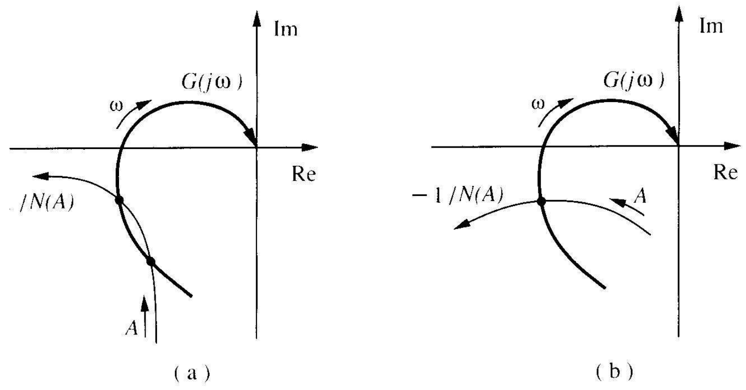
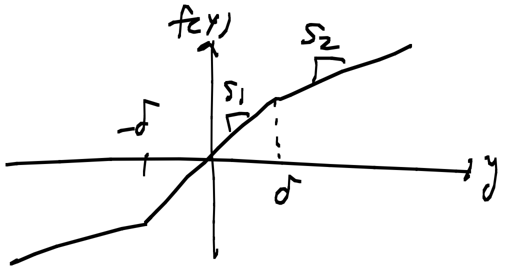


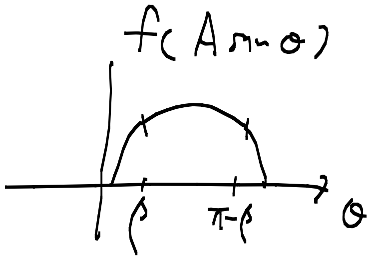
Figure G.

Up to now, we have studied an approximate method of describing function. The approximation mainly came from the fact that we only considered the fundamental frequency of the signal in the loop assuming that $G(s)$ acts like a low-pass filter. If we analyze the error caused by this approximation, we can get a sufficient condition and a necessary condition for the existence of the limit cycle although there exists some gap between the two. Also, there are many extensions in the literature that can handle the TV case, multiple nonlinearity and so on.

Example 7.7

Nonlinear function $f(y)$





f : odd $\rightarrow$ $a_0 = 0,\ a_1 = 0.$

If $A \le \delta$, f is linear $\Rightarrow$ $b_1 = A s_1$

If $A > \delta$,

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(A\sin\theta)\sin\theta\, d\theta \qquad \theta = \omega t$$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(A\sin\theta)\sin\theta\, d\theta$$

$$= \frac{4}{\pi}\int_0^{\beta} s_1 A\sin^2\theta\, d\theta + \frac{4}{\pi}\int_{\beta}^{\frac{\pi}{2}}\left(\delta s_1 + s_2(A\sin\theta - \delta)\right)\sin\theta\, d\theta$$

$$\beta = \sin^{-1}\left(\frac{\delta}{A}\right)$$

$$= A\frac{2}{\pi}s_1\left(\beta - \frac{1}{2}\sin 2\beta\right)$$
$$+ \frac{4\delta(s_1-s_2)}{\pi}\left(\cos\beta - \cos\frac{\pi}{2}\right) + A\frac{2s_2}{\pi}\left(\frac{\pi}{2} - \frac{1}{2}\sin\pi - \beta + \frac{1}{2}\sin 2\beta\right)$$

$$= A\frac{2(s_1-s_2)}{\pi}\left(\beta + \frac{\delta}{A}\cos\beta\right) + As_2$$

$$\therefore N(A,\omega) = \begin{cases} s_1, & 0 \le A \le \delta \\ \dfrac{2(s_1-s_2)}{\pi}\left[\sin^{-1}\left(\dfrac{\delta}{A}\right) + \dfrac{\delta}{A}\sqrt{1-\left(\dfrac{\delta}{A}\right)^2}\right] + s_2, & A > \delta \end{cases}$$

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