"Singular Perturbation Methods in Control: Analysis and Design" Kokotovic, Khalil, O'Reilly. 1986. Academic Press.

Chap. 11. Singular Perturbation

Ex: $\begin{cases} \dot{x} = z & x(0) = x_0 \\ \dot{z} = -1000z - x, & z(0) = z_0 \end{cases}$

$z(t) \longrightarrow -\frac{1}{1000}x(t)$, $\quad \dot{x} = -\frac{1}{1000}x$, $\quad x(t) \to 0$
$z(t) \to 0$.

Standard model:

$$\begin{cases} \dot{x} = f(t, x, z, \varepsilon) & x \in R^n \\ \varepsilon\dot{z} = g(t, x, z, \varepsilon) & z \in R^m \end{cases}$$

Ass: If we let $\varepsilon = 0$, $\Rightarrow$ $0 = g(t, x, z, 0)$
$\exists$ <u>an</u> (isolated) sol. $z = h(t, x)$

Ass: $\varepsilon\dot{z} = g(t, x, z, \varepsilon)$ is stable for each fixed (t, x, ε).

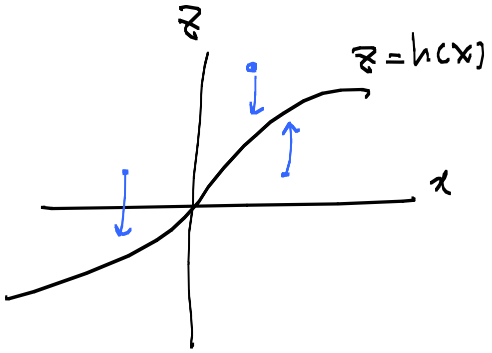
Approximation: $\begin{cases} \dot{x} = f(t, x, h(t,x), 0) \\ z = h(t, x) \end{cases}$

"reduced model", "quasi-steady-state model"
"slow model"

In fact, $\frac{dz(t)}{dt} = \frac{1}{\varepsilon} g(\underbrace{t, x(t), z(t)}_{\text{almost constant}}, \varepsilon)$

$\Rightarrow z(t) \to h(t, x(t))$

TI case :



Difference between original and approximation = fast transient

Ex 11.1.
$$J \frac{d\omega}{dt} = k i$$
$$L \frac{di}{dt} = -k\omega - Ri + u \, .$$

L: small

$R > 0$

If we let $L = 0$, $i = \dfrac{u - k\omega}{R}$

$$\Rightarrow J \dot{\omega} = -\frac{k^2}{R}\omega + \frac{k}{R}u$$; reduced model

※ $\dfrac{di}{dt} = -\dfrac{R}{L}i - \dfrac{k}{L}\omega + \dfrac{1}{L}u$; fast (stable) response

Ex 11.2 Fast actuator case.

$\boxed{11.2}$ Time-scale Properties of the Standard Model

$$\begin{cases} \dot{x} = f(t, x, z, \varepsilon), & x(t_0) = \xi(\varepsilon), \\ \varepsilon \dot{z} = g(t, x, z, \varepsilon), & z(t_0) = \eta(\varepsilon). \end{cases}$$

Ass: $\quad Re\left[\lambda\left(\frac{\partial g}{\partial z}(t, x, h(t,x), 0)\right)\right] \leq -c < 0$

$\forall (t, x) \in [0, t_1] \times D_x$

sol : $x(t,\varepsilon)$, $z(t,\varepsilon)$

$$\begin{cases} \dot{x} = f(t, x, h(t,x), 0), & x(t_0) = \xi_0 = \xi(0) \\ \text{sol: } \bar{x}(t), & \bar{z}(t) = h(t, \bar{x}(t)) \end{cases}$$

$(\bar{z}(t_0) \neq \eta(\varepsilon))$

THM 11.1. (Tikhonov's THM)

$z(t,\varepsilon) - h(t, \bar{x}(t)) = O(\varepsilon)$, $\forall t \in [t_b, t_1]$, $t_b > t_0$

$x(t, \varepsilon) - \bar{x}(t) = O(\varepsilon)$, $\forall t \in [0, t_1]$

and

$(z(t,\varepsilon) - \bar{z}(t)) \to 0$ during $[t_0, t_b]$

Or, $z(t,\varepsilon) - h(t,\bar{x}(t)) - \hat{y}(t/\varepsilon) = O(\varepsilon)$, $\forall t \in [t_0, t_1]$.

\#

Let $y \triangleq z - h(t, x)$.

