Chp. 9

9.5 의 식

$$\ell = \int_a^b |\vec{r}'| \, dt$$

$$\sqrt{\vec{r}' \cdot \vec{r}'} \qquad \vec{r}' = \frac{d\vec{r}}{dt}$$

$$S(t) = \int_a^t \sqrt{\vec{r}' \cdot \vec{r}'} \, d\hat{t}$$

$$\frac{dS(t)}{dt} = |\vec{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$= d\vec{r} \cdot d\vec{r} = |d\vec{r}|^2$$

$$\therefore |r'(s)| = 1$$

$$\vec{u}(s) = \frac{\vec{r}'(s)}{|\vec{r}'(s)|} = \vec{r}'(s)$$

(unit tangent vector.

Note) tangent $\vec{q}(w) = \vec{r} + w\vec{r}'$

Velocity. Acceleration.

$$\vec{v}(t) = \vec{r}'(t)$$

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$$

$$= \vec{a}_{tan} + \vec{a}_{norm}$$

$\vec{a}_{tan}$: tangential acceleration

$\vec{a}_{norm}$: normal acceleration

from t to s

$$v(t) = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \vec{u}(s)\cdot\frac{ds}{dt}$$

$\vec{u}(s)$: unit tangent vector

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\vec{u}(s)\cdot\frac{ds}{dt}\right) = \frac{d\vec{u}}{ds}\left(\frac{ds}{dt}\right)^2 + \vec{u}(s)\cdot\frac{d^2s}{dt^2}$$

$\frac{d\vec{u}}{ds}$: normal, $\vec{u}(s)$: tangential

$$= \vec{a}_{norm} + \vec{a}_{tan}$$

projection of $\vec{a}$ in the direction of $\vec{v}$

$$\frac{\vec{a}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\times\vec{v}$$

$\Rightarrow$

$$|\vec{a}_{tan}| = \frac{a\cdot v}{|v|}$$

Curvature & Torsion

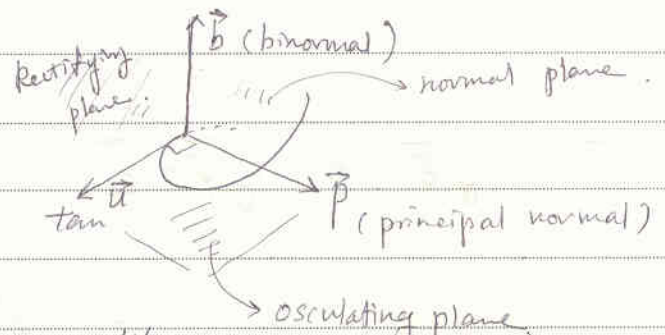
curvature $K(s) = |u'(s)| = |\gamma''(s)|$

torsion $\tau(s)$ of curve C at P

비틀기

osculate : 접촉하다

rectify : (곡선의 길이를 구하다



$\vec{p} = \frac{u'}{\kappa}$ (unit principal normal)

$\vec{b} = \vec{u} \times \frac{\vec{u}'}{\kappa}$ (unit binormal)

$|\tau(s)| = |\vec{b}'(s)|$ ∴ the rate of change of osculating plane (O)

접촉면

deviation of C at P from a plane

$\tau(s) = (-)p(s) \cdot b'(s)$

$(\because \vec{b}' = -\tau\vec{p})$

right-handed helix positive.

cross product 의 geometry

$$V = \frac{1}{6}(\vec{a} \times \vec{b}) \cdot \vec{c}$$

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

$$\vec{B} = \frac{\vec{v}_1}{c} \times \frac{\vec{E}}{c}$$

9.7 gradient of a scalar field

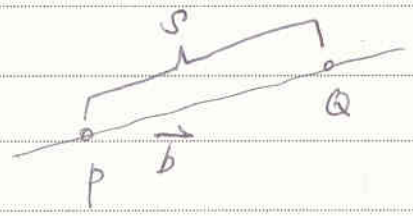
$$\nabla f = \text{grad} f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right]$$

$$= \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Directional derivative

$$D_{\underline{b}} f = \frac{df}{ds} = \lim_{s \to 0} \frac{f(Q) - f(P)}{s}$$



$$D_{\underline{b}} f = \frac{df}{ds} = \underline{b} \cdot \nabla f$$

$$= \frac{\underline{a}}{|\underline{a}|} \cdot \nabla f$$

$$= \frac{\partial f}{\partial x} \cdot x' + \frac{\partial f}{\partial y} \cdot y' + \frac{\partial f}{\partial z} \cdot z'$$

$$= \frac{\partial f}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial f}{\partial y} \cdot \frac{dy}{ds} + \frac{\partial f}{\partial z} \cdot \frac{dz}{ds}$$

MOOKEUK

Ex1. $f = 2x^2 + 3y^2 + z^2$ at $P(2, 1, 3)$

in the direction of $\underline{a} = [1, 0, -2]$

$\nabla f = [4x, 6y, 2z]$

$D_{\underline{a}} f(P) = \frac{1}{\sqrt{5}} [1, 0, -2] \cdot [8, 6, 6]$

$= -\frac{4}{\sqrt{5}}$

✓ Gradient is a vector : Max increase of f at P

$D_{\underline{b}} f = |\underline{b}| \cdot |\nabla f| \cdot \cos\gamma$

when $\gamma = 0$, max increase.

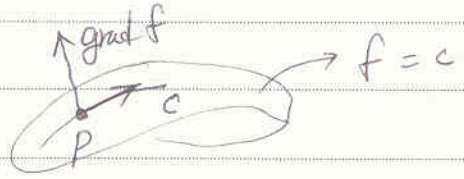
$\underline{b} \parallel \nabla f$: max directional derivative

(remind $\underline{b} = \frac{d\underline{r}}{ds}$)

✓ Gradient as Surface Normal Vector

$f(x(t), y(t), z(t)) = c$

see Fig. 2.14



then $\nabla f \cdot \frac{d\underline{r}}{dt} = 0$

↳ all vector in the tangent plane.

MOOKEUK

Vector field that are grad of scalar fields

v conservative Potential

$$\vec{f} = -\nabla U$$

$$= -\frac{c}{r^3}\vec{r} = -c\left[\frac{x-x_0}{r^3}, \frac{y-y_0}{r^3}, \frac{z-z_0}{r^3}\right]$$

(for gravitational force)

→ Laplace eqn. $\nabla^2 U = 0$

conservative.

nabla squared

delta

Laplace operator

for Coulombic interaction

$$\vec{f} = \frac{c}{r^3}\cdot\vec{r}$$

ex)

$$-\nabla U = \vec{F}$$

$$-\nabla\varphi = \vec{E} \qquad U = q\vec{\ell}\cdot\nabla\varphi_{ext}$$

$$= \vec{p}\cdot\nabla\varphi$$

$$-\mu_0\nabla\varphi^* = \vec{B} \qquad = -\vec{p}\cdot\vec{E}$$

↓ magnetic scalar potential

MOOKEUK

Divergence of a Vector field.

$$\operatorname{div}\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= \nabla\cdot\vec{v}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(v_1\hat{i} + v_2\hat{j} + v_3\hat{k}\right)$$

$$\int_V \nabla\cdot\vec{v}\,dV = \int_S \vec{v}\cdot\hat{n}\,dA$$

Note) $\nabla\cdot(\nabla f) = \nabla^2 f$

Ex1.

Ex2. $\vec{u} = \rho\vec{v}$ ($\vec{u}$: mass flux, $\vec{v}$: linear velocity)

2.

then, $\left(\frac{\Delta u_1}{\Delta x} + \frac{\Delta u_2}{\Delta y} + \frac{\Delta u_3}{\Delta z}\right)\cdot\Delta V\cdot\Delta t$

: total loss of mass in $\Delta V\,(=\Delta x\cdot\Delta y\cdot\Delta z)$ during the time interval Δt

$\sim \frac{\partial(\rho\cdot\Delta x\Delta y\Delta z)}{\partial t}\cdot\Delta t$ 1.

$$\therefore \operatorname{div}\vec{u} = \nabla\cdot(\vec{u}) = \nabla\cdot(\rho\vec{v}) = -\frac{\partial\rho}{\partial t}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \quad \text{(continuity equ)}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$$

incompressible $\nabla\cdot\vec{v} = 0$

another example) $\oint_S \vec{E}\cdot\hat{n}\,dA = \int_V \nabla\cdot\vec{E}\,dV = \frac{\int\rho\,dv}{\epsilon_0} = \frac{\Sigma q}{\epsilon_0}$

MOOKEUK

9.9 Curl of a Vector Field

$$\text{curl}\,\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

ex1. $\vec{v} = [yz,\ 3zx,\ z]$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix} = -3x\hat{i} + y\hat{j} + 2z\hat{k}$$

$\vec{v} = \vec{\omega} \times \vec{r}$

ex2. $\vec{v} = \vec{\omega} \times \vec{r} = \omega\hat{k} \times \vec{r} = -y\omega\hat{i} + x\omega\hat{j} + 0\hat{k}$

$[0\ 0\ \omega]$
$[x\ y\ z]$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y\omega & x\omega & 0 \end{vmatrix} = 2\omega\hat{k} = 2\vec{\omega}$$

$\therefore\ \nabla \times \vec{v} = 2\vec{\omega}$

Thm 2. $\nabla \times (\nabla f) = \vec{0}$ (gradient fields are irrotational.)

$\nabla \cdot (\nabla \times \vec{v}) = 0$ [irrotational when $\nabla \times \vec{v} = \vec{0}$ ($\because \vec{\omega} = 0$)]

$$\int_S (\nabla \times \vec{F}) \cdot \hat{n}\, dA = \int_C \vec{F} \cdot d\vec{r}$$

MOOKEUK

NO.

$\nabla \times \vec{E} = \vec{0}$ (why?

$\because \vec{E} = -\nabla\varphi$: conservative field.

$\nabla \times (\nabla\varphi) = \vec{0}$

$W = -q\int_i^f \nabla\varphi \, dl = -q(\varphi_f - \varphi_i)$

path indep.

$\nabla \cdot \vec{B} = 0$ (no magnetic monopoles)

$\nabla \times \vec{B} = \mu_0 \vec{J}$

$\nabla \cdot \vec{J} = 0 = \nabla \cdot (\nabla \times \vec{B}/\mu_0) = \nabla \cdot (\nabla \times \vec{B})$

MOOKEUK

Chp. 10

10.1 Line Integrals

Curve C.

$$\vec{r}(t) = [x(t), y(t), z(t)]$$
$$= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

piecewise smooth: finitely many smooth curves

- Definition & Evaluation of Line Integrals

$$\int_C \underbrace{\vec{F}(\vec{r}) \cdot d\vec{r}}_{\text{scalar integrand}} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

a line integral of a vector ftn

$\underbrace{\frac{\vec{F} \cdot \vec{r}'}{|\vec{r}'|}}_{\text{tangential comp. of } \vec{F}}$

$$= \int_C F_1\, dx + F_2\, dy + F_3\, dz$$

$$= \int_a^b (F_1 x' + F_2 y' + F_3 z')\, dt$$

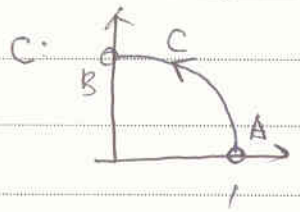
- Motivation of the Line Integral:

Work done by a force

$$\Delta W_m = \vec{F}(\vec{r}(t_m)) \cdot [\vec{r}(t_{m+1}) - \vec{r}(t_m)]$$
$$\approx \vec{F}(\vec{r}(t_m)) \cdot \vec{r}'(t_m) \cdot \Delta t_m$$

MOOKEUK

Ex 1. $\vec{F}(\vec{r}) = [-y, -x \cdot y]$



$\vec{r}(t) = [\cos t, \sin t]$

$0 \le t \le \frac{\pi}{2}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \int_0^{\frac{\pi}{2}} [-\sin t, -\cos t \cdot \sin t] \circ [\sin t, \cos t] \, dt$$

$$= \int_0^{\frac{\pi}{2}} (-\sin^2 t \cdot \cos t - \cos^2 t \cdot \sin t) \, dt$$

$$= \int_0^1 (+\sin^2 t - \sin t) \, d \sin t$$

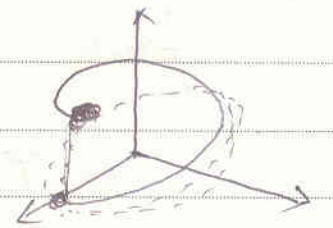
$$= \int_0^1 (p^2 - p) \, dp =$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 t \, dt + \int_1^0 \cos^2 t \cdot d(\cos t)$$

$$= \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2t}{2} \, dt = \int_0^1 e^2 \, de$$

$$= \frac{\pi}{4} - 0 - \frac{1}{3}$$

Ex 2. $\vec{F}(\vec{r}) = z\hat{i} + x\hat{j} + y\hat{k}$



$\vec{r}(t) = [\cos t, \sin t, 3t]$

$\vec{r}'(t) = [-\sin t, \cos t, 3]$

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$$

$$= \int_0^{2\pi} (-3t\sin t + \cos^2 t + 3\sin t)\, dt = 7\pi$$

$\frac{1+\cos 2t}{2}$

$= 6\pi + \pi$

$$\left[t\cdot(-\cos t) - 1\cdot(-\sin t) \right]_0^{2\pi}$$

Ex 3. & Ex 4.

for scalar ftn.

$$\int_C f(\vec{r})\, dt = \int_a^b f(\vec{r}(t))\, dt$$

than 2. path dep.

$\int_C F\,dr$ is not path dep only when $F\,dr = \underset{\sim}{dF}$

MOOKEUK

Path Indep of Line Integrals

i) $\vec{F} = \nabla f$

ii) integration around closed curves $\oint_C = 0$

iii) $\nabla \times \vec{F} = 0$

Thm 1. $\vec{F} = \nabla f$

$$F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}$$

$$\int_C (F_1 dx + F_2 dy + F_3 dz) = \int_C \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz\right)$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}\right) dt$$

$$= \int_a^b \frac{df}{dt}\cdot dt$$

$$= f(x(t), y(t), z(t))\Big|_{t=a}^{t=b}$$

MOOKEUK

Thm 2. 생략

Path indep. and Exactness of Differential Forms

$$\vec{F} \cdot d\vec{r} = F_1 dx + F_2 dy + F_3 dz$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$$= \nabla f \cdot d\vec{r}$$

$$\vec{F} = \nabla f \quad \text{thus} \quad F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}$$

Thm. 3. $\nabla \times \vec{F} = \vec{0}$

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

exactness $\int (F_1 dx + F_2 dy + F_3 dz) = \int df$

Ex 3. Using thm 3, show that the differential form under the integral sign of I is exact.

MOOKEUK

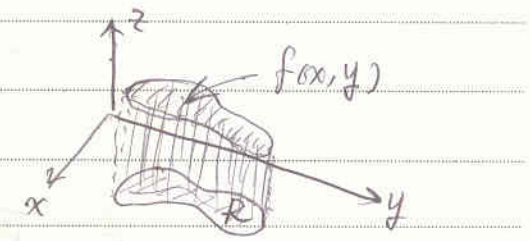
10.3 Double Integrals or Surface Integral

$$\iint_R f(x,y)\,dx\,dy \quad \text{or} \quad \iint_R f(x,y)\,dA$$

applications of Double Integrals

area $A = \iint_R dx\,dy$

volume $V = \iint_R f(x,y)\,dx\,dy$



Mass $M = \iint \rho\,dx\,dy$

ρ → mass per unit area

center of gravity

$$\bar{x} = \frac{1}{M}\iint_R x\cdot\rho\cdot dx\,dy$$

Moment of Inertia

$$I_x = \iint_R y^2\cdot f(x,y)\cdot dx\,dy$$

MOOKEUK

Change Variables : Jacobian

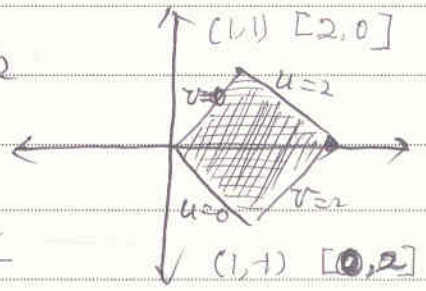
$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(x(w))\cdot\frac{dx}{du}\,du$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R^*} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

Jacobian

Ex 1. $\iint_R (x^2+y^2)\,dx\,dy$



$$= \int_{-1}^{1}\int_{f(y)}^{g(y)} (x^2+y^2)\,dx\,dy$$

$$\begin{cases} x+y=u \\ x-y=v \end{cases} \qquad \begin{cases} x=\frac{u+v}{2} \\ y=\frac{u-v}{2} \end{cases}$$

$$J = \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \int_0^1\int_0^2 \frac{u^2+v^2}{2}\cdot\frac{1}{2}\,du\,dv = \frac{8}{3}$$

Note) for polar coordinates : r & θ

$$J = r$$

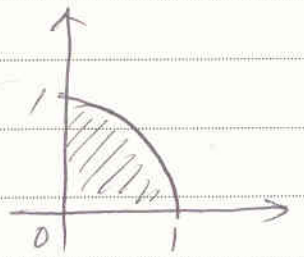
MOOKEUK

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Ex) 2 $f(x,y) = 1$ mass density



$$M = \iint_R dx\,dy = \int_0^{\pi/2}\int_0^1 r\,dr\,d\theta = \frac{\pi}{4}$$

$$x = \frac{\iint_R x\,dx\,dy}{M} = \frac{4}{\pi}\int_0^{\pi/2}\int_0^1 r\cos\theta \cdot r\,dr\,d\theta = \frac{4}{3\pi}$$

$$\left(= \frac{\iint x f\,dx\,dy}{M} \right)$$

$$y = \frac{4}{3\pi}$$

$$I_x = \iint y^2 f\,dx\,dy = \frac{\pi}{16}$$

$$I_y = \frac{\pi}{16}$$

10.4 Green's Thm

in the plane

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \oint_C (F_1\,dx + F_2\,dy)$$

Ex (2), ~~3~~, 4

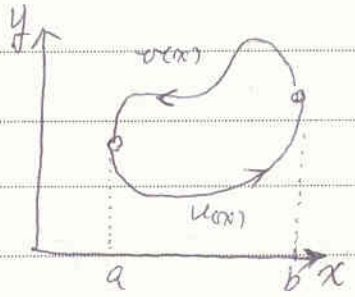
숙제 2, 4, 14, 18

$$\iint_R (\nabla \times \vec{F}) \cdot \vec{k}\, dx\,dy = \oint_C \vec{F} \cdot d\vec{r}$$

proof) $$\iint_R \frac{\partial F_1}{\partial y}\,dx\,dy = \int_a^b \left[\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y}\,dy\right] dx$$

$$F_1(x,y)\Big|_{u=y}^{v=y} = F_1(x,v) - F(x,u)$$

$$= \int_a^b F_1(x,v)\,dx - \int_a^b F_1(x,u)\,dx$$



$$= -\int_b^a F_1(x,v)\,dx - \int_a^b F_1(x,u)\,dx = -\oint_C F_1\,dx$$

similarly, $$\iint_R \frac{\partial F_2}{\partial x}\,dx\,dy = \oint_C F_2\,dy$$

$$\therefore \iint \nabla \times \vec{F} \cdot \hat{k}\,dx\,dy = \oint \vec{F} \cdot d\vec{r}$$

Ex 2. Area

(if we choose $F_1 = 0$, $F_2 = x$, $\iint_R dx\,dy = \oint_C x\,dy$

or $F_1 = -y$, $F_2 = 0$, $\iint_R dx\,dy = \oint_C -y\,dx$

$\therefore A = \frac{1}{2}\oint (x\,dy - y\,dx)$

for ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(x = a\cos t,\ y = b\sin t)$

$$A = \frac{1}{2}\int_0^{2\pi} (x y' - y x')\,dt = \frac{1}{2}\int_0^{2\pi} [ab\cos^2 t - (-ab\sin^2 t)]\,dt = \pi ab$$

