Class Note: Averaging Theory

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I. Averaging for periodic systems

Consider a system given by

$$\dot{x} = \epsilon f(t, x, \epsilon) \tag{1}$$

where $f(\cdot, x, \epsilon)$ is T-periodic regardless of ϵ and $f(\cdot, \cdot, \cdot)$ is sufficiently smooth. Also, consider a system

$$\dot{z} = \epsilon f_{av}(z) \tag{2}$$

(which is autonomous), where

$$f_{av}(x) := \frac{1}{T}\int_0^T f(\tau, x, 0)d\tau.$$

The goal of this note is to see how we can approximate the solution of (1) by the solution of (2). Note that, when f_{av} is defined, we set $\epsilon = 0$ inside the function f. In fact, the quantity ϵ inside f is not important for the averaging argument and it is treated just as a small perturbation to the system. What is important is the ϵ outside f of (1).

Example 1. More intuitive motivation might be the following example. Consider a system

$$\dot{x} = g(x) + h(x)(u(t) + c\sin(wt))$$

where $w >> 1$, so that the input u is perturbed by quickly changing periodic perturbation $c\sin(wt)$ that has zero mean. We may want to analyze the effect of the perturbation, and want to conclude that, if the perturbation is fast enough then its effect is negligible. In this case, we can transform this system into the standard form (1) as follows. Let $wt = \tau$. Then,

$$\dot{x} = \frac{dx}{dt} = w\frac{dx}{d\tau} = g(x) + h(x)\left(u\left(\frac{\tau}{w}\right) + c\sin(\tau)\right)$$

which leads to

$$\frac{dx}{d\tau} = \frac{1}{w}[g(x) + h(x)\left(u(\tau/w) + c\sin(\tau)\right)].$$

Defining $\epsilon := 1/w$ and rewrite the equation with t instead of τ, we have

$$\dot{x} = \epsilon(g(x) + h(x)(u(\epsilon t) + c\sin(t)).$$

Therefore, small ϵ implies large w. //

The main claim of averaging analysis is that the solution of (2) is a good approximate of the solution of (1) if ϵ is sufficiently small. To see this, we first define

$$h(t, x) := f(t, x, 0) - f_{av}(x) \tag{3}$$

which is T-periodic and has zero mean. Also, let

$$u(t, x) := \int_0^t h(\tau, x)d\tau \tag{4}$$

which is still T-periodic and has zero mean. (Since the function u has the same property as h, one may wonder why we define the function u. It will turn out that the function u is *a part of coordinate transformation* so that its time derivative (i.e., h) is T-periodic. In this regard, we will rely on the fact that $\frac{\partial u}{\partial t}(t,x) = h(t,x)$ and $\frac{\partial u}{\partial x} = \int_0^t \frac{\partial h}{\partial x}(\tau, x) d\tau$ are T-periodic.)

Let

$$x = y + \epsilon u(t,y).$$

This defines a new coordinates of y although it is defined implicitly (i.e., not defined like $y = ...(x)$). Note that the Jacobian is given by $[I + \epsilon \frac{\partial u}{\partial y}(t,y)]$ which is nonsingular for sufficiently small ϵ on a local region of y, i.e., with bounded $\|y\|$. This implies the above coordinate transformation is locally valid with small ϵ.

Now, we obtain that $\dot{x} = \dot{y} + \epsilon \frac{\partial u}{\partial t} + \epsilon \frac{\partial u}{\partial y} \dot{y}$, which leads to

$$\begin{aligned}\left[I + \epsilon \frac{\partial u}{\partial y}(t,y)\right] \dot{y} = \dot{x} - \epsilon \frac{\partial u}{\partial t} &= \epsilon f(t, y + \epsilon u(t,y), \epsilon) - \epsilon [f(t,y,0) - f_{av}(y)] \\ &= \epsilon f_{av}(y) + \epsilon [f(t, y + \epsilon u(t,y), \epsilon) - f(t,y,0)] = \epsilon f_{av}(y) + \epsilon^2 p(t,y,\epsilon)\end{aligned}$$

in which we have used the fact that there exists a function p such that

$$[f(t, y + \epsilon u(t,y), \epsilon) - f(t,y,0)] = p(t,y,\epsilon)\epsilon.$$

On the other hand, note that

$$\left[I + \epsilon \frac{\partial u}{\partial y}(t,y)\right]^{-1} = I + \mathcal{O}(\epsilon)$$

which follows from the Taylor expansion. Therefore, we finally obtain that

$$\dot{y} = (I + \mathcal{O}(\epsilon))(\epsilon f_{av}(y) + \epsilon^2 p(t,y,\epsilon)) = \epsilon f_{av}(y) + \epsilon^2 q(t,y,\epsilon),$$

where q is suitably defined with the properties that q, $\frac{\partial q}{\partial t}$, $\frac{\partial q}{\partial y}$, $\frac{\partial q}{\partial \epsilon}$ are uniformly locally bounded.