$\dot{x} = f(t, x, y + h(t,x), \varepsilon)$, $\quad x(t_0) = \xi(\varepsilon)$

$\varepsilon \dot{y} = g(t, x, y + h(t,x), \varepsilon) - \varepsilon \frac{\partial h}{\partial t} - \varepsilon \frac{\partial h}{\partial x} f(t, x, y + h(t,x), \varepsilon)$

$y(t_0) = \eta(\varepsilon) - h(t_0, \xi(\varepsilon))$

$$\tau \triangleq \frac{t - t_0}{\varepsilon}, \qquad t = t_0 + \varepsilon\tau$$

$$\therefore \quad \varepsilon \frac{dy}{dt} = \frac{dy}{d\tau}$$

when $\varepsilon = 0$,

$$\frac{dy}{d\tau} = g(t_0, \xi_0, y + h(t_0, \xi_0), 0)$$

$$y(0) = z(0) - h(t_0, \xi_0) = \eta_0 - h(t_0, \xi_0)$$

$$\Rightarrow \frac{dy}{d\tau} = g(t, x, y + h(t, x), 0) \; ; \text{ "boundary-layer model"}$$

Ex 11.5. (Tikhonov 예)

$$\dot{x} = z, \quad x(0) = \xi_0$$

$$\varepsilon \dot{z} = -x - z + t, \quad z(0) = \eta_0, \qquad h(t, x) = t - x$$

$$y = z - \underbrace{h(t, x)}_{\text{상수로 보면 boundary layer model 유도.}}$$

$$\frac{dy}{d\tau} = \varepsilon \dot{y} = \varepsilon \dot{z} = -x - z + t = -y$$

$$y(0) = z(0) + x(0) = \eta_0 + \xi_0$$

↑ GES

$$\Rightarrow \hat{y}(\tau) = (\eta_0 + \xi_0)\, e^{-\tau}$$

Reduced sys.

$$\dot{x} = -x + t, \quad x(0) = \xi_0$$

$$\bar{x}(t) = e^{-t} x(0) + \int_0^t e^{-(t-\tau)} \tau \, d\tau$$

$$= e^{-t}\xi_0 + e^{-t}\Big\{ \underbrace{e^{\tau}\tau \Big|_0^t - \int_0^t e^{\tau} d\tau}_{= e^t t - e^t + 1} \Big\}$$

$$= e^{-t}\xi_0 + t - 1 + e^{-t}$$

$$= t - 1 + (1 + \xi_0)\, e^{-t}$$

$\Rightarrow$ By Tikhonov's THM,

$$x(t) - [t - 1 + (1+\zeta_0)e^{-t}] = O(\varepsilon), \quad \forall t \in [0, T]$$

$$z(t) - [\underbrace{(\eta_0 + \zeta_0)e^{-\frac{t}{\varepsilon}}}_{\hat{y}} + \underbrace{1 - (1+\zeta_0)e^{-t}}_{h(t,x) = t - x}] = O(\varepsilon)$$

※ $A = \begin{bmatrix} 0 & 1 \\ -1/\varepsilon & -1/\varepsilon \end{bmatrix}$. e.v. $= \begin{cases} -1 + O(\varepsilon) \\ -\frac{1}{\varepsilon} + 1 + O(\varepsilon) \end{cases}$

$$\frac{-1 \pm \sqrt{1 - 4\varepsilon}}{2\varepsilon}, \quad \sqrt{1-4\varepsilon} = 1 + \frac{-4}{2\sqrt{1-4\varepsilon}}\Bigg|_{\varepsilon=0} \cdot \varepsilon + \cdots$$

$$= 1 - 2\varepsilon + \cdots$$

$$\begin{cases} \dfrac{-2 + 2\varepsilon + \cdots}{2\varepsilon} = -\dfrac{1}{\varepsilon} + 1 + O(\varepsilon) \\[2ex] \dfrac{-1 + 1 - 2\varepsilon + \cdots}{2\varepsilon} = -1 + O(\varepsilon) \end{cases}$$

#

11.4 Manifold Interpretation

TI, f, g are smooth, asymptotic stability of fast dynamics

$$\dot{x} = f(x,z) \qquad x \in R^n$$
$$\varepsilon \dot{z} = g(x,z) \qquad z \in R^m$$

$\underline{z = h(x)}$ s.t. $g(x, h(x)) = 0$

① n-dim. manifold

② invariant for the system with $\varepsilon = 0$

(may not be invariant with $\varepsilon \neq 0$)

And, the motion in this invariant set is described by

$$\dot{x} = f(x, h(x))$$

Is there an invariant manifold nearby $z = h(x)$ when $\varepsilon \neq 0$ but small?