Ex 3. 동일

Ex 4. if we set $F_1 = -\frac{\partial w}{\partial y}$ & $F_2 = +\frac{\partial w}{\partial x}$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \nabla^2 w$$

$$\oint_C (F_1\,dx + F_2\,dy) = \oint \left(F_1 \frac{dx}{ds} + F_2 \frac{dy}{ds}\right) ds = \oint_C \left(-\frac{\partial w}{\partial y}\cdot\frac{dx}{ds} + \frac{\partial w}{\partial x}\cdot\frac{dy}{ds}\right)\cdot ds$$

$\nabla w \cdot \vec{n}$

$$= \left[\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right]\cdot\left[\frac{dy}{ds}, \frac{-dx}{ds}\right]$$

$$= \frac{\partial w}{\partial x}\cdot\frac{dy}{ds} - \frac{\partial w}{\partial y}\cdot\frac{dx}{ds}$$

$$\Rightarrow \iint_R \nabla^2 w \cdot \hat{k}\,dx\,dy = \oint_C (\nabla w \cdot \vec{n})\,ds$$

$$= \oint_C \frac{\partial w}{\partial n}\cdot ds$$

MOOKEUK

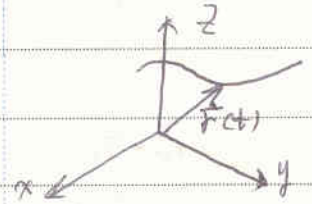
10.5 Surfaces for Surface Integrals 10.5-1

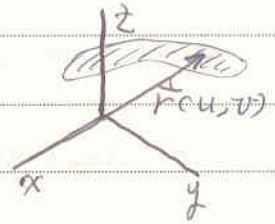
1. Representa of Surfaces

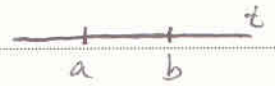
$z = f(x,y)$ or $g(x,y,z) = 0$

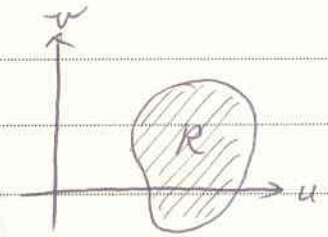
숙제. 2, 10

12, 16, 18









$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

Ex1. cylinder $x^2+y^2 = a^2$ $(-1 \le z \le 1)$

$$\vec{r}(u,v) = [a\cos u,\ a\sin u,\ v] \qquad \begin{matrix} 0 \le u \le 2\pi \\ -1 \le v \le 1 \end{matrix}$$

Ex2. Sphere $x^2+y^2+z^2 = a^2$

$$\vec{r}(u,v) = [a\cos u\cdot\cos v,\ a\sin u\cdot\cos v,\ a\sin v]$$

$$\begin{matrix} 0 \le u \le 2\pi \\ -\frac{\pi}{2} \le v \le \frac{\pi}{2} \end{matrix}$$

Ex3. Cone $z^2 = x^2+y^2$

$z = u$

$$\vec{r}(u,v) = [u\cos v,\ u\sin v,\ u] \qquad \begin{matrix} 0 \le u \le H \\ 0 \le v \le 2\pi \end{matrix}$$

MOOKEUK

2. tangent plane and surface normal.

$\vec{r}(u(t), v(t))$

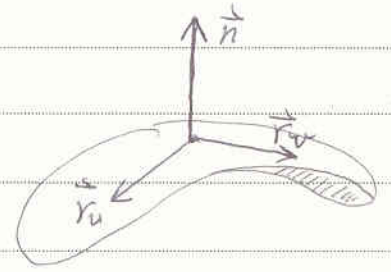
$$\Rightarrow \vec{r}'(u(t), v(t)) = \frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial u} \cdot u' + \frac{\partial \vec{r}}{\partial v} \cdot v'$$

$$= \vec{r}_u \cdot u' + \vec{r}_v \cdot v'$$

$\vec{r}_u$, $\vec{r}_v$ → tangential to S

normal vector $\vec{N} = \vec{r}_u \times \vec{r}_v$

unit normal vector $\vec{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$



$$\vec{n} = \frac{\nabla g}{|\nabla g|}$$

Thm 2. Uniqueness

Ex 4. Unit Normal vector of a sphere

$$g(x,y,z) = x^2+y^2+z^2-a^2 = 0$$

$$\vec{n} = \left[\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right]$$

Ex 5. Unit normal vector of a cone

$$g(x,y,z) = -z + \sqrt{x^2+y^2} = 0$$

$$\vec{n} = \left[\frac{x}{\sqrt{2(x^2+y^2)}}, \frac{y}{\sqrt{2(x^2+y^2)}}, \frac{-1}{\sqrt{2}}\right]$$

MOOKEUK

10.6. Surface Integrals

flux integral

$$\iint_S \underbrace{\vec{F}}_{\text{vector fn}} \cdot \vec{n}\, dA = \iint_R \vec{F}(\vec{r}(u,v)) \cdot \vec{N}(u,v)\, du\, dv$$

숙제 연습문제 10.1, 10.2
prob. 4, 6
14, 16

$$\vec{n}\, dA = N\, du\, dv$$
$$= n|N|\, du\, dv$$

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \iint_R (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma)\, dA$$

$$= \iint_R (F_1 N_1 + F_2 N_2 + F_3 N_3)\, du\, dv$$

$$= \iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

$$\checkmark \begin{cases} \cos\alpha\, dA = dy\, dz \\ \cos\beta\, dA = dz\, dx \\ \cos\gamma\, dA = dx\, dy \end{cases}$$

Ex 1. Flux thru a surface.

water flux through the parabolic cylinder $S: y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$

$v = F = [3x^2, 6, 6xz]$, $F = \rho v$ in general → here $\rho = 1$

sol) $\iint_S \vec{F} \cdot \vec{n}\, dA$

Let $x = u$, $z = v$

$\vec{r} = [u, u^2, v]$

$\vec{N} = \vec{r}_u \times \vec{r}_v = [1, 2u, 0] \times [0, 0, 1] = [2u, -1, 0]$

$$\iint_S \vec{F} \cdot \vec{n}\, dA = \iint_R (3v^2, 6, 6uv) \cdot (2u, -1, 0)\, du\, dv$$

$$= \int_0^3 \int_0^2 (6uv^2 - 6)\, du\, dv$$

$$= \int_0^3 (3v^2 \times 4 - 6 \times 2)\, dv = 27 \cdot 4 - 6 \cdot 6$$

$$= 72 \ m^3/sec$$

Ex 2. 생략

✓ Orient'n of Surfaces

- flux integral is dep. on unit normal vector.
  → an integral over an oriented surface S.

thm 1. $\vec{n} \to -\vec{n}$, multiplication of integral by '-1'.

Ex3. Ex1. 에서.

$\vec{N} = [2u, -1, 0]$ 대신 u, v 바꾸어 $\vec{N} = [-2v, 1, 0]$

$(\vec{r} = [v, v^2, u])$ 순간

~~then~~ $\vec{F} = [3z^2, 6, 6xz]$

then $\iint_R \vec{F} \cdot \vec{n} \, d\vec{A} = -72$.

✓ orient'n of smooth surface.

✓ piecewise smooth → orientable.

⇓

not hold for Möbius strip.

✓ Surface Integrals w/o rt orient'n

disregarding orient'n $\iint_S G(\vec{r}) dA = \iint_R G(\vec{r}(u,v)) \cdot |\vec{N}(u,v)| du dv$

area $A(S) = \iint_S dA = \iint_R |\vec{r}_u \times \vec{r}_v| du dv$

Ex 4. Area of sphere.

$\vec{r}(u,v) = [a\cos v\cos u,\ a\cos v\sin u,\ a\sin v]$

$0 \le u \le 2\pi$

$-\frac{\pi}{2} \le v \le \frac{\pi}{2}$

$\vec{r}_u \times \vec{r}_v = [a^2\cos^2 v\cos u,\ a^2\cos^2 v\sin u,\ a^2\cos v\cdot\sin u]$

$|\vec{r}_u \times \vec{r}_v| = a^2|\cos v|$

$A(S) = a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\pi}|\cos v|\,du\,dv = 2\pi a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos v\,dv$

$= 4\pi a^2$

Ex 5. Torus Surface

$\vec{r} = [(a+b\cos v)\cos u,\ (a+b\cos v)\sin u,\ b\sin v]$

$\vec{r}_u = [(a+b\cos v)(-\sin u),\ (a+b\cos v)\cdot\cos u,\ 0]$

$\vec{r}_v = [-b\sin v\cdot\cos u,\ -b\sin v\cdot\sin u,\ b\cos v]$

$\vec{r}_u \times \vec{r}_v =$ ...

$A = \int_0^{2\pi}\int_0^{2\pi} b(a+b\cos v)\,du\,dv = 4\pi^2 ab$

Ex 6. Moment of Inertia: 관성모멘트 $I = \iint_S \mu D^2\,dA$

✓ Representing $z = f(x,y)$ $\xrightarrow{\text{then}}$ $\vec{r} = [u, v, f]$

$|\vec{N}| = |\vec{r}_u \times \vec{r}_v| = |[1,0,f_u] \times [0,1,f_v]|$

$= |[-f_u, -f_v, 1]| = \sqrt{1+f_u^2+f_v^2}$

$\therefore \iint_S G(\vec{r})\,dA = \iint_{R^*} G(x,y,f(x,y))\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$

$A(S) = \iint_{R^*}\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}\,dx\,dy$

MOOKEUK

10.7 Triple Integrals

Divergence Thm of Gauss

10.7

ex 2

6, 12, 18, 20

$$\iiint_T \nabla \cdot \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, dA$$

$$\cos\alpha = \frac{\hat{n}\cdot\hat{i}}{|\hat{n}||\hat{i}|} = \hat{n}\cdot\hat{i}$$

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz = \iint_S (F_1\cos\alpha + F_2\cos\beta + F_3\cos\gamma)\, dA$$

$$= \iint_S (F_1\, dy\,dz + F_2\, dz\,dx + F_3\, dx\,dy)$$

$$\nabla\cdot(\rho\vec{v}) + \frac{\partial \rho}{\partial t} = 0$$

$$\nabla\cdot(\rho\vec{v}) = -\frac{\partial \rho}{\partial t}$$

$$\frac{\partial(\rho v_1)}{\partial x} + \frac{\partial(\rho v_2)}{\partial y} + \frac{\partial(\rho v_3)}{\partial z} = -\frac{\partial \rho}{\partial t}$$

ΔV 유체면

좌변만 고려.

$(\rho\vec{v})\cdot\hat{n}\, dA$ or $\nabla\cdot(\rho\vec{v})\cdot dV$

$$\oint_S (\rho\vec{v})\cdot\hat{n}\, dA = \int_V \nabla\cdot(\rho\vec{v})\cdot dV$$

ex for electro statics

$$\oint_S \vec{E}\cdot\hat{n}\, dA = \frac{\Sigma q}{\epsilon_0}$$

$$= \int_V \nabla\cdot\vec{E}\, dV = \frac{\int_V \rho\, dV}{\epsilon_0}$$

Ex 1. evaluate $I = \iint_S (x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy) = \oint_S \vec{F}\cdot\vec{n}\,dA$

↓

cylinder $x^2+y^2=a^2 \quad (0\le z\le b)$

Sol) $F_1 = x^3$, $F_2 = x^2y$, $F_3 = x^2z$

Hence $\nabla\cdot\vec{F} = 3x^2 + x^2 + x^2 = 5x^2$

$$\int_V 5x^2\,dV = \int_{z=0}^{b}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} 5r^2\cos^2\theta \cdot r\,dr\,d\theta\,dz$$

$$= 5\int_{z=0}^{b}\int_{\theta=0}^{2\pi}\frac{a^4}{4}\cdot\cos^2\theta\,d\theta\cdot dz$$

$$= \frac{5a^4b\pi}{4}$$

Ex 2. $\iint_S (7x\hat{i} - z\hat{k})\cdot\vec{n}\,dA$ (a) by div thm (b) directly

↘ $x^2+y^2+z^2 = 4 = 2^2$

$\vec{r} = [2\cos v\cos u,\ 2\cos v\sin u,\ 2\sin v]$

a) by div thm, $\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = 7-1 = 6$

$$\int_V 6\cdot dV = 6\cdot\frac{4\pi}{3}2^3 = 64\pi$$

b) get $\vec{N} = r_u \times r_v = [-2\cos v\cdot\sin u,\ 2\cos v\cdot\cos u,\ 2\sin v]$

$\times\ [-2\sin v\cdot\cos u,\ -2\sin v\sin u,\ 2\cos v]$

$\vec{F} = [14\cos v\cdot\cos u,\ 0,\ -2\sin v]$

$\vec{F}\cdot\vec{N} = 56\cos^3 v\cdot\cos^2 u - 8\cos v\cdot\sin^2 v$

MOOKEUK

10.8 Further Applications of the Divergence thm

숙제

Ex 2, 4 Ex1. 생략

2, 6 Ex2. heat flow. Heat or Difusion eqn.

$$\vec{v} = -K \cdot \nabla U$$

heat flow velocity

thermal conductivity

Temp.

$$\iint_S \vec{v} \cdot \vec{n}\, dA = \iiint_T \nabla \cdot \vec{v}\, dV$$

$$= \iiint_T \nabla \cdot (-K \nabla U)\, dV$$

total amount of heat in T

$$H = \iiint_T \sigma \rho U\, dV$$

specific heat

density

$$-\frac{\partial H}{\partial t} = -\iiint_T \sigma \rho \frac{\partial U}{\partial t}\, dV = \iiint \nabla \cdot (-K \nabla U)\, dV$$

$$-K \nabla^2 U = -\sigma \rho \frac{\partial U}{\partial t}$$

$$\frac{\partial U}{\partial t} = \frac{\sigma \rho}{K} \nabla^2 U \quad , \quad c^2 = \frac{K}{\sigma \rho}$$

$$= c^2 \nabla^2 U$$

thermal conductivity

Potential theory, Harmonic Ftn ($\nabla^2 f = 0$ harmonic ftn.

$\hookrightarrow \vec{E} = -\nabla\varphi$

Ex3. if $\vec{F} = \nabla f$

then $\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \nabla\cdot\nabla f = \nabla^2 f$

$$\vec{F}\cdot\vec{n} = \nabla f\cdot\vec{n} = \frac{\partial f}{\partial n}$$

$$\int_V \nabla^2 f \, dV = \int_S \frac{\partial f}{\partial n}\cdot dA$$

HW Ex 4. Let $\vec{F} = f\nabla g$

then $\nabla\cdot\vec{F} = \nabla\cdot(f\cdot\nabla g) = \nabla\cdot\left(f\frac{\partial g}{\partial x}, f\frac{\partial g}{\partial y}, f\frac{\partial g}{\partial z}\right)$

$$= \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial x^2}\right) + \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial y^2}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + f\frac{\partial^2 g}{\partial z^2}\right)$$

$$= f\nabla^2 g + \nabla f\cdot\nabla g$$

$$\vec{F}\cdot\vec{n} = \vec{n}\cdot(f\nabla g)$$

$$= (n\cdot\nabla g)\cdot f$$

$$\iiint_T (f\nabla^2 g + \nabla f\cdot\nabla g)\, dV = \iint_S f\cdot\frac{\partial g}{\partial n}\, dA \quad -①$$

or

$$\iiint_T (g\nabla^2 f + \nabla f\cdot\nabla g)\, dV = \iint_S g\cdot\frac{\partial f}{\partial n}\, dA \quad -②$$

① - ② results in

$$\iiint_T (f\nabla^2 g - g\nabla^2 f)\, dV = \iint_S \left(f\cdot\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n}\right)\cdot dA$$

MOOKEUK

10.9 Stokes's thm.

Stokes's thm $\iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dA = \oint \vec{F} \cdot \vec{r}'(s) \cdot ds$

숙제: Ex 1, 2, 5

6, 14

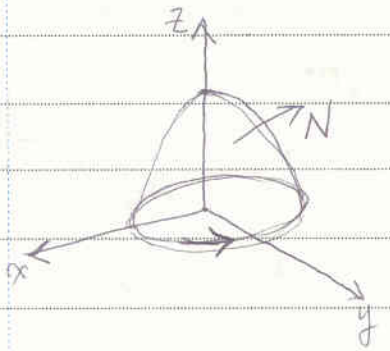
Ex2. Green thm $\iint_S (\nabla \times \vec{F}) \cdot \vec{k}\, dA = \oint \vec{F} \cdot d\vec{r}$

$= \oint (F_1 dx + F_2 dy)$

Ex1. $\vec{F} = [y, z, x]$

$z = f(x,y) = 1 - (x^2+y^2)$, $z \geq 0$

(paraboloid) Fig 2t.



$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(s)) \cdot \vec{r}'(s)\, ds$

$\vec{r}(s) = [\cos s, \sin s, 0]$

$\vec{r}'(s) = [-\sin s, \cos s, 0]$

$\vec{F} = [y, z, x] = [\sin s, 0, \cos s]$

$\oint_C \vec{F} \cdot d\vec{r}$

$= \int_0^{2\pi} \vec{F}(\vec{r}(s)) \cdot \frac{d\vec{r}(s)}{ds} \cdot ds$

$= \int_0^{2\pi} [(\sin s)(-\sin s) + 0 + 0]\, ds = -\pi$

$\int_S \nabla \times \vec{F} \cdot n\, dA = \iint_R (-2x - 2y + 1)\, dx\, dy = \int_0^{2\pi} \int_{r=0}^{1} (-2r(\cos\theta + \sin\theta) - 1)\, r\, dr\, d\theta$

$= -\pi$

$\nabla \times \vec{F} = D \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = [-1, -1, -1]$

$N = \nabla(z - f(x,y)) = [2x, 2y, 1]$,

$(\nabla \times \vec{F}) \cdot \vec{N} = -2x - 2y - 1$

MOOKEUK

Ex3. $\oint_C \vec{F} \cdot d\vec{r} = ?$

C: $x^2+y^2=4$, $z=-3$

$\vec{F} = [y, xz^3, -zy^3] = y\hat{i} + xz^3\hat{j} - zy^3\hat{k}$

$\hat{n} = \hat{k}$,

$\nabla \times \vec{F} \cdot \hat{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = -27 - 1 = -28$

$\int_S \nabla \times \vec{F} \cdot \hat{n}\, dA = \int_S -28 \cdot dA = -28 \cdot 4\pi = -112\pi$

Ex.4 Physical Meaning of the Curl in Fluid Motion

→ Circulation

due to mean value thm

Curl $\vec{F} \cdot n(P^*)$

$= (\nabla \times \vec{F}) \cdot \hat{n}(P^*) = \frac{1}{A_{R_0}} \oint_{C_{R_0}} \vec{F} \cdot \vec{r}'\, ds$

if $\vec{F} = \vec{v}$, $\oint_{C_{R_0}} \vec{v} \cdot \vec{r}'\, ds$

↳ circulation.

(which fluid motion?
a rotation around circle C_0

Ex5 $\vec{f}$ (force) work? 0

$W = \oint \vec{f} \cdot d\vec{r} = \int_S \nabla \times \vec{f} \cdot \hat{n}\, dA$ since $\vec{f} = -\nabla U$, $\nabla \times \vec{f} = \vec{0}$

(Section 9.9 $\nabla \times (-\nabla U) = \vec{0}$

$= 0$

Path Indep. $\nabla \times \vec{F} = \vec{0}$

Chp. 11.