Let $s = \epsilon t$. Then,

$$\frac{dy}{ds} = f_{av}(y) + \epsilon q(\frac{s}{\epsilon}, y, \epsilon)$$

where q is now ϵT-periodic. We compare the solution $y(s)$ with the solution $z(s)$ of (2), that is,

$$\frac{dz}{ds} = f_{av}(z).$$

Indeed, if $z(s)$ is contained in a compact set $\mathcal{D}$ for all $s \in [0,b]$ and if $z(0) - y(0) = \mathcal{O}(\epsilon)$, then (by Theorem 3.5 of Khalil's Third Edition) $z(s) - y(s) = \mathcal{O}(\epsilon)$ for all $s \in [0,b]$. This implies that, (noting also that $z(\epsilon t)$ is the solution of (2))

$$\begin{aligned}x(\epsilon t) &= y(\epsilon t) + \epsilon u(\epsilon t, y(\epsilon t)) \\ &= z(\epsilon t) + \mathcal{O}(\epsilon) + \epsilon u(\epsilon t, z(\epsilon t) + \mathcal{O}(\epsilon)) \\ &= z(\epsilon t) + \mathcal{O}(\epsilon)\end{aligned}$$

for all $t \in [0, b/\epsilon]$. Also, by Theorem 9.1 of Khalil's Third Edition, this approximation holds for all $t \in [0, \infty)$ if, in addition, the averaged system (2) is locally exponentially stable at the origin.

II. General Averaging

Now we deal with systems (1) that is not periodic, and assert that the solution of (1) can still be approximated by the solution of (2) where f_{av} is defined differently. This time we suppose that the system (1) has an average system (2) in the sense that the limit

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T}\int_t^{t+T} f(\tau, x, 0)d\tau \tag{5}$$

exists. Note that the left hand side is independent of t. Furthermore, to be more specific on the convergence of the equation (5), we assume that there exist a constant $k > 0$ and a strictly decreasing continuous function $\sigma : [0,\infty) \to [0,\infty)$ such that $\sigma(\infty) = 0$ and that

$$\left\| \frac{1}{T}\int_t^{t+T} f(\tau, x, 0)d\tau - f_{av}(x)\right\| \le k\sigma(T) \tag{6}$$

for all t and x of interest. The function σ is called the *convergence function.*

Example 2. Let $f(t, x, \epsilon) = \frac{1}{1+t}h(x)$. Then $f_{av}(x) = 0$, and

$$\begin{aligned}
\left\| \frac{1}{T}\int_t^{t+T} f(\tau, x, 0)d\tau - f_{av}(x)\right\| &= \left\| \frac{1}{T}\int_t^{t+T} \frac{1}{1+\tau}h(x)d\tau \right\| \\
&= \left\| \frac{h(x)}{T}\ln\frac{1+t+T}{1+t}\right\| \\
&= \left\| \frac{h(x)}{T}\ln\left(1+\frac{T}{1+t}\right)\right\| \\
&\le \frac{\|h(x)\|}{T}\ln(1+T) \le k\frac{\ln(1+T)}{T}
\end{aligned}$$

where k is a local bound of $\|h(x)\|$ on the region of interest. Therefore, the convergence function is given by $\sigma(T) = \ln(1 + T)/T$.

Example 3. When the function $f(t, x, \epsilon)$ is periodic with a period T_p, its convergence function is given by $\sigma(T) = 1/(T + 1)$.

The rest of argument for 'general averaging' is similar to the 'periodic averaging', but there is an important difference. We seek again for a coordinate change so that the system (1) can be seen as a perturbed system of (2). So, we define $h(t, x)$ just as (3), but instead of $u(t, x)$ defined in (4) we will use the function $w(t, x, \eta)$ that is defined as

$$w(t, x, \eta) := \int_0^t h(\tau, x)\exp[-\eta(t-\tau)]d\tau. \tag{7}$$

Here, the exponential term is added compared to (4).Note just that, when $\eta = 0$, the function $w(t, x, 0)$ is the same as $u(t, x)$ of (4).

We omit the details.