Ex 11.10, $\dot{x} = -x + z$

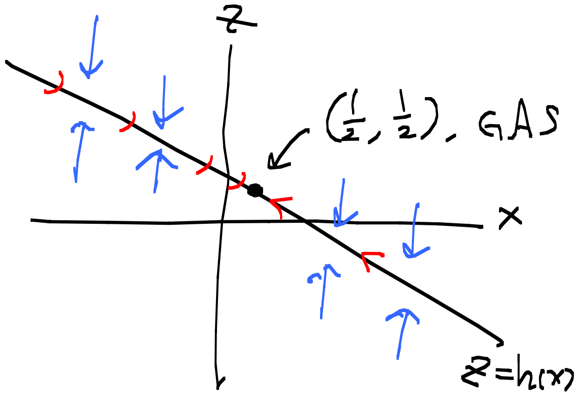
$\varepsilon \dot{z} = \tan^{-1}(1 - z - x)$

$\varepsilon = 0 \Rightarrow z = h(x) = 1 - x$

slow model : $\dot{x} = -x + (1-x)$

$= -2x + 1$

GAS at $x = \frac{1}{2}$



Let $\tau = \frac{t}{\varepsilon}$.

$$\frac{dz}{d\tau} = \tan^{-1}(1 - z - x(\varepsilon\tau)), \quad \varepsilon = 0 \Rightarrow \frac{dz}{d\tau} = \tan^{-1}(1 - z - x(0))$$

GAS at $z = 1 - x(0)$

Compare the true figure (Fig. 11.8) with $\varepsilon = 0.1$.

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* Exact invariant manifold

$z = H(x, \varepsilon)$; exact slow manifold

$$\dot z = \frac{1}{\varepsilon} g(x, H(x,\varepsilon)) = \frac{\partial H}{\partial x}(x,\varepsilon) f(x, H(x,\varepsilon)) + \underbrace{\frac{\partial H}{\partial \varepsilon}\cdot\dot\varepsilon}_{0}$$

$$\Rightarrow \quad g(x, H(x,\varepsilon)) - \varepsilon \frac{\partial H}{\partial x}(x,\varepsilon) f(x, H(x,\varepsilon))$$; invariant manifold condition

(for $\varepsilon = 0$, the solution of Φ is $h(x)$)

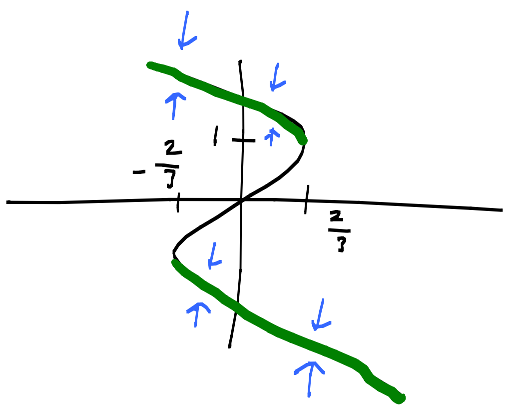
* For bounded x, $\exists\ \varepsilon^* > 0$ & a sol. $H(\cdot,\cdot)$ s.t

$$H(x,\varepsilon) - h(x) = O(\varepsilon) \quad \forall \varepsilon \in [0, \varepsilon^*].$$

(Proof is omitted.)

Ex 11.11

$$\dot x = z$$
$$\varepsilon \dot z = -x + z - \frac{1}{3} z^3$$
$$\Rightarrow \quad x = -\frac{1}{3} z^3 + z$$



11.5 Stability Analysis

$$\begin{cases} \dot{x} = f(x, z) \\ \varepsilon \dot{z} = g(x, z), \end{cases} \qquad f(0) = g(0) = 0, \quad z = h(x), \quad h(0) = 0$$

$$y \triangleq z - h(x)$$

$$\Rightarrow \quad \dot{x} = f(x, y + h(x))$$
$$\varepsilon \dot{y} = g(x, y + h(x)) - \varepsilon \frac{\partial h}{\partial x} f(x, y + h(x))$$

($H(x, \varepsilon)$는 쓴다면, $y = 0$ 일때 $\dot{y} = 0$ 이겠으나 그렇게 하지 않았다.)

Ass: $\begin{cases} \dot{x} = f(x, h(x)) & ; \text{AS} \\ \dfrac{dy}{d\tau} = g(x, y + h(x)) & ; \text{UAS for fixed } x \end{cases}$

Is the overall system AS ?

① $\exists$ p.d $V(x)$, $W(x, y)$ s.t $W_1(y) \le W(x, y) \le W_2(y)$

② $\dfrac{\partial V}{\partial x} f(x, h(x)) \le -\alpha_1 \psi_1^2(x)$, $\quad \psi_1, \psi_2$: p.d. functions

(ex) $\psi_1(x) = \|x\|$
$\psi_2(y) = \|y\|$

③ $\dfrac{\partial W}{\partial y} g(x, y + h(x)) \le -\alpha_2 \psi_2^2(y)$

$$\Rightarrow \quad \nu(x, y) = (1 - d) V(x) + d\, W(x, y), \quad (0 < d < 1)$$

$$\dot{\nu} = (1 - d) \frac{\partial V}{\partial x} f(x, y + h(x)) + \frac{d}{\varepsilon} \frac{\partial W}{\partial y} g(x, y + h)$$
$$- d \frac{\partial W}{\partial y} \frac{\partial h}{\partial x} f(x, y + h) + d \frac{\partial W}{\partial x} f(x, y + h)$$