11.1 Fourier Series

숙제 Eqn 6 증명.

periodic ftn. $f(x+\underset{\text{period}}{\underline{p}}) = f(x)$

$$f(x+np) = f(x)$$

$1, \cos x, \sin x, \cos 2x, \ldots$

trigonometric series

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

coefficients

a given ftn of period : 2π

Fourier coefficients of $f(x)$: a_i, b_i

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin nx\,dx$$

why fourier series? 주기함수

$$e^{i\theta} = \cos\theta + i\sin\theta$$

여러가지 진동문제가 간편히 표시됨.

MOOKEUK

NO.

Ex 1. $f(x) = \begin{cases} -k, & -\pi < x < 0 \\ k, & 0 < x < \pi \end{cases}$ & $f(x+2\pi) = f(x)$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\cdot\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= 0$$

$$a_0 = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\cdot\sin nx\,dx + \int_{0}^{\pi} k\cdot\sin nx\,dx\right]$$

$$= \frac{2k}{n\pi}(1-\cos n\pi)$$

$$= \frac{2k}{n\pi}(1-(-1)^n)$$

$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

복잡해 보이나 주기함수를 표시하면 유용한 것 발견.

Thm 1. Orthogonality (for the derivation of Euler formulas)

$$\int_{-\pi}^{\pi} \cos nx\cdot\cos mx\,dx = 0 \quad (n\neq m)$$

$$\int_{-\pi}^{\pi} \sin nx\cdot\sin mx\,dx = 0 \quad (n\neq m)$$

$$\int_{-\pi}^{\pi} \sin nx\cdot\cos mx\,dx = 0 \quad (n\neq m)$$

NO.

11.2 Ftns of Any period $p = 2L$

Eng Ex3

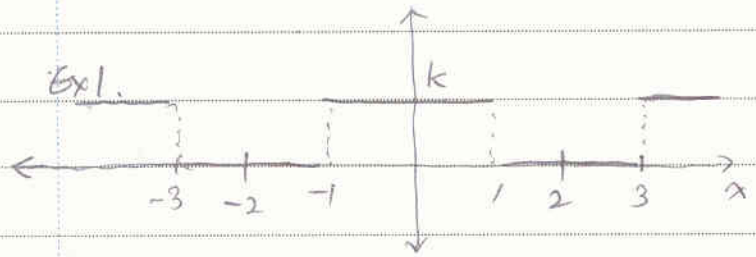
$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\sin\frac{n\pi x}{L}\,dx$$

Ex1.



$b_n = 0$ (why?)

$$a_0 = \frac{k}{2}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot\cos\frac{2\pi x}{L}\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

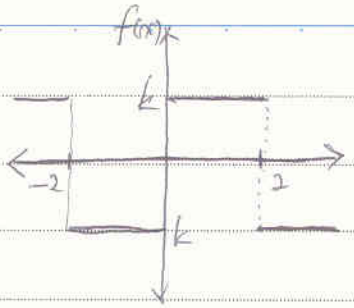
$$= \frac{1}{2}\int_{-1}^{1} k\cdot\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{2k}{n\pi}\left[\sin\frac{n\pi x}{2}\right]_{-1}^{1} =$$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - \cdots\right)$$

MOOKEUK

Ex 2.



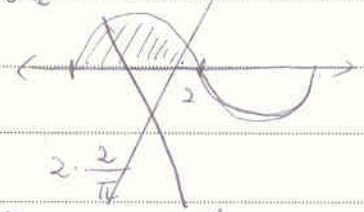
$$f(x) = \begin{cases} -k, & -2 < x < 0 \\ k, & 0 < x < 2 \end{cases} \qquad p = 2L = 4$$

$a_0 = 0$

$a_n = 0$ (why ?)

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\cdot \sin\frac{n\pi x}{2}\cdot dx = \frac{1}{2}\int_{0}^{2} k\cdot \sin\frac{n\pi x}{2}\cdot dx$$

$$-\frac{1}{2}\int_{-2}^{0} k\cdot \sin\frac{n\pi x}{2}\,dx$$



$$= \frac{k}{2}\cdot\left[-\cos\frac{n\pi x}{2}\right]_0^2 \cdot\frac{2}{n\pi} - \frac{k}{2}\cdot\frac{2}{n\pi}\left[-\cos\frac{n\pi x}{2}\right]_{-2}^{0}$$

$$= \frac{k}{n\pi}\left(2 - 2\cos n\pi\right) = \begin{cases} 4k/n\pi & \text{if } n=1,3,\cdots \\ 0 & \text{if } n=2,4,\cdots \end{cases}$$

Ex 3. 숙제

11.3 Even and Odd ftns. Half-Range Expansions

NO.

even ftn: $g(x) = g(-x)$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\frac{n\pi x}{L}\,dx$$

odd ftn: $g(x) = -g(-x)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Ex 1.

Ex 2. ~~V(t)~~ $-V(t)$: even ftn.

Ex 3 $f(x) = x + \pi$, $-\pi < x < \pi$ $\rightarrow$ $\underset{\sim}{f_1} + \underset{\pi}{f_2} = x + \pi$

$f(x) = f(x + 2\pi)$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f_1 \cdot \sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cdot\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

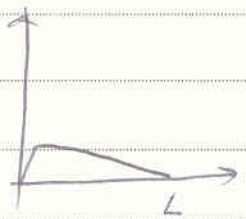
$$= \frac{-2(-1)^n}{n}$$

$$\therefore f(x) = \pi + 2\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right)$$

MOOKEUK

NO.

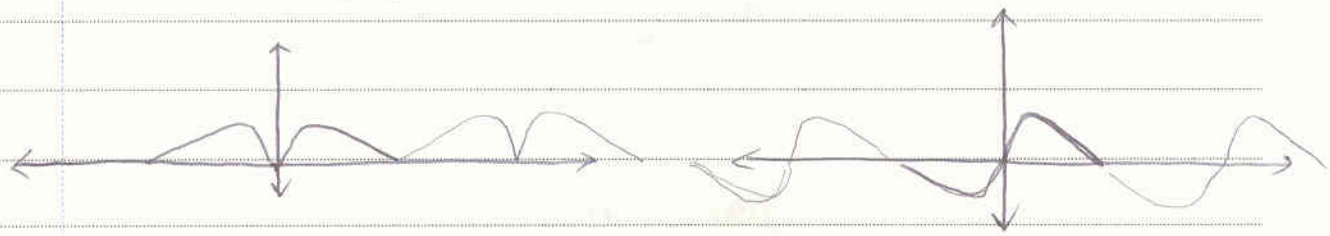
Half - Range Expansions



given fn $f(x)$

odd extension

even extension



Ex 4.

$$f(x) \begin{cases} \frac{2kx}{L} & 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \frac{L}{2} < x < L \end{cases}$$

a) Even periodic expansions

MOOKEUK

11.4 Complex Fourier Series

NO.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$\cos nx \rightarrow \dfrac{e^{inx} + e^{-inx}}{2}$, $\sin nx \rightarrow \dfrac{e^{inx} - e^{-inx}}{2i}$

$$f(x) = \underbrace{c_0}_{= a_0} + \sum_{n=1}^{\infty} (c_n e^{inx} + k_n \cdot e^{-inx})$$

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot (\cos nx - i \sin nx)\, dx$$

$\cos nx - i\sin nx = \cos(-nx) + i\sin(-nx)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot e^{-inx}\, dx$$

from n to $-n$

$$k_n = \frac{a_n + ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot (\cos nx + i \sin nx)\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot e^{inx}\, dx$$

SO!

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \cdot e^{-inx}\, dx$$

주기 : 2π

주기 : $2L$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cdot e^{i\frac{n\pi x}{L}}$$

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x) \cdot e^{-i\frac{n\pi x}{L}} \cdot dx$$

MOOKEUK

NO.

Ex.1. $f(x) = e^x$, $-\pi < x < \pi$

$f(x+2\pi) = f(x)$

Sol) $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x \cdot e^{-inx}\,dx$

$= \frac{1}{2\pi}\cdot\frac{\left[e^{(1-in)x}\right]_{-\pi}^{\pi}}{1-in} = \frac{1}{2\pi}\cdot\frac{1}{1-in}\left(e^{\pi}-e^{-\pi}\right)(-1)^n$

$= \frac{\sinh\pi}{\pi}\cdot(-1)^n\cdot\frac{1+in}{1+n^2}$

$\therefore\ e^x = \frac{\sinh x}{\pi}\sum_{n=-\infty}^{\infty}(-1)^n\cdot\frac{1+in}{1+n^2}\cdot e^{inx}$

$(1+in)\cdot e^{inx} = (1+in)\cdot(\cos nx + i\sin nx)$

$= (\cos nx - n\sin nx) + i(n\cos nx + \sin nx)$

w/ conjugate

$2\left[(\cos nx) - n\sin nx\right]$

$e^x = \frac{2\sinh\pi}{\pi}\cdot\left[\frac{1}{2} - \frac{1}{1+1^2}\cdot(\cos x - \sin x) + \frac{1}{1+2^2}(\cos 2x - 2\sin 2x) - \cdots\right]$

MOOKEUK

11.5 Forced Oscillations

NO.

$my'' + cy' + ky = r(t)$ spring-dashpot-mass

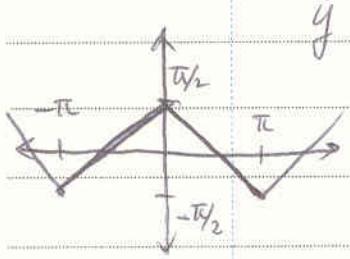
$LI'' + RI' + \frac{I}{C} = E'(t)$ R-L-C circuit

Ex 1. Forced Oscillations under a non-sinusoidal periodic driving force.

$m = 1$ g, $c = 0.05$ g/sec, $k = 25$ g/sec^2

$y'' + 0.05y' + 25y = r(t)$

$r(t)$ $\begin{cases} t + \frac{\pi}{2}, & -\pi < t < 0 \\ -t + \frac{\pi}{2}, & 0 < t < \pi \end{cases}$



$\begin{cases} p = 2\pi \\ f(t+2\pi) = f(t) \end{cases}$

Find the steady state soln $y(t)$.

Sol) $r(t)$ by a Fourier series

$$r(t) = \frac{4}{\pi}\left(\cos t + \frac{1}{3^2}\cdot\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$

$$y'' + 0.05y' + 25y = \frac{4}{n^2\pi}\cos nt \quad (n = 1, 3, 5 \cdots)$$

$y_n = A_n \cos nt + B_n \sin nt$

$A_n = \frac{4(25-n^2)}{n^2\pi \cdot D_n}$, $B_n = \frac{0.2}{n\pi D_n}$, where $D_n = (25-n^2)^2 + (0.05n)^2$

$y = y_1 + y_3 + \cdots$

MOOKEUK

11.6 Approximation by Trigonometric Polynomials

NO.

$$f(x) \sim a_0 + \sum_{n=1}^{N} (a_0 \cdot \cos nx + b_n \sin nx)$$

best app. $\Rightarrow F(x) = A_0 + \sum_{1}^{N} (A_n \cdot \cos nx + B_n \sin nx)$

trigonometric polynomial of degree, N

square error $E = \int_{-\pi}^{\pi} (f - F)^2 dx$

$$= \int_{-\pi}^{\pi} f^2 dx \qquad = \int_{-\pi}^{\pi} f^2 dx$$

$$-2\int_{-\pi}^{\pi} f \cdot F \, dx \qquad -2\pi \left[2A_0 a_0 + \sum (A_n a_n + B_n b_n) \right]$$

$$+ \int_{-\pi}^{\pi} F^2 dx \qquad + \pi \left[2A_0^2 + \sum (A_n^2 + B_n^2) \right]$$

$$\pi \left(2A_0 a_0 + A_1 a_1 + \cdots + A_N a_N + B_1 b_1 + \cdots + B_N b_N \right)$$

$$\int_{-\pi}^{\pi} \left[A_0 + \sum_{n=1}^{N} (A_n \cdot \cos nx + B_n \sin nx) \right]^2 dx$$

$$= \pi \left(2A_0^2 + A_1^2 + \cdots + A_N^2 + B_1^2 + \cdots + B_N^2 \right)$$

Taking $A_n = a_n$ & $B_n = b_n$,

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi \left(2a_0^2 + \sum_{1}^{N} (a_n^2 + b_n^2) \right) \geq 0$$

↳ Bessel's ineq.

Note, $E \geq E^* \geq 0$

MOOKEUK

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f(v)\,dv + \frac{1}{L}\sum \left(\int_{-L}^{L} f_L(v)\cos\omega_n v\,dv\right)\cos\omega_n x + \left(\int_{-L}^{L} f_L(v)\cdot \sin\omega_n v\,dv\right)\sin\omega_n x$$

representation of $f(x)$ by F integral,

$$f(x) = \int_0^{\infty} \left(A(\omega)\cdot\cos\omega x + B(\omega)\sin\omega x\right) d\omega$$

($f(x) = \int_0^{\infty} A(\omega)\cos\omega x\,d\omega$ Fourier cos integral

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cdot\cos\omega x\,dx,\quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv \xrightarrow{\text{if } f \text{ even}} A(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\cos\omega v\,dv$$

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cdot\sin\omega x\,dx,\quad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cdot\sin\omega v\,dv \xrightarrow{\text{if } f \text{ odd}} B(\omega) = \frac{2}{\pi}\int_0^{\infty} f(v)\sin\omega v\,dv$$

(Fourier sin integral $f(x) = \int_0^{\infty} B(\omega)\sin\omega x\,d\omega$

$$f(x) = \frac{\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} \hat{f}_c(\omega)\cdot\cos\omega x\,d\omega$$

$$f(x) = \frac{\sqrt{2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} \hat{f}_s(\omega)\cdot\sin\omega x\,d\omega$$

$$f_L(x) = \sum_{-\infty}^{\infty} C_n\cdot e^{i n a x/L}$$

$$C_n = \frac{1}{2L}\int_{-L}^{L} f_L(x)\cdot e^{-\frac{i n a x}{L}}\,dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\cdot e^{-i\omega v}\,dv\right)\cdot e^{+i\omega x}\,d\omega$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{-i\omega x}\,dx \quad , \text{ F.T}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}\cdot e^{i\omega x}\,d\omega \quad , \text{ i F.T}$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\left(\int_{-L}^{L} f_L(v)\cdot\cos\omega_n v\,dv\right)\cos\omega_n x + \left(\int_{-L}^{L} f_L(v)\sin\omega_n v\,dv\right)\sin\omega_n x\right]$$

$L\to\infty$, $f_L(x) \to f(x)$

$$\frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv \to 0$$

since $\omega_n = \frac{n\pi}{L}$, $\frac{\Delta\omega}{\pi} = \frac{1}{L}$.

(therefore $L\to\infty$, $\frac{\Delta\omega}{\pi} = \frac{d\omega}{\pi}$

$$f(x) = \int_0^{\infty}\cos\omega x\left(\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv\right)\frac{d\omega}{\pi} + \int_0^{\infty}\sin\omega x\left(\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv\right)\frac{d\omega}{\pi}$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[A(\omega)\cos\omega x + B(\omega)\sin\omega x\right]d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v\,dv \qquad \frac{2}{\pi}\int_0^{\infty} f(v)\cdot\cos\omega v\,dv$$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v\,dv \qquad \frac{2}{\pi}\int_0^{\infty} f(v)\cdot\sin\omega v\,dv$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{1}^{\infty}\left[\left(\int_{-L}^{L} f_L(v)\cdot\cos\omega_n v\,dv\right)\cdot\cos\omega_n x + \left(\int_{-L}^{L} f_L(v)\cdot\sin\omega_n v\,dv\right)\cdot\sin\omega_n x\right]$$

$\downarrow 0$

when $L\to\infty$,

$$\int_0^{\infty}\left[\left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cdot\cos\omega v\,dv\right)\cdot\cos\omega x + \left(\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cdot\sin\omega v\,dv\right)\cdot\sin\omega x\right]d\omega$$

Scaling

$$\mathcal{F}(f(ax)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) \cdot e^{-i\omega x} dx \qquad ax = p$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p) \cdot e^{-i\left(\frac{\omega}{a}\right)p} \cdot dp$$

$$= \frac{1}{a} \hat{f}\left(\frac{\omega}{a}\right)$$

Convolution

$$h(x) = f * g(x) = \int_{-\infty}^{\infty} f(p) \cdot g(x-p)\, dp$$

$$= \int_{-\infty}^{\infty} f(x-p) \cdot g(p) \cdot dp$$

$$\mathcal{F}(f * g) = \sqrt{2\pi} \cdot \mathcal{F}(f) \cdot \mathcal{F}(g)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p) \cdot g(x-p) \cdot dp \cdot e^{-i\omega x} dx \qquad x - p = q$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(p) \cdot g(q) \cdot e^{-i\omega(p+q)} \cdot dp \cdot dq$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(p) \cdot e^{-i\omega p} dp \, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(q) \cdot e^{-i\omega q} dq \cdot \sqrt{2\pi}$$

$$= \mathcal{F}(f) \cdot \mathcal{F}(g) \sqrt{2\pi}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) \cdot e^{-i\omega v} dv \right) \cdot e^{i\omega x} d\omega$$

F.T $\quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} dt$

i F.T $\quad f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) \cdot e^{i\omega t} dt$

t or x $\xrightarrow{\text{F.T}}$ ω or q

$\xleftarrow{\text{i F.T}}$

Linearity

differentiation

$$\mathcal{F}(f'(x)) = i\omega \mathcal{F}(f(x))$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) \cdot e^{-i\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\left[f(x) \cdot e^{-i\omega x} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) (-i\omega) e^{-i\omega x} dx \right)$$

($\left[f(x) \cdot e^{-i\omega x} \right]_{-\infty}^{\infty} \to 0$)

shift $\quad \mathcal{F}(f(x-a)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) \cdot e^{-i\omega x} dx \qquad x - a = t$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} \cdot e^{-i\omega a} \cdot dt$$

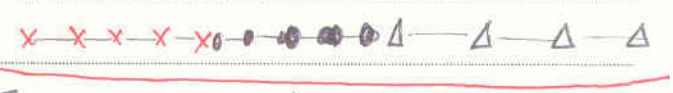
$$= e^{-i\omega a} \cdot \mathcal{F}(f(x-a))$$

11.7 Fourier Integral.

Ex 1 & Fig 277

physical meaning!

$L \to \infty$



From Fourier Series to Fourier Integral

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{1}^{\infty}\left[\left(\int_{-L}^{L} f_L(v)\cdot\cos\omega_n v\,dv\right)\cdot\cos\omega_n x + \left(\int_{-L}^{L} f_L(v)\cdot\sin\omega_n v\,dv\right)\cdot\sin\omega_n x\right]$$

finite.

so, $\frac{\text{finite}}{L} \to 0$ (when $L\to\infty$)

$\omega_n = \frac{n\pi}{L}$ $\to$ $\Delta\omega = \omega_{n+1} - \omega_n = \frac{\pi}{L}$

if $L\to\infty$ $\left(f(x) = \lim_{L\to\infty} f_L(x)\right)$ $\omega_n \to \omega$, $\frac{\pi}{L}(=\Delta\omega) \to d\omega$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos\omega x\int_{-\infty}^{\infty} f(v)\cdot\cos\omega v\cdot dv + \sin\omega x\cdot\int_{-\infty}^{\infty} f(v)\cdot\sin\omega v\,dv\right]\cdot d\omega$$

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cdot\cos\omega v\,dv \xrightarrow{\text{if } f \text{ is even}} \frac{2}{\pi}\int_0^{\infty} f(v)\cdot\cos\omega v\,dv$$

Fourier cosine Integ

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cdot\sin\omega v\cdot dv \xrightarrow{\text{if } f \text{ is odd}} \frac{2}{\pi}\int_0^{\infty} f(v)\cdot\sin\omega v\,dv$$

Fourier sine Integ

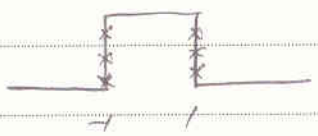
$$f(x) = \int_0^{\infty}\left[A(\omega)\cdot\cos\omega x + B(\omega)\cdot\sin\omega x\right]\cdot d\omega$$

: representation of $f(x)$ by a Fourier integral.

Ex 2. Single pulse. Sine integral.

Find the Fourier integral representation of the following fcn.

$$f(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases}$$



sol) $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cdot \cos wv\, dv$

$$= \frac{1}{\pi}\int_{-1}^{1} \cos wv\, dv = \frac{\sin wv\,|_{-1}^{1}}{\pi w} = \frac{2\sin w}{\pi w}$$

$B(w) = 0$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\cos wx \cdot \sin w}{w}\, dw$$

from $A(w)$

$$\int_0^{\infty} \frac{\cos wx \cdot \sin w}{w}\cdot dw = \begin{cases} \pi/2 & 0\le x<1 \\ \pi/4 & x=1 \\ 0 & x>1 \end{cases}$$

when $x=0$, $\int_0^{\infty} \frac{\sin w}{w}\cdot dw = \frac{\pi}{2}$

$$\underbrace{Si(u)}_{\text{sin integral}} = \int_0^{u} \frac{\sin w}{w}\cdot dw \qquad \text{Fig 279}$$

partial sum

$$\frac{2}{\pi}\int_0^{a} \frac{\cos wx \cdot \sin w}{w}\, dw \qquad \text{Fig 280}$$

MOOKEUK

if f is an even ftn,

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cdot\cos\omega v\,dv$$

$$f(x) = \int_0^\infty A(\omega)\cdot\cos\omega x\,d\omega$$ Fourier cos integral representation

if f is an odd ftn,

$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cdot\sin\omega v\,dv$$

$$f(x) = \int_0^\infty B(\omega)\cdot\sin\omega x\,d\omega$$ Fourier sin integral representation

Setting $A(\omega) = \sqrt{\frac{2}{\pi}}\cdot\hat{f}_c(\omega)$

$$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cdot\cos\omega x\,dx$$ Fourier cos transform

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}(\omega)\cdot\cos\omega x\,d\omega$$ inverse F.C.T

$$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cdot\sin\omega x\,dx$$ Fourier sin transform

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{-i\omega x}\,dx$$ F.T

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)\cdot e^{+i\omega x}\,d\omega$$ iFT

$$f_L(x) = \sum_{-\infty}^{\infty} C_n\cdot e^{i n\pi x/L}$$

$$C_n = \frac{1}{2L}\int_{-L}^{L} f_L(x)\cdot e^{-i\frac{n\pi x}{L}}\cdot dx$$

$L\to\infty$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x)\cdot e^{-i\omega x}\,dx\right)\cdot e^{i\omega x}\cdot d\omega$$

MOOKEUK

Fourier cos & sin Integral

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cdot\cos\omega v\,dv$$

Fourier cos int. $f(x) = \int_0^\infty A(\omega)\cdot\cos\omega x\,d\omega$

$B(\omega) =$

Fourier sin int $f(x) = \int_0^\infty A(\omega)\cdot\sin\omega x\,d\omega$

Ex3. Find the Fourier cos int and sin int.

$f(x) = e^{-kx} \quad (x>0,\ k>0)$

$$A(\omega) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cdot\cos\omega v\,dv$$

$$\int e^{-kv}\cdot\cos\omega v\,dv = -\frac{k}{k^2+\omega^2}\cdot e^{-kv}\cdot\left(-\frac{\omega}{k}\cdot\sin\omega v + \cos\omega v\right)$$

$e^{v(-k+i\omega)}$

$$\mathrm{Re}\left[\int_0^\infty e^{-kv}\cdot e^{i\omega v}\,dv = \frac{\left[e^{(-k+i\omega)v}\right]_0^\infty}{-k+i\omega} = \frac{0-1}{-k+i\omega}\right]$$

$$= \frac{k}{k^2+\omega^2}$$

$$\rightarrow A(\omega) = \frac{2}{\pi}\cdot\frac{k}{k^2+\omega^2}$$

$$f(x) = e^{-kx} = \int_0^\infty A(\omega)\cdot\cos\omega x\,dx \quad \text{F.C.I representation}$$

$$e^{-kx} = \int_0^\infty \frac{2}{\pi}\cdot\frac{k\cos\omega x}{k^2+\omega^2}\,dx \quad (x>0,\ k>0)$$

MOOKEUK

$$B(\omega) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cdot \sin\omega v\, dv$$

$$\mathrm{Im}\left(\int_0^\infty e^{-kv}\cdot e^{i\omega v}\,dv\right) = \mathrm{Im}\left(\left[\frac{e^{(-k+i\omega)v}}{-k+i\omega}\right]_0^\infty\right)$$

$$= \mathrm{Im}\left(\frac{-1}{-k+i\omega}\right)$$

$$= \mathrm{Im}\left(\frac{k+i\omega}{k^2+\omega^2}\right) = \frac{\omega}{k^2+\omega^2}$$

$$\rightarrow B(\omega) = \frac{2}{\pi}\left(\frac{\omega}{k^2+\omega^2}\right)$$

$$f(x) = e^{-kx} = \int_0^\infty \frac{2}{\pi}\left(\frac{\omega}{k^2+\omega^2}\right)\cdot \sin\omega x\, d\omega$$

$$\left(k>0,\ x>0\right)$$

$$e^{-kx} = \int_0^\infty \frac{2}{\pi}\,\frac{\omega\sin\omega x}{k^2+\omega^2}\,d\omega$$

MOOKEUK

11.8 Fourier Cos and Sine Transforms 11.8-1

from $f(x) = \int_0^\infty A(\omega)\cdot\cos\omega x\, d\omega$, $A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cdot\cos\omega v\, dv$

HW Ex 1.

Ex 3.

let $A(\omega) = \sqrt{\frac{2}{\pi}}\cdot\hat{f}_c(\omega)$ ← cosine.

$\hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cdot\cos\omega x\, dx$ Fourier cosine transform of $f(x)$

$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\omega)\cdot\cos\omega x\, d\omega$ inverse Fourier cosine transform of $\hat{f}_c(\omega)$

Similarly,

$\hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\cdot\int_0^\infty f(x)\cdot\sin\omega x\, dx$ Fourier sine transform.

$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\omega)\cdot\sin\omega x\, d\omega$ inverse Fourier cos transform.

Ex 2 $\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cdot\cos\omega x\, dx$

$= \sqrt{\frac{2}{\pi}}\cdot\frac{\left[e^{-x}(-\cos\omega x + \omega\sin\omega x)\right]_0^\infty}{1+\omega^2}$

$= \frac{\sqrt{2/\pi}}{1+\omega^2}$

MOOKEUK

Linearity, Transforms of Derivatives

i) $$\mathcal{F}_c(af+bf) = \sqrt{\frac{2}{\pi}}\int_0^\infty [af(x)+bf(x)]\cdot\cos\omega x\,dx$$

$$= a\mathcal{F}_c(f) + b\mathcal{F}_c(g)$$

ii) $$\mathcal{F}_c\{f'(x)\} = \omega\mathcal{F}_s\{f(x)\} - \sqrt{\frac{2}{\pi}}f(0)$$

$$\mathcal{F}_s\{f'(x)\} = -\omega\cdot\mathcal{F}_c\{f(x)\}$$

$$\mathcal{F}_c(f'(x)) = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cdot\cos\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[f(x)\cdot\cos\omega x\Big|_0^\infty + \omega\int_0^\infty f\cdot\sin\omega x\,dx\right]$$

$$= \omega\cdot\mathcal{F}_s(f(x)) - \sqrt{\frac{2}{\pi}}\cdot f(0)$$

$$\mathcal{F}_s(f'(x)) = \sqrt{\frac{2}{\pi}}\cdot\int_0^\infty f'(x)\cdot\sin\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\cdot\left[f(x)\cdot\sin\omega x\Big|_0^\infty - \int_0^\infty f(x)\cdot\omega\cdot\cos\omega x\,dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}\cdot\omega\int_0^\infty f(x)\cdot\cos\omega x\,dx = -\omega\cdot\mathcal{F}_c(f(x))$$

Ex3

MOOKEUK

11.9 Fourier Transform.

Discrete and fast Fourier transform.

$$f(x) = \int_0^\infty [A(\omega)\cdot\cos\omega x + B(\omega)\cdot\sin\omega x]\,dx$$

HW. Ex1.
Ex2.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)\cdot e^{-i\omega v}\,dv\right]\cdot e^{i\omega x}\cdot d\omega$$

complex Fourier integral,

Prob 4.8

Fourier Transform

$$\hat{f}(\omega) = \mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\cdot\int_{-\infty}^{\infty} f(x)\cdot e^{-i\omega x}\cdot dx$$

$$t(\text{or } x) \underset{i\mathcal{F}T}{\overset{FT}{\rightleftarrows}} \omega(\text{or } q)$$

inverse F.F

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)\cdot e^{i\omega x}\cdot d\omega$$

$$= \mathcal{F}^{-1}(\hat{f})$$

Ex2. $\mathcal{F}(e^{-ax})$ (a>0) $= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-ax}\cdot e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\cdot\frac{\left[e^{-(a+i\omega)x}\right]_0^\infty}{-(a+i\omega)}$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{a+i\omega}$$

$$f(x)\begin{cases} e^{-ax} & x\geq 0 \\ 0 & x<0 \end{cases}$$

Physical interpretation

total E of vib. system (osc.) = $\int_{-\infty}^{\infty}|\hat{f}(\omega)|^2 d\omega$ → $I(\omega)$

ex $m\ddot{y}+ky=0$ $E_0 = \frac{mv^2}{2} + \frac{ky^2}{2} = 2k|a_1|^2$

$x\dot{y}$ → $m\cdot\dot{y}\ddot{y}+k\dot{y}y=0$ ← integration.

MOOKEUK

Linearity.

$$\mathcal{F}(af+bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

F.T of derivative

$$\mathcal{F}\{f'(x)\} = i\omega\,\mathcal{F}\{f(x)\}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'(x)\cdot e^{-i\omega x}\,dx = \frac{1}{\sqrt{2\pi}}\left[f\cdot e^{-i\omega x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f\cdot(-i\omega)\cdot e^{-i\omega x}\,dx \right]$$

$f \to 0$ 적분존재의해

$$= i\omega\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{-i\omega x}\,dx$$

$$= i\omega\cdot\mathcal{F}(f(x))$$

$$\mathcal{F}\{f''(x)\} = -\omega^2\mathcal{F}(f(x))$$

Ex 3. $\mathcal{F}(x\cdot e^{-x^2}) = ?$ from table 3.

Sol) from $\mathcal{F}(e^{-x^2}) = \frac{1}{\sqrt{2}}\cdot e^{-\frac{\omega^2}{4}}$

$$\mathcal{F}(-2x\cdot e^{-x^2}) = i\omega\cdot e^{-\frac{\omega^2}{4}}$$

$$\therefore \mathcal{F}(x\cdot e^{-x^2}) = \frac{-i\omega}{2\sqrt{2}}\cdot e^{-\frac{\omega^2}{4}}$$

Shift

$$\mathcal{F}(f(x-a)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)\cdot e^{-i\omega x}\,dx$$

$x-a=p$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(p)\cdot e^{-i\omega(p+a)}\cdot dp$$

$$= e^{-i\omega a}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(p)\cdot e^{-i\omega p}\cdot dp$$

$$= e^{-i\omega a}\cdot\mathcal{F}(f(x))$$

Scaling

$$\mathcal{F}(f(ax)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\cdot e^{-i\omega x}\,dx$$

$p = ax$, (for $a>0$)

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(p)\cdot e^{-i\omega\cdot\left(\frac{x}{a}\right)}\cdot d\left(\frac{x}{a}\right)$$

$$= \frac{1}{a}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(p)\cdot e^{-i\left(\frac{\omega}{a}\right)x}\cdot dx$$

$$= \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right)$$

Convolution.

$$h(x) = f * g(x) = \int_{-\infty}^{\infty} f(p) \cdot g(x-p)\, dp$$

$$= \int_{-\infty}^{\infty} f(x-p) \cdot g(p)\, dp$$

Convolution thm.

$$\mathcal{F}(f * g) = \sqrt{2\pi} \cdot \mathcal{F}(f) \cdot \mathcal{F}(g)$$

$$f * g = \int_{-\infty}^{\infty} \hat{f}(\omega) \cdot \hat{g}(\omega) \cdot e^{i\omega x} \cdot d\omega$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(p) \cdot g(x-p)\, dp \right) \cdot e^{-i\omega x}\, dx$$

let $x-p=q$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(p) \cdot g(q) \cdot e^{-i\omega(p+q)} \cdot dp \right) dq$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{\infty} f(p) \cdot e^{-i\omega p} dp \right) \left(\int_{-\infty}^{\infty} g(q) \cdot e^{-i\omega q} \cdot dq \right)$$

$$= \sqrt{2\pi} \cdot \mathcal{F}(f(p)) \cdot \mathcal{F}(g(q))$$

MOOKEUK

Discrete Fourier Transform (DFT)

Fast Fourier Transfor (FFT)

$$\hat{f}_n = \sum_{k=0}^{N-1} f_k \cdot e^{-inx_k}$$

$$f_k = f(x_k) = \sum_{n=0}^{N-1} c_n \cdot e^{inx_k}$$

Ex4. 생략

Fourier matrix $F_N = [e_{nk}]$

$$\hat{f} = F_N \cdot f$$

NO.

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{-i\omega x} dx \qquad \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx \qquad \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x \, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \cdot e^{+i\omega x} dx \qquad f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(\omega) \cdot \cos \omega x \, dx$$

i) L·C

ii) differential

iii) shift

iv) scaling

v) convolution $\sqrt{2\pi}\, F \cdot G$

11.8-2

i) linearity

ii) differential

$$\mathcal{F}_c\{f'(x)\} = \omega \mathcal{F}_s(f(x)) - \sqrt{\frac{2}{\pi}} f(0)$$

$$\mathcal{F}_s\{f'(x)\} = -\omega \cdot \mathcal{F}_c\{f(x)\}$$

Ex 2.

comp Ex2 in 11.9

MOOKEUK

Chp. 12 PDE

- 시험문제 + be creative (positive, persistent, aggressive.
  + 연구분야 (why nano?)
  + creative (then responsible)

Ex1.

B.C. and I.C

Thm 1.

for homogeneous linear PDE, linearity

why superposition?

Ex 2. & Ex 3

ODE 처럼 풀면 됨

12.2 Modeling : Vibrating String, Wave Eqn.

$$T_1 \cos\alpha = T_2 \cos\beta = T = \text{constant}$$

$$T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \cdot \frac{\partial^2 u}{\partial t^2}$$

↳ from $ma = m\ddot{x}$

$$\frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\left.\frac{\partial u}{\partial x}\right|_x = \tan\alpha \quad \& \quad \left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} = \tan\beta$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

12.3 Soln by Separating Variables

Use of Fourier Series

$$\ddot{u} = c^2 u''$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{T}{\rho}$$ ← acoustic modulus

$$u(0,t) = 0$$
$$u(L,t) = 0 \quad \text{for all } t$$

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x) \quad (0 \le x \le L)$$

Step 1. Method of separating variables

Step 2. BC & IC

Step 3. Fourier Series

Step 1.

$$u(x,t) = F(x) \cdot G(t)$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 F \ddot{G} \quad \& \quad \frac{\partial^2 u}{\partial x^2} = c^2 F'' G$$

$$F\ddot{G} = c^2 F'' G$$

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k$$

$$\begin{cases} F'' - kF = 0 \\ \ddot{G} - c^2 k G = 0 \end{cases}$$

Step 2. Satisfying the B.C.

$u(0,t) = F(0) \cdot G(t) = 0$ for all t

$u(L,t) = F(L) \cdot G(t) = 0$

$F(0) = 0, \quad F(L) = 0$ ⟶

for positive k

$F = A \cdot e^{\mu x} + B \cdot e^{-\mu x}$ ⟶ $F = 0$ from B.C

since $k = -p^2$

$F(x) = A \cos px + B \sin px$

$F(0) = A = 0$ and then $F(L) = B \sin pL = 0$

We should take $B \neq 0$ otherwise $F = 0$

Hence $\sin pL = 0$. Thus, $pL = n\pi$.

so that $p = \frac{n\pi}{L}$

∨ Setting $B = 1$

$F_n(x) = \sin \frac{n\pi x}{L}$

for G

$$\dot{G} + \lambda_n^2 G = 0 \quad \text{where} \quad \lambda_n = cp = \frac{cn\pi}{L}$$

A general soln is

$$G_n(t) = B_n \cos\lambda_n t + B_n^* \sin\lambda_n t$$

eigen ftns

$$u_n(x,t) = \left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right) \sin\frac{n\pi}{L}x$$

$(n = 1, 2, \ldots)$

eigenftns

$\lambda_n = cn\pi/L$ eigenvalues

$\sin\frac{n\pi x}{L} = 0$ at $x = \frac{L}{n}, \frac{2L}{n}, \ldots \frac{n+1}{n} L$

See Fig 284

Fig 285.