$\rightarrow$

$$= (1-d)\frac{\partial W}{\partial x} f(x,h) + \frac{d}{\varepsilon}\frac{\partial W}{\partial y} g(x, y+h)$$

$$+ (1-d)\frac{\partial V}{\partial x}\left[f(x, y+h) - f(x,h)\right]$$

$$+ d\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h}{\partial x}\right] f(x, y+h)$$

Ans:

④ $\frac{\partial V}{\partial x}\left(f(x, y+h) - f(x,h)\right) \leq \beta_1 \psi_1(x)\psi_2(y)$; Interconnection condition

⑤ $\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h}{\partial x}\right] f(x, y+h(x)) \leq \beta_2 \psi_1(x)\psi_2(y) + \gamma\psi_2^2(y)$

$$\leq -(1-d)\alpha_1\psi_1^2 - \frac{d}{\varepsilon}\alpha_2\psi_2^2$$

$$+ (1-d)\beta_1\psi_1\psi_2 + d\beta_2\psi_1\psi_2 + d\gamma\psi_2^2$$

$$= -\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix}^T \begin{bmatrix}(1-d)\alpha_1 & -\frac{1}{2}(1-d)\beta_1 - \frac{1}{2}d\beta_2 \\ -\frac{1}{2}(1-d)\beta_1 - \frac{1}{2}d\beta_2 & d\left(\frac{\alpha_2}{\varepsilon} - \gamma\right)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$d(1-d)\alpha_1\left(\frac{\alpha_2}{\varepsilon} - \gamma\right) > \frac{1}{4}\left[(1-d)\beta_2 + d\beta_2\right]^2$$

$$\Leftrightarrow \quad \varepsilon < \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \frac{1}{4d(1-d)}\left((1-d)\beta_1 + d\beta_2\right)^2} \leqq \varepsilon_d$$

maximize for d

$$= \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \beta_1\beta_2}$$

$\Rightarrow$ $\boxed{\text{THM } 11.3}$

* Sufficient condition for ④, ⑤

$$\left\|\frac{\partial v}{\partial x}\right\| \le k_1 \psi_1(x), \qquad \| f(x, h(x)) \| \le k_2 \psi_1(x)$$

$$\| f(x, y+h(x)) - f(x, h(x)) \| \le k_3 \psi_2(y)$$

$$\left\|\frac{\partial w}{\partial y}\right\| \le k_4 \psi_2(y), \qquad \underline{\left\|\frac{\partial w}{\partial x}\right\| \le k_5 \psi_2(y)}$$

$$y \le k \psi_2^2(y) \ \text{"}$$

Note: In ⑤,

$$\Big[\underbrace{\frac{\partial w}{\partial x}}_{\le \psi_2} - \underbrace{\frac{\partial w}{\partial y}}_{\le \psi_2} \underbrace{\frac{\partial h}{\partial x}}_{\text{const.}}\Big] \underbrace{f(x, y+h(x))}_{[\underbrace{f(x,y+h)-f(x,h)}_{\le \psi_2} + \underbrace{f(x,h)}_{\le \psi_1}]}$$

So, ⑤ holds under the sufficient condition.

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Ex 11.12 DIY.

Ex 11.14

$$\begin{cases} \dot{x} = f(t, x, v) \\ \varepsilon \dot{z} = Az + Bu \\ v = Cz \end{cases} \rightarrow \dot{z} = \frac{A}{\varepsilon} z + \frac{B}{\varepsilon} u = \bar{A} z + \bar{B} u.$$

fast actuator

Ass: ① $-C\bar{A}^{-1}\bar{B} = I$

② $\det(sI - \bar{A}) = 0 \Rightarrow \det(\varepsilon s I - A) = 0$

$\bar{A}$: Hurwitz, $s \to -\infty$ as $\varepsilon \to 0$.

$$\dot{x} = f(t, x, v), \qquad v = \gamma(t, x)$$

$$\rightarrow \begin{cases} \dot{x} = f(t, x, v) \\ \dot{z} = \bar{A} z + \bar{B} u, \qquad u = \gamma(t, x) \\ v = Cz \end{cases}$$

$$\text{or } \begin{cases} \varepsilon \dot{z} = Az + Bu \\ v = Cz \end{cases} \Rightarrow \begin{array}{l} z = h(x, u) = -A^{-1}Bu \\ v = -CA^{-1}Bu = -u \end{array}$$

$\therefore$ reduced sys : $\dot{x} = f(t, x, \gamma(t, x))$

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