MOOKEUK

Soln of entire prob. Fourier Series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos\lambda_n t + B_n^* \sin\lambda_n t\right) \sin\frac{n\pi x}{L}$$

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x = f(x)$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} B_n^* \cdot \lambda_n \cdot \sin\frac{n\pi x}{L} = g(x)$$

$$= \left[\sum_{n=1}^{\infty}\left(-B_n\lambda_n \sin\lambda_n t + B_n^*\lambda_n \cos\lambda_n t\right)\cdot \sin\frac{n\pi x}{L}\right]_{t=0}$$

$$B_n^* = \frac{2}{c_n\pi}\int_0^L g(x)\cdot \sin\frac{n\pi x}{L}\,dx$$

→ ~~$u(x,t)$~~

Setting the initial velocity $g(x) \equiv 0$ for simplicity then $B_n^* = 0$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos\lambda_n t \cdot \sin\frac{n\pi x}{L}, \quad \lambda_n = \frac{c_n\pi}{L}$$

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \cdot \sin\left\{\frac{n\pi}{L}(x-ct)\right\}$$

$$+ \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x+ct)\right\}$$

$$u(x,t) = \frac{1}{2}\left\{f^*(x-ct) + f^*(x+ct)\right\}$$

MOOKEUK

$$u_{tt} = c^2 u_{xx}$$

$v = x + ct$

$w = x - ct$

HW

12.3 e1

prob 14

$$u_x = u_v v_x + u_w w_x = u_v + u_w$$

$$u_{xx} = (u_v + u_w)_x = (u_v + u_w)_v v_x + (u_v + u_w)_w w_x$$

$$= u_{vv} + 2u_{vw} + u_{ww}$$

results in

$$u_{tt} = c^2 (u_{vv} - 2u_{vw} + u_{ww})$$

$$\rightarrow \quad u_{vw} \equiv \frac{\partial^2 u}{\partial w \, \partial v} = 0$$

$$\frac{\partial u}{\partial v} = h(v) \quad \text{and} \quad u = \int h(v)\, dv + \psi(w)$$

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

d'Alembert's soln

w/ I.C

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

$u(x,t) = \phi(x+ct) + \psi(x-ct)$

$u_t = c\phi'(x+ct) - c\psi'(x-ct)$

✓

$u(x,0) = \phi(x) + \psi(x) = f(x)$

$u_t(x,0) = c\phi'(x) - c\psi'(x) = g(x)$

$\phi = \psi = \frac{k}{2} + \frac{1}{c}\int_{x_0}^{x} g(s)\,ds$

$k(x_0) = \phi(x_0) - \psi(x_0)$

$\phi(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_{x_0}^{x} g(s)ds + \frac{1}{2}k(x_0)$ → $(x \to x+ct)$

$\psi(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_{x_0}^{x} g(s)ds - \frac{1}{2}k(x_0)$ → $(x \to x-ct)$

$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$

if $u(0,0) = 0$

then, $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]$

MOOKEUK

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u_x, u_y)$$

v quasilinear.

three types of PDEs, dep. on $AC - B^2$

$v = \Phi$, $w = \psi$	hyperbolic	$AC - B^2 < 0$	wave eqn	$u_{vw} = F_1$
$v = x$, $w = \Phi = \psi$		$= 0$	heat eqn	$u_{ww} = F_2$
$v = \frac{\Phi + \psi}{2}$, $w = \frac{\Phi - \psi}{2i}$	Elliptic	> 0	Laplace eqn.	$u_{vv} + u_{ww} = F_3$

12.5 Heat Eqn: Soln by Fourier Series

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u, \quad c^2 = \frac{K}{\sigma \rho}$$

K: thermal conductivity
σ: specific heat
ρ: density

Ex 3.4
prob. 6, 18, 30

B.C $u(0,t) = 0$
$u(L,t) = 0$ for all t

I.C $u(x,0) = f(x)$

$$u = F \cdot G = F(x) \cdot G(t)$$

$$F\dot{G} = c^2 F'' G$$

$$\frac{\dot{G}}{c^2 G} = \frac{F''}{F} = -p^2$$

$$\begin{cases} F'' + p^2 F = 0 \\ \dot{G} + c^2 p^2 G = 0 \end{cases}$$

$$F(x) = A\cos px + B\sin px$$

w/ B.C,

$\sin pL = 0$ hence $p = \frac{n\pi}{L}$, $n = 1, 2 \ldots$

$$F_n(x) = \sin\frac{n\pi x}{L}, \quad n = 1, 2 \ldots$$

$$\dot{G} + \lambda_n^2 G = 0 \quad \text{where } \lambda_n = \frac{cn\pi}{L}$$

$$G_n(t) = B_n \cdot e^{-\lambda_n^2 t}$$

$$u_n(x,t) = F_n(x)\cdot G_n(x) = B_n \sin\frac{n\pi x}{L} \cdot e^{-\lambda_n^2 t}$$

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\frac{n\pi x}{L} \cdot e^{-\lambda_n^2 t}$$

$$\left(\lambda_n = \frac{cn\pi}{L}\right)$$

w/ I.C

$$f(x) = \sum_{n=1}^{\infty} B_n \cdot \sin\frac{n\pi x}{L}$$

$$\rightarrow B_n = \frac{2}{L}\int_0^L f(x)\cdot \sin\frac{n\pi x}{L}\,dx$$

Ex 1. $L = 80$ cm.

I. Temp. $= 100 \sin(\pi x/80)$ °C

ends $= 0$°C

How long for the max T to 50°C?

given ρ, σ, K.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \cdot \sin\frac{n\pi x}{80} = f(x) = 100 \sin\frac{\pi x}{80}$$

$B_1 = 100$, $B_2 = B_3 = \cdots = 0$

$\lambda_1^2 = c^2\pi^2/L^2 = 0.001785$ sec^{-1}

$$\rightarrow c^2 = \frac{K}{\sigma\rho}$$

$$\therefore u(x,t) = 100 \sin\frac{\pi x}{80} \cdot e^{-0.001785t}$$

Ex 2. HW

from $n=1$ to $n=3$

Ex 3.

MOOKEUK

$$B_n = \frac{2}{L}\left(\int_0^{L/2} x \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}(L-x)\sin\frac{n\pi x}{L}\,dx\right)$$

for even n, $B_n = 0$

$n = 1, 5, 9 \quad B_n = \frac{4L}{n^2\pi^2}$

$n = 3, 7, 11 \quad B_n = -\frac{4L}{n^2\pi^2}$

$$\therefore u(x,t) = \frac{4L}{\pi^2}\left[\sin\frac{\pi x}{L}\exp\left[-\left(\frac{c\pi}{L}\right)^2 t\right] - \frac{1}{9}\sin\frac{3\pi x}{L}\exp\left[-\left(\frac{3c\pi}{L}\right)^2 t\right] + \cdots\right]$$

Ex 4. HW

insulated : $u_x(0,t) = 0$ or $u_x(L,t) = 0$.

Ex 5.

$$u_n(x,t) = F_n(x)\cdot G_n(x) = A_n \cdot \cos\frac{n\pi x}{L}\cdot e^{-\lambda_n^2 t}$$

$$u(x,t) = \sum u_n = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\cdot e^{-\lambda_n^2 t}$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

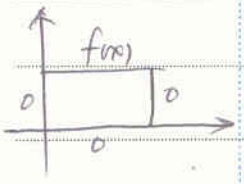
$$A_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2\cdots$$

$$A_n = \frac{2}{L}\left[\int_0^{L/2} x\cdot\cos\frac{n\pi x}{L}\,dx + \int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}\,dx\right] = \frac{2L}{n^2\pi^2}\left(2\cos\cdots\right)$$

$$u(x,t) = \sum u_n$$

Steady 2-d heat Prob. Laplace prob.

B.C.



$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

steady state app. $\frac{\partial u}{\partial t} = 0$

$\nabla^2 u = 0 \Rightarrow$ Sep. var. $u = F(x)\,G(y)$ $\quad = -k^2$

$\Rightarrow u_{xx} = -u_{yy} \Rightarrow \frac{F_{xx}}{F} = -\frac{G_{yy}}{G}$

$\Rightarrow \frac{d^2F}{dx^2} + k^2 F = 0, \quad \frac{d^2G}{dx^2} - k^2 G = 0$

First BVP - Dirichlet Prob.

u is prescribed on C.

Second BVP - Neumann Prob.

$u_n = \frac{\partial u}{\partial n}$ is on C.

normal derivative

3rd BVP - Mixed BVP, Robin Prob.

part of u on C

part of u_n on the rest of C

$$\frac{d^2F}{dx^2} + k^2 F = 0, \quad k = \frac{n\pi}{a}$$

if B.C is $F(0) = 0, \; F(a) = 0.$

$$F(x) = F_n(x) = \sin \frac{n\pi}{a} x$$

MOOKEUK

$$\frac{d^2G}{d^2y} - \left(\frac{n\pi}{a}\right)^2 G = 0$$

$$G(y) \equiv G_n(y) = B_n \cdot e^{n\pi y/a} + B_n \cdot e^{-n\pi y/a}$$

$$G_n(0) = 0, \quad G_n(0) = A_n + B_n = 0$$

$$G_n(y) = \underbrace{2 A_n}_{A_n^*} \cdot \sinh \frac{n\pi y}{a}$$

$$= A_n^* \sinh \frac{n\pi y}{a}$$

$$u_n = F_n(x) \cdot G_n(y) = A_n^* \cdot \sin \frac{n\pi x}{a} \cdot \sinh \frac{n\pi y}{a}$$

$$u = \sum u_n = \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \cdot \sinh \frac{n\pi b}{a}$$

another B.C., $u(x, b) = f(x)$

$$= \sum_{n=1}^{\infty} A_n^* \cdot \sin \frac{n\pi x}{a} \cdot \sinh \frac{n\pi b}{a}$$

$$b_n = A_n^* \cdot \sinh \frac{n\pi b}{a}$$

$$= \frac{2}{a} \int_0^a f(x) \cdot \sin \frac{n\pi x}{a} dx$$

$$\therefore u(x, y) = \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} A_n^* \cdot \sin \frac{n\pi x}{a} \cdot \sinh \frac{n\pi y}{a}$$

$$A_n^* = \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \cdot \sin \frac{n\pi x}{a} dx$$

NO.

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \omega_n x + b_n \cdot \sin \omega_n x \right)$$

$$\omega_n = \frac{n\pi}{L}$$

$$f_L(x) = \frac{1}{2L}\int_{-L}^{L} f_L(v)\,dv + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos \omega_n x \cdot \int_{-L}^{L} f_L(v) \cdot \cos \omega_n v \, dv \right.$$

$$\left. + \sin \omega_n x \cdot \int_{-L}^{L} f_L(v) \sin \omega_n v \, dv \right]$$

$$\Delta \omega_n = \frac{\pi}{L}, \qquad \omega_n = \frac{n\pi}{L}$$

$L \to \infty$

$$f(x) = \int_0^{\infty} \left[A(\omega) \cdot \cos \omega x + B(\omega) \cdot \sin \omega x \right] d\omega$$

$$A(\omega) = \int_{-\infty}^{\infty} f(v) \cdot \cos \omega v \, dv$$

$$B(\omega) = \int_{-\infty}^{\infty} f(v) \cdot \sin \omega v \, dv$$

MOOKEUK

12.6 Heat eqn.

Soln by Fourier Integrals & Transforms

Ex ~~prob~~ 3, 4
prob 2

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

only I.C., but no B.C.

$u(x,0) = f(x)$

then, f doesn't have to be periodic ftn.

$$F'' + p^2 F = 0$$

$$\dot{G} + c^2 p^2 G = 0$$

$$F(x) = A\cos px + B\sin px$$

$$G(t) = e^{-c^2 p^2 t}$$

$$u = FG = (A\cos px + B\sin px)\cdot e^{-c^2 p^2 t}$$

since f doesn't have to be periodic,

then,

$$u = \int_0^\infty u(x,t;p)\,dp = \int_0^\infty [A(p)\cos px + B(p)\sin px] \cdot e^{-c^2 p^2 t}\,dp$$

from the IC,

$$u(x,0) = \int_0^\infty (A(p)\cdot\cos px + B(p)\cdot\sin px)\,dp = f(x)$$

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cdot\cos pv\,dv$$

$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cdot\sin pv\,dv$$

$$u(x,0) = \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cdot(\cos pv\cos px + \sin pv\cdot\sin px)\,dv\right]dp$$

$$= \frac{1}{\pi}\int_0^\infty \left[\int_{-\infty}^{\infty} f(v)\cdot\cos(px - pv)\,dv\right]dp$$

MOOKEUK

$$u(x,t) = \frac{1}{\pi}\int_0^{\infty}\left[\int_{-\infty}^{\infty} f(v)\cdot\cos(px-pv)\cdot e^{-c^2p^2t}\,dv\right]dp$$

reversing the order of integration

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left(\int_0^{\infty}\cos(px-pv)\cdot e^{-c^2p^2t}\,dp\right)dv$$

using $\int_0^{\infty} e^{-s^2}\cdot\cos 2bs\,ds = \frac{\sqrt{\pi}}{2}\cdot e^{-b^2}$

Setting $p = \frac{s}{c\sqrt{t}}$ and $b = \frac{x-v}{2c\sqrt{t}}$

then, $\int_0^{\infty} e^{-c^2p^2t}\cos(px-pv)\,dp = \frac{\sqrt{\pi}}{2c\sqrt{t}}\cdot\exp\left\{-\frac{(x-v)^2}{4c^2t}\right\}$

$\therefore\ u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(v)\cdot\exp\left\{-\frac{(x-v)^2}{4c^2t}\right\}dv$

$z = \frac{v-x}{2c\sqrt{t}}$

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2cz\sqrt{t})\cdot e^{-z^2}\cdot dz$$

Ex 1. $f(x) = \begin{cases} u_0 : \text{const} & \text{if } |x|<1 \\ 0 & \text{if } |x|>1 \end{cases}$

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{\frac{-1-x}{2c\sqrt{t}}}^{\frac{+1-x}{2c\sqrt{t}}} u_0\cdot e^{-z^2}\,dz$$

Use of Fourier Transforms

Ex 2. I.C. $f(x) = u_0 = \text{const}$ if $|x| < 1$ even ftn.

0 otherwise.

$$\mathcal{F}(u_t) = c^2 \mathcal{F}(u_{xx}) = c^2 \cdot (-\omega^2) \cdot \mathcal{F}(u) = -c^2 \omega^2 \hat{u}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\omega x} dx$$

$$= \frac{\partial}{\partial t} \cdot \mathcal{F}(u) = \frac{\partial}{\partial t} \hat{u}$$

$$\Rightarrow \quad \frac{\partial \hat{u}}{\partial t} = -c^2 \omega^2 \hat{u}$$

$$\hat{u} = D(\omega) \cdot e^{-c^2 \omega^2 t} = \hat{f}(\omega) \cdot e^{-c^2 \omega^2 t}$$

$$\mathcal{F}^{-1}(\hat{u}) = u = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \cdot e^{-c^2 \omega^2 t} \cdot e^{i\omega x} d\omega$$

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) \cdot e^{-i\omega v} dv$$

$$u(x,t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} f(v) \cdot \left(\int_{-\infty}^{\infty} e^{-c^2 \omega^2 t} \cdot e^{i\omega(x-v)} d\omega \right) dv$$

Since $f(v)$ is even ftn, only cos ftn (or even ftn survives or real part)

$$\therefore \quad u(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cdot \left[\int_{0}^{\infty} e^{-c^2 \omega^2 t} \cdot \cos(\omega x - \omega v) d\omega \right] dv$$

NO.

Ex3. by convolution

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{\infty} \hat{f}(w) \cdot e^{-c^2w^2t} \cdot e^{iwx} dw$$

$$= f * g = \int_{-\infty}^{\infty} \hat{f}(w) \cdot \hat{g}(w) \cdot e^{iwx} dw \Rightarrow \mathcal{F}(f) \cdot \mathcal{F}(g)$$

$$\hat{g}(w) = \frac{1}{\sqrt{2\pi}} \cdot e^{-c^2t \cdot w^2}$$

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} e^{-w^2/4a} \qquad (a = \tfrac{1}{4c^2t})$$

$$g(x) = \frac{1}{\sqrt{2c^2t}\sqrt{2\pi}} e^{-\frac{x^2}{4c^2t}}$$

$$\left(= \sqrt{2a} \cdot \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1}(e^{-c^2tw^2}) \right)$$

$$u(x,t) = f * g = \int_{-\infty}^{\infty} f(p) \cdot g(x-p)\, dp$$

$$= \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(p) \cdot \exp\left\{ -\frac{(x-p)^2}{4c^2t} \right\} \cdot dp$$

12.8 Rectangular Membrane NO. 12.8.-1

Double Fourier Series

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

$u = 0$ on the boundary

$u(x, y, 0) = f(x, y)$

$u_t(x, y, 0) = g(x, y)$

sol) $u(x, y, t) = F(x, y) \cdot G(t)$

$$F \cdot \ddot{G} = c^2(F_{xx} \cdot G + F_{yy} \cdot G)$$

$$\frac{\ddot{G}}{c^2 G} = \frac{F_{xx} + F_{yy}}{F} = -\nu^2$$

($\ddot{G} + c^2\nu^2 G = 0$ time ftn

$F_{xx} + F_{yy} + \nu^2 F = 0$ amplitude ftn

$F = H(x) \cdot Q(y)$

$$\frac{d^2H}{dx^2} \cdot Q + H \cdot \frac{d^2Q}{dy^2} + \nu^2 H \cdot Q = 0$$

$$\underbrace{\frac{1}{H} \cdot \frac{d^2H}{dx^2}}_{-k^2} + \underbrace{\frac{1}{Q} \cdot \frac{d^2Q}{dy^2} + \nu^2}_{k^2} = 0$$

$$\frac{d^2H}{dx^2} + k^2 H = 0 \quad , \quad \frac{d^2Q}{dy^2} + \underbrace{(\nu^2 - k^2)}_{p^2} Q = 0$$

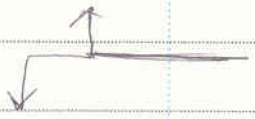
MOOKEUK

$$H(x) = A\cos kx + B\sin kx, \quad \begin{cases} H(0) = H(a) = 0 \\ k = \frac{n\pi}{a}, \ A = 0 \end{cases}$$

$$Q(y) = C\cos py + D\sin py, \quad \begin{cases} Q(0) = Q(b) = 0 \\ p = \frac{n\pi}{b}, \ C = 0 \end{cases}$$

$$T_{mn}(x,y) = H_m(x) \cdot Q_n(y)$$

$$= \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b} \quad \begin{cases} m = 1, 2, \cdots \\ n = 1, 2, \cdots \end{cases}$$



$$\lambda = cv = c\sqrt{k^2 + p^2}$$

$$k = \frac{m\pi}{a}, \quad p = \frac{n\pi}{b}$$

$$\lambda = \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \quad G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B_{mn}^{*}\sin\lambda_{mn}t$$

$$u_{mn}(x,y,t) = \left(B_{mn}\cos\lambda_{mn}t + B_{mn}^{*}\sin\lambda_{mn}t\right)\sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}$$

Ex1. ~~$\lambda_{mn} = \lambda_{nm}$~~

ex) $\lambda_{12} = \lambda_{21} = c\pi\sqrt{5}$ → see Fig 300

$$u_{12} = \left(B_{12}\cos c\pi\sqrt{5}t + B_{12}^{*}\sin c\pi\sqrt{5}t\right)F_{12}$$

$$u_{21} = \left(B_{21}\cos c\pi\sqrt{5}t + B_{12}^{*}\sin c\pi\sqrt{5}t\right)\cdot F_{12}$$

MOOKEUK

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(B_{mn}\cdot\cos\lambda_{mn}t + B_{mn}^{*}\cdot\sin\lambda_{mn}t\right)\cdot\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}$$

at $t=0$, $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\cdot\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b} = f(x,y)$

(double Fourier series of $f(x,y)$

setting $K_m(y) = \sum_{n=1}^{\infty} B_{mn}\cdot\sin\frac{n\pi y}{b}$

then $f(x,y) = \sum_{m=1}^{\infty} K_m(y)\cdot\sin\frac{m\pi x}{a}$

$$K_m(y) = \frac{2}{a}\int_0^a f(x,y)\cdot\sin\frac{m\pi x}{a}\,dx$$

$$B_{mn} = \frac{2}{b}\int_0^b K_m(y)\cdot\sin\frac{n\pi y}{b}\,dy$$

then, $$B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\cdot\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\cdot dx\cdot dy$$

$m = 1, 2 \cdots$

$n = 1, 2 \cdots$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum\sum B_{mn}^{*}\cdot\lambda_{mn}\cdot\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b} = g(x,y)$$

$$B_{mn}^{*} = \frac{4}{ab\lambda_{mn}}\int_0^b\int_0^a g(x,y)\cdot\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}\,dx\,dy$$

Ex 2. Vib of a rectangular memb.

$a = 4\,ft,\ b = 2\,ft,\ T = 12.5,\ \rho = 2.5$

$\dot{u}_{t=0} = 0 \rightarrow B^*_{mn} = 0$

$u_{t=0} = f(x,y)$

$$\rightarrow B_{mn} = \frac{4}{ab}\int_0^b\int_0^a f \cdot \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}\,dx\,dy$$

$$= \frac{1}{20}\int_0^4\left(\int_0^2 (4x - x^2)\cdot \sin\frac{m\pi x}{4}\,dx\right)\cdot(2y - y^2)\sin\frac{n\pi y}{2}\,dy$$

For even m or n, the integral is '0'.

For odd m, $\dfrac{256}{m^3\pi^3}$

odd n, $\dfrac{32}{n^3\pi^3}$

$$\rightarrow B_{mn} \simeq \frac{0.426}{m^3n^3}$$

$$\therefore u = 0.426 \sum_{m,n\ \text{odd}}\sum \frac{1}{m^3n^3}\cos\left(\frac{\sqrt{5}\pi}{4}\cdot\sqrt{m^2+4n^2}\right)t \cdot \sin\frac{m\pi x}{4} \cdot \sin\frac{n\pi y}{2}$$

12.9 Laplacean in Polar Coordinates

Circular Membrane

prob. 4.18 Fourier - Bessel Series

$$u_{tt} = c^2 \nabla^2 u$$

$$u(r,\theta)_x = u_r \cdot r_x + u_\theta \cdot \theta_x$$

$$\begin{aligned} u_{xx} &= (u_r \cdot r_x)_x + (u_\theta \cdot \theta_x)_x \\ &= (u_r)_x \cdot r_x + u_r \cdot r_{xx} + (u_\theta)_x \cdot \theta_x + u_\theta \cdot \theta_{xx} \\ &= (u_{rr} r_x + u_{r\theta} \cdot \theta_x) \cdot r_x + u_r \cdot r_{xx} \\ &\quad + (u_{\theta r} \cdot r_x + u_{\theta\theta} \cdot \theta_x) \cdot \theta_x + u_\theta \cdot \theta_{xx} \end{aligned}$$

$$r = \sqrt{x^2+y^2}, \quad r_x = \frac{x}{r}, \quad r_{xx} = \frac{r - x \cdot r_x}{r^2} = \frac{y^2}{r^3}$$

$$\tan\theta = \frac{y}{x}, \quad \theta_x = -\frac{y}{r^2}, \quad \theta_{xx} = \frac{2xy}{r^4}$$

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$$

2-d wave eqn in polar coordinates

$$u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} \cdot u_{\theta\theta} \right)$$

for radially symmetric sol'n, $u_{\theta\theta} = 0$

$$u_{tt} = c^2 \left(u_{rr} + \frac{1}{r} u_r \right)$$

B.C. $u(R,t) = 0$

I.C. $u(r,0) = f(r)$

$u_t(r,0) = g(r)$

MOOKEUK

$W(r) \cdot G(t)$

Let $u = WG$

$$\frac{\ddot{G}}{c^2 G} = \frac{1}{W}\left(W'' + \frac{1}{r}W'\right) = -k^2$$

$$G_{tt} + c^2 k^2 \cdot G = 0$$

$$W_{rr} + \frac{1}{r}W_r + k^2 W = 0$$

let $s = kr$

$W_{rr} = W_{ss} \cdot k^2$, $W_r = W_s \cdot k$

$$W_{ss} + \frac{1}{s}W_s + W = 0$$

Bessel's eqn $(\nu = 0)$

$$W(r) = J_0(s) = J_0(kr)$$

Bessel fun 1st kind.

~~$W(R) = J_0(kR) = 0$~~

$kR = \alpha_m$ thus $k = k_m = \frac{\alpha_m}{R}$

$$W_m(r) = J_0(k_m r) = J_0\left(\frac{\alpha_m}{R} \cdot r\right)$$

MOOKEUK

$G_m(t) = A_m \cos\lambda_m t + B_m \sin\lambda_m t$

$u_m(r,t) = W_m(r) \cdot G_m(t) = (A_m \cos\lambda_m t + B_m \sin\lambda_m t) \cdot J_0(k_m r)$

Eigen ftns.

λ_m : eigen-values

u_m : mth normal mode

freq $\lambda_m / 2\pi$

Fig 3.05 , zeros of the J_0 are not regularly spaced.

(note. zeros of sin fn are regularly spaced)

The sound of a drum is entirely different.

Soln of the Entire problem

$$u(r,t) = \sum_{m=1}^{\infty} u_m = \sum_{m=1}^{\infty} W_m(r) \cdot G_m(t)$$

$$= \sum_{m=1}^{\infty} (A_m \cos\lambda_m t + B_m \cdot \sin\lambda_m t) \cdot J_0\left(\frac{\alpha_m}{R} \cdot r\right)$$

$t=0$, then $u(r,0) = \sum_{m=1}^{\infty} A_m \cdot J_0\left(\frac{\alpha_m}{R} \cdot r\right) = f(r)$

$$A_m = \frac{2}{R^2 \cdot J_1^2(\alpha_m)} \cdot \int_0^R r \cdot f(r) \cdot J_0\left(\frac{\alpha_m}{R} r\right) dr$$

Fourier-Bessel series

12.11 Soln by L.T.

Ex 1. Semi-infinite string.

I.C $t=0$, $w=0$, $w_t(x,0)=0$

B.C. $\begin{cases} t>0,\ w(0,t)=f(t)=\begin{cases} \sin t & 0\le t\le 2\pi \\ 0 & \text{otherwise} \end{cases} \\ \lim_{x\to\infty} w = 0 \end{cases}$

Soln) $w_{tt} = c^2 w_{xx}$

↓ L.T. wrt time

$$\mathcal{L}\{w_{tt}\} = s^2\cdot\mathcal{L}(w) - s\,w(x,0) - w_t(x,0)$$

$$\mathcal{L}\{w_{xx}\} = \frac{d^2}{dx^2}\mathcal{L}(w) \quad \left(\because \int_0^\infty e^{-st}\frac{\partial^2 w}{\partial x^2}\,dt = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st}\cdot w\,dt\right)$$

$$s^2\cdot W = c^2\cdot\frac{\partial^2 W}{\partial x^2} \Rightarrow W_{xx} - \frac{s^2}{c^2}\cdot W = 0$$

$$W(x,s) = A(s)\,e^{\frac{s}{c}x} + B(s)\cdot e^{-\frac{s}{c}x}$$

$$W(0,s) = \mathcal{L}\{w(0,t)\} = \mathcal{L}\{f(t)\} = F(s)$$

considering $\lim_{x\to\infty} W(x,s) = \lim_{x\to\infty}\int_0^\infty e^{-st}\cdot w(x,t)\,dt$

$$= \int_0^\infty e^{-st}\cdot\lim_{x\to\infty} w(x,t)\,dt = 0$$

$\to A(s)=0$ since $c>0$.

$$W(0,s) = B(s) = F(s)$$

$W(x,s) = F(s) \cdot e^{-sx/c}$

$w(x,t) = f(t-\frac{x}{c}) \cdot u(t-\frac{x}{c})$

that is

$w(x,t) = \sin(t-\frac{x}{c})$ if $\frac{x}{c} < t < \frac{x}{c} + 2\pi$

or $ct > x > (t-2\pi)\cdot c$

MOOKEUK

12.10 Laplace Eqn in Cylindrical & Spherical Coordinates.

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0$$

eq) st. st. heat flow or fluid flow.

$$u = \frac{c}{r}$$

$$u = k \iiint_T \frac{\rho}{r} \, dx\,dy\,dz$$

u on S I) 1st boundary value problem or Dirichlet problem:

u_n on S II) 2nd boundary value problem or Neumann problem

u or u_n on S III) mixed b.v.p

$$\nabla^2 u = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} + u_{zz} \quad \text{cylindrical coordinate}$$

Spherical coordinate

$$= u_{rr} + \frac{2u_r}{r} + \frac{u_{\phi\phi}}{r^2} + \frac{\cot\phi \cdot u_\phi}{r^2} + \frac{u_{\theta\theta}}{r^2 \sin^2\phi}$$

or

$$= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi} \cdot \frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi} \cdot \frac{\partial^2 u}{\partial \theta^2}\right]$$

$$= \frac{1}{r^2}\left[(r^2 u_r)_r + \frac{(\sin\phi \cdot u_\phi)_\phi}{\sin\phi} + \frac{u_{\theta\theta}}{\sin^2\phi}\right]$$

MOOKEUK

BVP in spherical coordinate

$$\nabla^2 u = 0 = \frac{1}{r^2}\left[(r^2 u_r)_r + \frac{(\sin\phi\, u_\phi)_\phi}{\sin\phi} \right] = 0$$

$$u(R,\phi) = f(\phi)$$

$$\lim_{r\to\infty} u(r,\phi) = 0$$

sol) Sep. Var. $u(r,\phi) = G(r)\cdot H(\phi)$

$$\underbrace{\frac{(r^2 G_r)_r}{G}}_{k} + \underbrace{\frac{(\sin\phi\cdot H_\phi)_\phi}{H\cdot\sin\phi}}_{=-k} = 0$$

$$(r^2 G_r)_r = kG$$

$$\hookrightarrow r^2 G_{rr} + 2r G_r - kG = 0$$

$k = n(n+1)$

$$r^2 G_{rr} + 2r G_r - n(n+1)G = 0$$

~~Euler-Cauchy eqn.~~

$$G = r^a$$

$$a(a-1) + 2a - n(n+1) = 0. \qquad a = n \text{ and } -n-1$$

$$\therefore G_n(r) = r^n \text{ and } G_n^*(r) = \frac{1}{r^{n+1}}$$

MOOKEUK

$$\frac{(\sin\phi \cdot H_\phi)_\phi}{H \cdot \sin\phi} = -k$$

$$\cos\phi \cdot H_\phi + \sin\phi \cdot H_{\phi\phi} + kH\sin\phi = 0$$

$$H_{\phi\phi} + \frac{\cos\phi}{\sin\phi} \cdot H_\phi + kH = 0$$

$$\rightarrow \sin^2\phi \, H_{\omega\omega} - \omega H_\omega + kH = 0$$

Let $\cos\phi = \omega$

then $\frac{d}{d\phi} = \frac{d}{d\omega}\frac{d\omega}{d\phi} = -\sin\phi \cdot \frac{d}{d\omega}$

$$+\sin\phi \left[(\omega^2-1)\cdot H_\omega\right]_\omega = +kH\sin\phi$$

$$\left[(1-\omega^2)\cdot H_\omega\right]_\omega + kH = 0$$

$$\left[(1-\omega^2)\cdot H_\omega\right]_\omega + n(n+1)\cdot H = 0$$

Legendre's eqn,

$$(1-\omega)^2 \cdot H_{\omega\omega} - 2\omega \cdot H_\omega + n(n+1)\cdot H = 0$$

$$H = P_n(\omega) = P_n(\cos\phi)$$

$\therefore$ $u_n(r,\phi) = A_n \cdot r^n \cdot P_n(\cos\phi)$ inside the sphere

$u_n^*(r,\phi) = \frac{B_n}{r^{n+1}} \cdot P_n(\cos\phi)$ outside the sphere

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n \cdot r^n \cdot P_n(\cos\phi)$$

at $r=R$

$$u(R,\phi) = \sum_{n=0}^{\infty} A_n \cdot R^n \cdot P_n(\cos\phi) = f(\phi)$$

Fourier-Legendre Series

$$A_n R^n = \frac{2n+1}{2} \int_{-1}^{1} \tilde{f}(\omega) \cdot P_n(\omega)\, d\omega$$

$$A_n = \frac{2n+1}{2 \cdot R^n} \int_0^{\pi} f(\phi) \cdot P_n(\cos\phi) \cdot \sin\phi\, d\phi$$

outside the sphere

$$u(r,\phi) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} \cdot P_n(\cos\phi)$$

$$B_n = \frac{2n+1}{2} \cdot R^{n+1} \int_0^{\pi} f(\phi) \cdot P_n(\cos\phi) \cdot \sin\phi\, d\phi$$

Ex1. $f(\phi) = \begin{cases} 110 & \text{if } 0 \le \phi \le \pi/2 \\ 0 & \text{if } \pi/2 < \phi \le \pi \end{cases}$

$$A_n = \frac{2n+1}{2} \cdot 110 \int_0^{\pi/2} P_n(\cos\phi) \cdot \sin\phi\, d\phi$$

$$= \frac{2n+1}{2} \cdot 110 \int_0^{1} P_n(\omega)\, d\omega$$

$\vdots$

MOOKEUK

Bessel $r^2\ddot{y} + r\dot{y} + (r^2 - \nu^2)y = 0$ J_ν Y_ν

Legendre $(1-r^2)\ddot{y} - 2r\dot{y} + n(n+1)y = 0$ P_n (nth power)

$(C_1 y_1 + C_2 y_2)$ odd even

Euler-Cauchy $r^2\ddot{y} + 2r\dot{y} - n(n+1)y = 0$ $y = r^n$

Sturm-Liouville Prob.

$$[p(x)\cdot y']' + [q(x) + \lambda \cdot r(x)]y = 0$$

orthogonal $\int r y_m y_n = 0$

Rodrigues formula $P_n(x) = \frac{1}{2^n \cdot n!} \cdot [(x^2-1)^n]^{(n)}$

MOOKEUK

13.1, 13.2, 13.3

NO.

polar form

$$z = r(\cos\theta + i\sin\theta)$$

or

$$r\angle\theta$$

absolute value or modulus $|z| = \sqrt{z\bar{z}}$

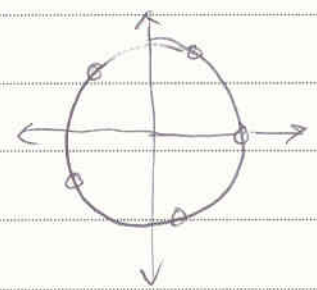
argument $\theta = \arg(z) = \arctan\frac{y}{x}$

triangular inequality

p611

$$z^n = r^n\angle n\theta$$

$$= r^n(\cos n\theta + i\sin n\theta)$$



NO.

analytic fn:

differentiable in some domain

(Note) $\bar{z}$ doesnot have a derivative.

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{\overline{z+\Delta z}+\bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

path dep.

13.4 Cauchy - Riemann Eqns.

Laplace eqn

* Cauchy - Riemann eqn : analyticity test

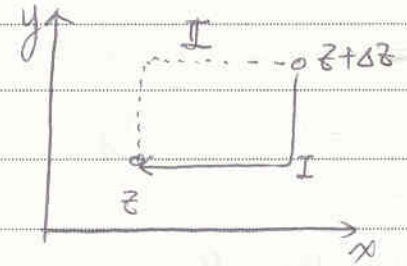
$$w = f(z) = u(x,y) + i\,v(x,y)$$

analytic in a domain D : $u_x = v_y$

$u_y = -v_x$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

path I or II ?



for path I , $\Delta z = \Delta x$ (1st $\Delta y \to 0$, 2nd $\Delta x \to 0$)

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= u_x + i\,v_x$$

for path II , $\Delta z = i\Delta y$ (1st $\Delta x \to 0$, 2nd $\Delta y \to 0$)

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x,y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x,y+\Delta y) - v(x,y)}{i\Delta y}$$

$$= v_y - i\,u_y$$

Ex 2. $f(z) = e^x(\cos y + i \sin y)$ analytic?

$u_x = e^x \cdot \cos y$, $v_y = e^x \cdot \cos y$

$u_y = -e^x \cdot \sin y$, $v_x = e^x \cdot \sin y$

Ex 3. if $|f(z)| = k = \text{const}$ in D, then $f(z) = \text{const}$

$|f|^2 = |u+iv|^2 = u^2 + v^2 = k^2$

$u \cdot u_x + v \cdot v_x = 0$

$u \cdot u_y + v \cdot v_y = 0$

$\longrightarrow$

$u \cdot u_x + v \cdot u_y = 0$ — ①

$u \cdot u_y + v \cdot u_x = 0$ — ②

① × u + ② × v = $(u^2 + v^2) \cdot u_x = 0$

① × v − ② × u = $-(u^2 + v^2) \cdot v_y = 0$

$u^2 + v^2 = 0$ or $u_x = u_y = 0$

$(v_x = v_y = 0)$

$u = v = 0$ | $u = \text{const}$ & $v = \text{const}$

$f = \text{const}$ | $f = \text{const}$

MOOKEUK

Thm 3. Laplace's Eqn.

if $f(z) = u + iv$ is analytic in a domain D, then both u and v satisfy

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$
$$\nabla^2 v = v_{xx} + v_{yy} = 0$$

→ sol'n: harmonic ftns

v is harmonic conjugate ftn of u.

proof) $(u_x)_x = (v_y)_x$ $\quad (u_x)_y = (v_y)_y$

$(u_y)_y = (-v_x)_y$ $\quad (u_y)_x = (-v_x)_x$

$u_{xx} + u_{yy} = 0$ $\quad v_{xx} + v_{yy} = 0$

Ex 4. verify if $u = x^2 - y^2 - y$ is harmonic or not. find a harmonic conjugate ftn v of u.

$\nabla^2 u = 0$ → harmonic.

$u_x = 2x = v_y$. $\quad v = 2xy + h(x)$

$v_x = -u_y = 2y + 1$ $\quad v_x = 2y + h'(x)$

$h'(x) = 1$

$h(x) = x + C$

$v = 2xy + x + C$

13.5 nothing

NO.

13.6 Trigonometric & Hyperbolic fns

$$e^{iz} = \cos z + i \sin z$$

$$\cos z = \cos x \cdot \cosh y - i \sin x \cdot \sinh y$$
$$\sin z = \sin x \cdot \cosh y + i \cos x \cdot \sinh y$$

$$\cosh z = \frac{e^z + e^{-z}}{2} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cosh iz = \cos z, \quad \sinh iz = i \sin z$$
$$\cos iz = \cosh z, \quad \sin iz = i \sinh z$$

MOOKEUK

13.7 Logarithm. General Power

$\ln z = ?$

$z = r \cdot e^{i\theta}$

✓ $\ln z = \ln r + i\theta$

✓ $\operatorname{Ln} z = \ln|z| + i\,\operatorname{Arg}(z)$ $\quad -\pi < \operatorname{Arg} z \le \pi$

✓ $\ln z = \operatorname{Ln} z \pm 2n\pi i$

~~for positive real r~~

✓ $\ln(e^z) = z \pm 2n\pi i, \quad n = 0, 1, 2, \cdots$

Ex 1.

$\ln 1 = 0, \pm 2\pi i, \pm 4\pi i \cdots$

$\operatorname{Ln} 1 = 0$

Ex 2. $\quad z_1 = z_2 = e^{\pi i} = -1$

$\ln(z_1 \cdot z_2) = \ln 1 = 2n\pi i \qquad \ln z_1 z_2 = \ln z_1 + \ln z_2$

$\operatorname{Ln}(z_1 \cdot z_2) = \operatorname{Ln} 1 = 0 \qquad \operatorname{Ln} z_1 z_2 \neq \operatorname{Ln} z_1 + \operatorname{Ln} z_2$

MOOKEUK

Thm 1. $(\ln z)' = \frac{1}{z}$, $z = x+iy$

$\ln z = \ln r + i(\theta + c)$ → $2n\pi$

$= \ln\sqrt{x^2+y^2} + i(\arctan\frac{y}{x} + c)$

$u_x = \frac{x}{x^2+y^2} = \frac{1}{1+(\frac{y}{x})^2}\times(\frac{1}{x})'$

$v_x = -u_y = -\frac{y}{x^2+y^2} = \frac{1}{1+(\frac{y}{x})^2}\times(-\frac{1}{y})'$

(y = const ($\Delta y = 0$), and then $\Delta x \to 0$

$(\ln z)' = u_x + i v_x$

$= \frac{x-yi}{x^2+y^2} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{z}$

Ex 3. $i^i = (e^{\ln i})^i = e^{(\frac{\pi}{2}\pm 2n\pi)i\cdot i} = e^{-(\frac{\pi}{2}\pm 2n\pi)}$

$\ln i = \ln(r\cdot e^{i\theta}) = \ln(1\cdot e^{i(\frac{\pi}{2}\pm 2n\pi)})$

$= \ln 1 + i(\frac{\pi}{2}\pm 2n\pi)$

$= 0 + (\frac{\pi}{2}\pm 2n\pi)i$

$e^{-\frac{\pi}{2}}$ — principal value

14. Complex Integration

14.1 Line Integral in the complex plane

$$\int_C f(z)dz$$

1. Linearity
2. Sense reversal
3. Partitioning of path

① Indefinite integration

Ex1. Ex3 참조

Ex 2. $\int_{-\pi i}^{\pi i} \cos z\, dz = \left[\sin z\right]_{-\pi i}^{\pi i} = 2\sin \pi i$

$$= 2\left(\frac{e^{(\pi i)i} - e^{-(\pi i)i}}{2i}\right)$$

$$= \frac{e^{-\pi} - e^{\pi}}{i} = \left(e^{\pi} - e^{-\pi}\right)i$$

when C is simply connected

(every simple closed curve encloses only points of D)

See 14.2.

Ex 4. $\int_{-i}^{i} \frac{dz}{z} = \operatorname{Ln} i - \operatorname{Ln}(-i) = \pi i$

$$\operatorname{Ln} i = \ln|i| + i\operatorname{Arg}(i) = 0 + \frac{\pi i}{2}$$

$$\operatorname{Ln}(-i) = \ln|i| + i\operatorname{Arg}(-i) = 0 - \frac{\pi i}{2}$$

NO.

② Use of a representation of a path.

$$\int_C f(z)\,dz = \int_a^b f(z(t))\cdot \dot{z}(t)\,dt$$

(when C is simply connected,) (no need to be analytic in D)

ex5.) $\oint_C \frac{dz}{z} = 2\pi i$, C unit circle.

$z = e^{it} \quad (0 \le t \le 2\pi)$

$$\int_0^{2\pi} e^{-it}\cdot(e^{it})'\,dt = \int_0^{2\pi} e^{-it}\cdot i\cdot e^{it}\,dt = 2\pi i$$

ex6) $1/z^m$

$$\oint_C (z-z_0)^m\,dz = ?$$

Let $z - z_0 = \rho\cdot e^{it}$

then $\oint_C (z-z_0)^m\,dz = \int_0^{2\pi} \rho^m e^{imt}\cdot(\rho\cdot e^{it})'\,dt$

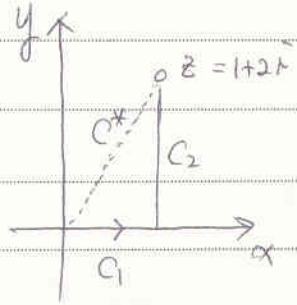
$$= \int_0^{2\pi} \rho^{m+1}\cdot i\cdot e^{it(m+1)}\,dt$$

$m = -1$ $\quad 2\pi i$

$m \neq -1$ $\quad 0$

ex.7)

$f(z) = \mathrm{Re}\, z = x$



a) $\int_{C^*} \mathrm{Re}\, z\, dz = \int_0^1 t \cdot (t+2ti)' \cdot dt = \int_0^1 t(1+2i) \cdot dt$

$C^* = t + 2ti \qquad = \frac{1}{2} + i$

b) $C_1 : t = z \quad 0 \le t \le 1$

$C_2 : 1+ti = z \quad 0 \le t \le 2$

$\int_{C_1} + \int_{C_2} = \int_0^1 t \cdot t' \cdot dt + \int_0^2 1 \cdot (1+ti)' dt$

$= \frac{1}{2} \qquad + 2i$

$= \frac{1}{2} + 2i$

~~~ different

path-dep.

MOOKEUK
~~~

NO.

ML - inequality

$$\left| \int_C f(z)\,dz \right| \leq M \cdot L$$

L: length of C

$|f(z)| \leq M$ everywhere on C.

14.2

proof of Cauchy integral thm

$$\oint_C f(z)\,dz = \oint_C (u\,dx - v\,dy) + i \oint (u\,dy + v\,dx)$$

$f(z) = u + iv$

$dz = dx + i\,dy$

$$\oint (u\,dx - v\,dy) = \iint_R (-v_x - u_y)\,dx\,dy = 0$$

$(\because u_y = -v_x)$

by Green thm

$$\oint (u\,dy + v\,dx) = \iint_R (u_x - v_y)\,dx\,dy = 0$$

$(\because u_x = v_y)$

$$\int_R (\nabla \times \vec{F}) \cdot \hat{k}\,dx\,dy = \oint (F_1\,dx + F_2\,dy)$$

$$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

MOOKEUK

14.2 Cauchy's Integral thm

NO.

1. Simple closed path ⟶ $\oint_C f(z)dz = 0$

2. simply connected domain (analytic and simple)

Fig 343.

if $f(z)$ is analytic and simply connected in domain D,

$f(z)$ is indep. of path.

Ex 1, 2, 3, 4, 5, 6

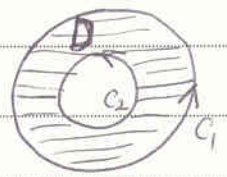
$\oint_C \frac{dz}{z^2} = 0$

analyticity (sufficient)

$\oint_C \frac{dz}{z} = 2\pi i$

analytic in $\frac{1}{2} < |z| < \frac{3}{2}$, not simply connected (necessary)

Cauchy Integral Thm for multiply connected domains

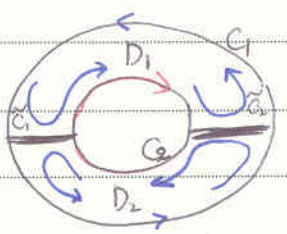


$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$

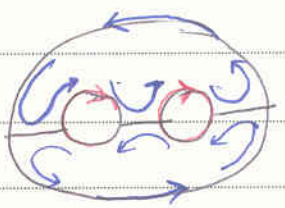
$\int_{D_1} f dz = \int_{D_2} f dz = 0$

$\int_{D_1} + \int_{D_2} = 0$

$= \oint_{C_1} - \oint_{C_2}$



doubly connected



triply connected

MOOKEUK

14.3 Cauchy's integral formula

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_C \frac{f(z_0)}{z-z_0}\,dz + \oint_C \frac{f(z)-f(z_0)}{z-z_0}\,dz$$

$$= f(z_0)\oint_C \frac{1}{z-z_0}\,dz + 0$$

$$= f(z_0)\cdot 2\pi i$$

$$= 2\pi i f(z_0)$$

z_0 is enclosed by C in D

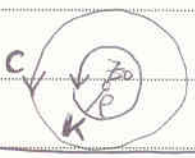
$$\oint \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

for all ϵ

$|f(z)-f(z_0)|<\epsilon$, $|z-z_0|<\delta$

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \frac{\epsilon}{\rho}$$

$$\rightarrow \oint_K \frac{f(z)-f(z_0)}{z-z_0}\,dz < \frac{\epsilon}{\rho}\cdot 2\pi\rho = 2\pi\epsilon$$



ex1. $\oint_C \frac{e^z}{z-2}\,dz = 2\pi i e^z\Big|_{z=2} = 2\pi i e^2$ if C enclose 2.

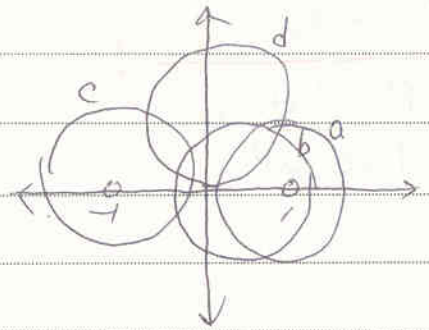
otherwise 0.

ex2. $\oint_C \frac{z^3-6}{2z-i}\,dz = \oint \frac{\frac{z^3}{2}-3}{z-\frac{i}{2}}\,dz = 2\pi i\cdot \frac{z^3}{2}-3\Big|_{z=\frac{i}{2}}$

$$= 2\pi i\cdot\left(\frac{-i}{16}-3\right)$$

$$= \frac{\pi}{8} - 6\pi i$$

Ex 3 $g(z) = \dfrac{z^2+1}{z^2-1} = \dfrac{z^2+1}{(z+1)(z-1)}$



a & b $\displaystyle\oint_a = \oint_b g(z)\,dz = 2\pi i \cdot \left.\frac{z^2+1}{z+1}\right|_{z=1} = 2\pi i$

c $\displaystyle\oint_c g(z)\,dz = 2\pi i \cdot \left.\frac{z^2+1}{z-1}\right|_{z=-1} = -2\pi i$

d $\displaystyle\oint_d g(z)\,dz = 0$

$$f'(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^2}\,dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\oint \frac{f(z)}{(z-z_0)^3}\,dz$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\cdot\oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$\Rightarrow \oint \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}\cdot f^{(n)}(z_0)$$

proof)

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(\alpha)}{(\alpha - z)}\,d\alpha$$

then $$\left.\frac{d^n f(z)}{dz^n}\right|_{z=z_0} = \left.\frac{n!}{2\pi i}\oint \frac{f(\alpha)}{(\alpha-z)^{n+1}}\,d\alpha\right|_{z=z_0}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint \frac{f(\alpha)}{(\alpha-z_0)^{n+1}}\,d\alpha$$

Ex1. $\oint_C \frac{\cos z\,dz}{(z-\pi i)^2} = \frac{2\pi i}{1!}\cdot(\cos z)'\Big|_{\pi i} = -2\pi i \cdot \sin \pi i = 2\pi i \cdot \frac{e^{(\pi i)i} - e^{-(\pi i)i}}{2i}$

$= \pi(-e^{-\pi} + e^{\pi})$

$= 2\pi \sinh \pi$

C : enclosing πi

Ex.2 $\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3}\,dz = \frac{2\pi i}{2!}(z^4 - 3z^2 + 6)''\Big|_{z=-i} = -18\pi i$

C : enclosing $-i$

Ex3. $\frac{2\pi i}{1!}\cdot\left(\frac{e^z}{z^2+4}\right)'\Big|_{z=1} = 2\pi i \cdot \frac{3e}{25} = \frac{6\pi e i}{25}$

C : enclosing $+1$

MOOKEUK

Chp. 15 power series, Taylor Series

NO. 15.1-1

$S = S_n + R_n$

R_n: Remainder

S: convergent series

S_n: partial sum.

Thm 2. $z_n = x_n + iy_n$ converges

$\downarrow$

$S = u + iv$

Thm 3. if a series $z_1 + z_2 + \cdots$ converges, then $\lim_{n\to\infty} z_n = 0$

Hence if this does not hold, the series diverges.

Thm 4.

Thm 6. Geometric Series

$$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \cdots$$

converges when $|q| < 1$ with the sum $\frac{1}{1-q}$

MOOKEUK

Thm 7. Ratio test

if a series $z_1 + z_2 + \cdots$ w/ $z_n \neq 0$ has a property

$$\left|\frac{z_{n+1}}{z_n}\right| \leq q < 1$$

then this series converges.

Thm 8 $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$

a) $L < 1$, converges

b) $L > 1$, diverges

c) $L = 1$, may converge or diverge

Thm 9. Root test

$|z_n|^{1/n} \leq q < 1$ converges.

Thm 10. If a series $z_1 + z_2 + \cdots$ $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$

a) $L < 1$, converges

b) $L > 1$, diverges

c) If $L = 1$, test fails

MOOKEUK

15.2 Power Series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Ex1. The geometric series $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$ converges when $|z|<1$.

Ex2. Maclaurin series

$$e^z = \sum_{n=0}^{\infty} z^n$$

will be convergent for every z.

by the ratio test.

$$\left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right| = \frac{|z|}{n+1} \to 0 \quad (\text{as } n \to \infty)$$

Ex3. Convergence only at the center. (Useless)

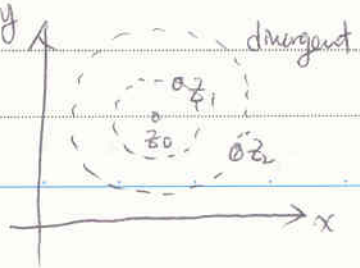
$$\sum_{n=0}^{\infty} (n!) \cdot z^n$$

$$\left| \frac{(n+1)!\, z^n}{n!\, z^n} \right| = (n+1)\cdot|z| \to \infty \quad \text{as } n \to \infty$$

Thm 1. (a) Every power series (1) converges at the center z_0

(b) ~~skip~~ $|z-z_0| < |z_1 - z_0|$, if (1) converges at $z = z_1 \neq z_0$, it converges absolutely for z closer to z_0.

(c) if (1) diverges at a $z = z_2$, it diverges for every z farther away from z_0 than z_2.



MOOKEUK

Radius of convergence

NO. 15.2-2

Ex 4. $\sum \frac{z^n}{n^2}$ converges everywhere since $\sum \frac{1}{n^2}$ converges

$\sum \frac{z^n}{n}$ converges at '-1' but diverges at 1

$\sum z_n$ diverges everywhere

Thm 2 Radius of Convergence R

$$R = \frac{1}{L^*} = \lim \left| \frac{a_n}{a_{n+1}} \right|$$

→ in the ratio test, $\left| \frac{a_{n+1}(z-z_0)^{n+1}}{a_n (z-z_0)^n} \right| = \left| \frac{a_{n+1}}{a_n} \right| (z-z_0) = L$

$= L^*(z-z_0)$

When $L = L^*|z-z_0| < 1$, we have convergence

Hence $|z-z_0| < \frac{1}{L^*} = R$

Ex 3. $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2} \cdot (z-3i)^n \Rightarrow R = \lim \frac{\frac{(2n)!}{(n!)^2}}{\frac{(2n+2)!}{[(n+1)!]^2}} = \lim \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4}$

Ex 6. $\sum_{n=0}^{\infty} \left[1 + (-1)^n + \left(\frac{1}{2}\right)^n \right] \cdot z^n \Rightarrow R = ?$

$R = 1/L$, $\tilde{L} = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

but, for odd n $\sqrt[n]{\left(\frac{1}{2}\right)^n} \Rightarrow \frac{1}{2}$

even n $\sqrt[n]{2+\left(\frac{1}{2}\right)^n} \Rightarrow 1$

which one? larger one

$\tilde{L} = 1 = k$

Therefore, the series converges for $|z| < 1$.

MOOKEUK

15.3 Fns Given by Power Series

main goal: power series represent analytic fns

Taking $z_0 = 0$, $\sum_{n=0}^{\infty} a_n z^n$ (for simplicity)

Terminology

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$$

$f(z)$ is represented by power series

developed in the power series

Thm 1. Continuity

When $f(z) = \sum a_n z^n$, then $f(z)$ is continuous at $z = 0$.

(w/ $R > 0$)

Thm 2. Identity thm.

When $a_0 + a_1 z + a_2 z^2 + \cdots = b_0 + b_1 z + b_2 z^2 + \cdots$ at all z $|z| < R$

then $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, ...

MOOKEUK

- Termwise addition or subtraction.
- multiplication (Cauchy product)

$$a_0b_0 + (a_0b_1 + a_1b_0)z + (a_0b_2 + a_1b_1 + a_2b_0)z^2 + \cdots$$
$$= \sum_{n=0}^{\infty} (a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)z^n$$

- Termwise differentiation & integration of power series is permissible.

$$\sum_{n=1}^{\infty} n \cdot a_n \cdot z^{n-1} = a_1 + 2a_2z + 3a_3z^2 + \cdots$$

Thm 3 The derived series of a power series has the same R of convergence as the original one.

proof) $\lim_{n\to\infty} \frac{n|a_n|}{(n+1)|a_{n+1}|} = \lim_{n\to\infty} \frac{n}{n+1} \cdot \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

Ex) R of $\sum_{n=2}^{\infty} \binom{n}{2} z^n = \sum_{n=2}^{\infty} {}_nC_2 \cdot z^n = f(z)$

$$R = \lim_{n\to\infty} \frac{\frac{n(n+1)}{2}}{\frac{(n+1)(n+2)}{2}} = 1$$

or

$$f''(z)\frac{z^2}{2} = f(z) \quad \text{Therefore, } R = 1$$

MOOKEUK

Thm 4. Termwise Integration of Power Series

$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z_0^{n+1}$ has the same radius of convergence as the original series.

Power Series Represent Analytic Ftns.

Thm 5. A power series w/ a non-zero radius of convergence R represents an analytic ftn.

→ derivatives of this ftn are obtained by differentiating the original series term by term. All the series thus obtained have the same radius of convergence as the original series.

Hence, each of them represents an analytic ftn.

MOOKEUK

15.4 Taylor & Maclaurin Series

Taylor Series

$$f(z) = \sum_{n=1}^{\infty} a_n (z-z_0)^n \quad , \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

or $$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)\,dz^*}{(z^*-z_0)^{n+1}}$$

Maclaurin series (is a Taylor series w/ center $z_0 = 0$)

Remainder $$R_n(z) = \frac{(z-z_0)^{n+1}}{2\pi i} \oint_C \frac{f(z^*)\,dz^*}{(z^*-z_0)^{n+1}(z^*-z)}$$

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z-z_0)^n}{n!} \cdot f^{(n)}(z_0) + R_n(z)$$

→ Taylor formular w/ remainder

Thm 1. Taylor's Thm

$|a_n| \le \frac{M}{r^n}$, M is a Max of $|f(z)|$ on a circle $|z-z_0| = r$

(note that $R_n \to 0$ when $n \to \infty$)

$$\frac{1}{z^*-z} = \frac{1}{z^*-z_0}\left[1 + \frac{z-z_0}{z^*-z_0} + \left(\frac{z-z_0}{z^*-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{z^*-z_0}\right)^n\right] + \frac{1}{z^*-z}\left(\frac{z-z_0}{z^*-z_0}\right)^{n+1}$$

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(z^*)}{z^*-z_0}dz^* + \frac{z-z_0}{2\pi i}\oint_C \frac{f(z^*)}{(z^*-z_0)^2}dz + \cdots + \frac{(z-z_0)^n}{2\pi i}\oint \frac{f(z^*)}{(z^*-z_0)^{n+1}}dz + R_n(z)$$

$\left|\frac{f(z^*)}{z^*-z}\right| \le \hat{M}$, ($z^*$ lies on C)

$\because |z-z_0| < |z^*-z_0|$

due to ML inequality (14.1)

$$|R_n| = \frac{|z-z_0|^{n+1}}{2\pi}\left|\oint \frac{f(z^*)}{(z^*-z_0)^{n+1}(z^*-z)}dz^*\right| \le \frac{|z-z_0|^{n+1}}{2\pi}\cdot\frac{\hat{M}\cdot 2\pi r}{r^{n+1}} = \hat{M}\cdot\left|\frac{z-z_0}{r}\right|^{n+1}\cdot r \to 0$$

Thm 2. A power series w/ a nonzero radius of convergence is the Taylor series of its sum.

Important Special Taylor Series

Ex 1. $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots$

Ex 2. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots$

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}$$

$$= \cos y + i \sin y$$

Ex 3. $\cos z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n}}{(2n)!}$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{z^{2n+1}}{(2n+1)!}$$

Ex 4. $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - + \cdots$

MOOKEUK

Practical Methods

Ex 5. Substitution.

$$f(z) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot z^{2n}$$

$$= 1 - z^2 + z^4 - z^6 \cdots$$

$(|z|<1)$

Ex 6. Integration

$$f(z) = \arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot z^{2n+1} \quad (|z|<1)$$

$\because f'(z) = \frac{1}{1+z^2}$

by integration w/ $f(0)=0$, we get $\arctan z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cdot z^{2n+1}$

Ex 7. Development by using the Geometric Series

$$\frac{1}{c-z} = \frac{1}{c-z_0-(z-z_0)} = \frac{1}{1-\frac{z-z_0}{c-z_0}} \cdot \frac{1}{c-z_0}$$

$$= \frac{1}{c-z_0} \cdot \sum_{n=0}^{\infty} \left(\frac{z-z_0}{c-z_0}\right)^n$$

convergent for $\left|\frac{z-z_0}{c-z_0}\right| < 1$

MOOKEUK

Ex 8. Binomial Series find a Taylor series w/ center $z_0 = 1$

$$f(z) = \frac{2z^2 + 9z + 15}{z^3 + z^2 - 8z - 12}$$

In a binomial series,

$$\frac{1}{(1+z)^m} = (1+z)^{-m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n$$

$$= 1 - mz + \frac{m(m+1)}{2!} z^2 - \frac{m(m+1)(m+2)}{3!} z^3 + \cdots$$

Then, $f(z) = \frac{1}{(z+2)^2} + \frac{2}{z-3}$

$$= \frac{1}{[3 + (z-1)]^2} - \frac{2}{2 - (z-1)}$$

$$= \frac{1}{9} \cdot \left(\frac{1}{\left(1 + \frac{(z-1)}{3}\right)^2} \right) - \frac{1}{1 - \frac{z-1}{2}}$$

$$= \frac{1}{9} \sum_{n=0}^{\infty} \binom{-2}{n} \cdot \left(\frac{z-1}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n \cdot (n+1)}{3^{n+2}} - \frac{1}{2^n} \right] \cdot (z-1)^n$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\binom{-m}{k} = (-1)^k \binom{m+k-1}{k}, \quad \binom{-2}{n} = (-1)^n \binom{2+n-1}{n} = (-1)^n \binom{n+1}{n}$$

$$\sqrt{\quad} \quad \frac{(-2)(-3)\cdots(-2-n+1)}{n!} = (-1)^n \cdot \frac{(n+1)\cdots 2}{n!}$$

MOOKEUK

Laurent Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} \cdot dz$$

$$b_n = \frac{1}{2\pi i} \oint_C (z^* - z_0)^{n-1} f(z^*) \, dz$$

MOOKEUK

Ex1. Find the Laurent series of z^{-5} w/ center 0

$$z^4 \cdot \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4}$$

$|z| > 0$

Ex2. substitution.

$$z^2 \cdot e^{1/z} = z^2 \left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots\right) \quad (|z| > 0)$$

Ex3. Development of $1/1-z$

a) in nonnegative

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$

b) negative powers of z

$$\frac{1}{1-z} = \frac{-1}{z(1-z^{-1})} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \quad (|z| > 1)$$

Ex4. Laurent Expansions in Different Concentric Annuli

$$\frac{1}{z^3 - z^4} = \sum_{n=0}^{\infty} z^{n-3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \cdots \quad (0 < |z| < 1)$$

(w/ center 0)

$$= -\sum_{n=0}^{\infty} \frac{1}{z^{n+4}} = -\frac{1}{z^4} - \frac{1}{z^5} - \cdots \quad (|z| > 1)$$

MOOKEUK

Ex 5.

find all Taylor and Laurent series of

$$f(z) = \frac{-2z+3}{z^2-3z+2} \quad (\text{w/ center } 0)$$

sol) $f(z) = -\frac{1}{z-1} - \frac{1}{z-2}$

$$-\frac{1}{z-2} = \frac{1}{2\left(1-\frac{z}{2}\right)} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \quad (|z|<2)$$

or

$$= \frac{-1}{z\left(1-\frac{2}{z}\right)} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \quad (|z|>2)$$

So, $f(z) = \sum_{n=0}^{\infty} \left(1+\frac{1}{2^{n+1}}\right) \cdot z^n = \frac{3}{2} + \frac{5}{4}z + \frac{9}{8}z^2 + \cdots$

(when $|z|<1$)

$$= \sum \frac{z^n}{2^{n+1}} - \sum \frac{1}{z^{n+1}}$$

(when $1<|z|<2$)

$$= -\sum_{n=0}^{\infty} (2^n+1) \frac{1}{z^{n+1}}$$

(when $|z|>2$)

16.2 Singularity & Zero.

singular or has a singularity

isolated singularies of $f(z)$ if $z=z_0$ has a neighborhood w/ further singularies of $f(z)$

ex) $\tan z$ has isolated singularies at $\pm\pi/2, \pm 3\pi/2, \cdots$

$\tan(1/z)$ has <u>non isolated</u> singularity at 0.

Isolated singularities of $f(z)$ at $z=z_0$ can be classified by the Laurent Series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

→ principal parts

valid in the immediate neighborhood of the singular pt $z=z_0$

w/ ~~finite~~ # of negative terms in principal parts,

$z=z_0$ is called a pole

m is called its order

(when $m=1$, simple poles)

MOOKEUK

Ex 1.

$$f(z) = \frac{1}{z(z-1)^5} + \frac{3}{(z-2)^2}$$

simple pole at $z=0$

fifth order at $z=2$

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n}$$

$$\sin\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\, z^{2n+1}}$$

isolated essential singularities at $z=0$

Ex 2. $f(z) = \frac{1}{z^2}$ has a pole at $z=0$, and $|f(z)| \to \infty$ as $z \to \infty$

Ex 3.

Thm 2. Picard's Thm

MOOKEUK

16.3 Residue Integration Method

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \cdot \frac{1}{(z-z_0)^n}$$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{\oint \frac{f(z^*)}{(z^*-z_0)^{n+1}} \cdot dz^*}{2\pi i}$$

$$b_n = \frac{\oint f(z^*) \cdot (z^*-z_0)^{n+1} dz^*}{2\pi i}$$

or

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

$$a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{\oint f(z) \cdot (z-z_0)^{-n-1} \cdot dz}{2\pi i}$$

background

$$\oint_C f(z)\,dz = 2\pi i\, b_1 = 2\pi i \operatorname*{Res}_{z=z_0} f(z) \quad \left(b_1 = \frac{\oint f(z)\,dz}{2\pi i} \right)$$

$$f(z) = \sum_{0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots$$

Ex1. $\oint_C \frac{\sin z}{z^4} dz = ?$

C : unit circle

$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots \quad (b_1)$$

$|z| > 0$, a pole of 3rd order at $z=0$

$$\therefore \oint_C \frac{\sin z}{z^4} dz = 2\pi i \left(-\frac{1}{3!}\right) = \frac{-\pi i}{3}$$

MOOKEUK

Ex 2.

$$\oint_C \frac{1}{z^3 - z^4}\,dz = ?$$

$C: |z| = \frac{1}{2}$ (clockwise integration)

$f(z)$ is singular at $z=0$ & $z=1$.

(But, $z=1$ lies outside C.)

$$\frac{1}{z^3 - z^4} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + \cdots \quad (0 < |z| < 1)$$

$\frac{1}{z}$ → $\operatorname{Res}_{z=0} f(z)$

$$\therefore \oint_C \frac{dz}{z^3 - z^4} = \underset{\text{clockwise}}{(-2\pi i)\cdot(1)} = -2\pi i$$

Formulas for Residues

$$\operatorname{Res}_{z=z_0} f(z) = b_1 = \lim_{z \to z_0} (z - z_0) f(z)$$ at a simple pole $z = z_0$

assuming that $f(z) = \frac{p(z)}{q(z)}$, $p(z_0) \neq 0$ &

$q(z)$ has a simple zero at z_0

(so that $f(z)$ has a simple pole at z_0)

✓ at a simple pole,

$$\operatorname{Res}_{z=z_0} f(z) = \operatorname{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

proof) $f(z) = \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$

as $b_1 \neq 0$, $\lim_{z \to z_0} (z - z_0) f(z) = b_1 + \lim_{z \to z_0} (z - z_0)[a_0 + a_1(z - z_0) + \cdots]$ $(0 < |z - z_0| < R)$

$= b_1$

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) \cdot \frac{p(z)}{q(z)} = \lim_{z \to z_0} \frac{(z - z_0)\,p(z)}{(z - z_0)[q(z_0) + (z - z_0)q'(z_0) + \cdots]} = \frac{p(z_0)}{q'(z_0)}$$

Ex3. Residue at a simple pole.

$f(z) = \dfrac{9z+i}{z^3+z}$ has a simple pole at i.

$$\operatorname*{Res}_{z=i} \frac{9z+i}{z(z^2+i)} = \lim_{z\to i} (z-i)\cdot \frac{9z+i}{z\cdot(z+i)(z-i)} = \left[\frac{9z+i}{z(z+i)}\right]_{z=i}$$

$$= -5i$$

Poles of any order

The residue of $f(z)$ at mth order pole at z_0 is

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\cdot \lim_{z\to z_0}\left\{\left((z-z_0)^m \cdot f(z_0)\right)^{(m-1)}\right\}$$

this is what we want!!!

(proof) $$f(z) = \frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \cdots + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$$(z-z_0)^m \cdot f(z) = b_m + b_{m-1}(z-z_0) + \cdots + b_1(z-z_0)^{m-1} + a_0(z-z_0)^m + \cdots$$

$$b_1 = \lim_{z\to z_0} \frac{g^{(m-1)}(z_0)}{(m-1)!}$$

$$= \lim_{z\to z_0} \frac{1}{(m-1)!}\cdot\left[(z-z_0)^m \cdot f(z)\right]^{(m-1)}$$

Ex4. $f(z) = \dfrac{5z}{z^3+2z^2-7z+4} = \dfrac{5z}{(z+4)(z-1)^2}$ ($z=1$, second order pole)

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \cdot f(z)\right] = \left(\frac{5z}{z+4}\right)'\Big|_{z=1} = \frac{20}{5^2} = 0$$

MOOKEUK

Several Singularities inside the Contour

- Residue Thm -

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

Ex 5 (a) $\oint \frac{4-3z}{z^2-z} dz$ (where C (counterclockwise) encloses 0 & 1)

$$\operatorname*{Res}_{z=0} \frac{4-3z}{z^2-z} = \left.\frac{4-3z}{z-1}\right|_{z=0} = -4$$

$$\operatorname*{Res}_{z=1} \frac{4-3z}{z^2-z} = \left.\frac{4-3z}{z}\right|_{z=1} = 1$$

$$\therefore \oint \frac{4-3z}{z^2-z} dz = 2\pi i(-4+1) = -6\pi i$$

(b) only 0 is enclosed

$$= 2\pi i(-4) = -8\pi i$$

(c) only 1 is included

$$= 2\pi i(1) = 2\pi i$$

(d) nothing is included

$$= 0$$

MOOKEUK

Ex6. $\oint \frac{\tan z}{z^2-1} dz = ?$ $\quad C: |z| = \frac{3}{2}$

$$\oint_C \frac{\tan z}{z^2-1} dz = 2\pi i \left(\operatorname{Res}_{z=1} \frac{\tan z}{z^2-1} + \operatorname{Res}_{z=-1} \frac{\tan z}{z^2-1} \right)$$

$$= 2\pi i \left(\frac{\tan z}{2z}\Big|_{z=1} + \frac{\tan z}{2z}\Big|_{z=-1} \right)$$

$$= 2\pi i \tan 1$$

Ex 7. Poles & Essential Singularities

$$\oint_C \left(\frac{z e^{\pi z}}{z^4-16} + z e^{\pi/z} \right) dz = ?$$

$C: 9x^2 + y^2 = 9$ (counterclockwise)

$\frac{z \cdot e^{\pi z}}{z^4-16}$

$z^4 = 16$, $(\pm 2i \;\&\; \pm 2)$ → outside ellipse

simple poles at $\pm 2i$ inside C

$$\operatorname{Res}_{z=2i} \frac{z e^{\pi z}}{z^4-16} = \frac{z e^{\pi z}}{4z^3}\Big|_{z=2i} = -\frac{e^{2\pi i}}{16} = -\frac{1}{16}$$

$$\operatorname{Res}_{z=-2i} \frac{z e^{\pi z}}{z^4-16} = \frac{z e^{\pi z}}{4z^3}\Big|_{z=-2i} = -\frac{1}{16}$$

$$\text{Sol)} \quad 2\pi i \left(-\frac{1}{16} - \frac{1}{16} + \frac{\pi^2}{2} \right) = \pi \left(\pi^2 - \frac{1}{4} \right) i$$

$$z e^{\pi/z} = z \left(1 + \frac{\pi}{z} + \frac{\pi^2}{2! z^2} + \cdots \right) = z + \pi + \frac{\pi^2}{2} \cdot \frac{1}{z} + \cdots \quad (|z| > 0)$$

16.4 Residue Integration of Real Integrals.

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + 1/z}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - 1/z}{2i}$$

$$J = \oint f(z)\frac{dz}{iz} = \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$$

$$\left(\text{let } z = e^{i\theta} \text{ then } dz = e^{i\theta} i\, d\theta\right)$$

Ex 1. $$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = 2\pi$$

$$= \oint_C \frac{1}{\sqrt{2} - \frac{z + 1/z}{2}} \cdot \frac{dz}{iz} = \oint_C \frac{dz}{-\frac{i}{2}(z^2 - 2\sqrt{2}z + 1)}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)}$$

inside $|z| < 1$, there is $\sqrt{2} - 1$ only.

$$\operatorname*{Res}_{z = z_2} \frac{1}{(z - \sqrt{2} - 1)(z - \sqrt{2} + 1)} = \left.\frac{1}{z - \sqrt{2} - 1}\right|_{z = \sqrt{2} - 1} = -\frac{1}{2}$$

$$\therefore -\frac{2}{i}\oint \frac{dz}{-\frac{i}{2}(z^2 - 2\sqrt{2}z + 1)} = \left(\frac{2}{i}\right) 2\pi i\left(-\frac{1}{2}\right) = (+2\pi i)\left(+\frac{2}{i}\right) = 2\pi$$

MOOKEUK

As another large class, $\int_{-\infty}^{\infty} f(x)dx = ?$

(improper integral?)

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx$$

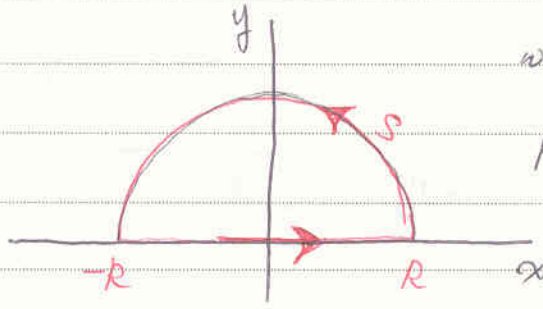
(Cauchy principal value)

$$\text{pr. v.} \int_{-\infty}^{\infty} f(x)\,dx$$

Assumption: $f(x)$ is a real rational ftn. whose denominator is different from '0' for all real x and is of degree at least two units higher than the degree of numerator.

Then, $\int_{-\infty}^{\infty} f(x)dx$ will exist.

$$\oint_C f(z)dz = \int_S f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum \text{Res}\, f(z)$$



when $R\to\infty$, $\int_S f(z)dz \to 0$

proof) $|f(z)| < \frac{k}{|z|^2}$ (from assumption)

$|z| = R > R_0$

$$\left|\int_S f(z)dz\right| < \frac{k}{R^2}\cdot \pi R = \frac{k\pi}{R}$$

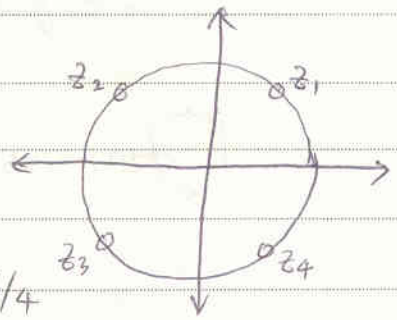
So, $R\to\infty$, $\int_S f(z)dz \to 0$

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum \text{Res}\, f(z)$$

MOOKEUK

Ex 2.

prove: $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$



four simple poles at

$z = e^{\pi i/4}, e^{3\pi i/4}, e^{-3\pi i/4}, e^{-\pi i/4}$

only these two lies in the upper half-plane.

$$\operatorname*{Res}_{z=z_1} f(z) = \left.\frac{1}{(1+z^4)'}\right|_{z=z_1} = \left.\frac{1}{4z^3}\right|_{z=e^{\pi i/4}} = \frac{e^{-3\pi i/4}}{4} = \frac{-e^{\pi i/4}}{4}$$

$$\operatorname*{Res}_{z=z_2} f(z) = \left.\frac{1}{4z^3}\right|_{z=e^{3\pi i/4}} = \frac{e^{-9\pi i/4}}{4} = \frac{e^{-\pi i/4}}{4}$$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = 2\pi i\left(\frac{e^{\pi i/4}}{4} + \frac{e^{-\pi i/4}}{4}\right) = -\pi i\left(\frac{e^{\pi i/4} - e^{-\pi i/4}}{2i}\right)$$

$$= -\pi i^2 \cdot 2\sin\frac{\pi}{4} = \frac{\pi}{\sqrt{2}}$$

$$\therefore \int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

~~Ex3~~. Fourier Integrals

(570)
$$\int_{-\infty}^{\infty} f(x)\cdot\cos sx\,dx = -2\pi \sum \mathrm{Im}\cdot \mathrm{Res}\left[f(z)\cdot e^{isz}\right]$$
$$\int_{-\infty}^{\infty} f(x)\cdot\sin sx\,dx = 2\pi \sum \mathrm{Re}\cdot \mathrm{Res}\left[f(z)\cdot e^{isz}\right]$$

from $\oint_C f(z)\cdot e^{isz}\,dz$

$$\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\,dx = 2\pi i \sum \mathrm{Res}\left[f(z)\cdot e^{isz}\right] \quad (570)$$

As $R\to\infty$, $\int_S f(z)\cdot e^{isz}\,dz \to 0$

$$|e^{isz}| = |e^{is(x+yi)}| = |e^{isx}|\cdot|e^{-sy}| = e^{-sy} \le 1$$

So,
$$|f(z)\cdot e^{isz}| = |f(z)|\cdot|e^{isz}| \le |f(z)| \quad (s>0,\ y\ge 0)$$

therefore, $\left|\int_S e^{isz} f(z)\,dz\right| \le \frac{k\pi}{R}\left(=\frac{k}{R^2}\cdot\pi R\right)$

Ex3. $\int_{-\infty}^{\infty}\frac{\cos sx}{k^2+x^2}dx = 2\pi i \operatorname*{Res}_{z=ik}\frac{e^{isz}}{k^2+z^2} = \left.\frac{e^{isz}}{2z}\right|_{z=ik} = \frac{e^{is\cdot ik}}{2ik} = \frac{e^{-sk}}{2ik}\cdot 2\pi i$

$\int_{-\infty}^{\infty}\frac{\sin sx}{k^2+x^2}dx = \frac{\pi}{k}\cdot e^{-ks}$ ↗

$\int_{-\infty}^{\infty}\frac{e^{isz}}{k^2+x^2}dx = \frac{\pi\cdot e^{-ks}}{k}$

$\therefore \int_{-\infty}^{\infty}\frac{\cos sx}{k^2+x^2}dx = \frac{\pi\cdot e^{-ks}}{k}$

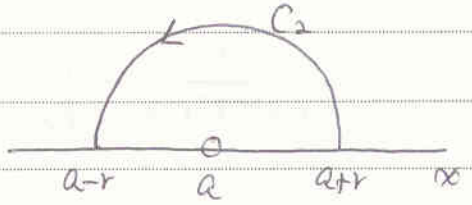
$$\int_{-\infty}^{\infty}\frac{\sin sx}{k^2+x^2}dx = 0$$

MOOKEUK

Another kind of Improper Integral

If $f(z)$ has a simple pole at $z=a$ on the real axis, then

$$\lim_{r\to 0}\int_{C_2} f(z)\,dz = \pi i \operatorname{Res} f(z)$$



by the definition of simple pole (16.2), the integrand $f(z)$ has for $0<|z-a|<R$ the Laurent series

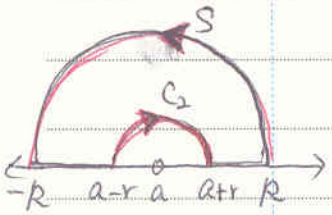
$$f(z) = \frac{b_1}{z-a} + g(z), \quad \text{where } b_1 = \operatorname*{Res}_{z=a} f(z).$$

Here, $g(z)$ is analytic on the semicircle of integration

$$C_2:\ z = a + r\cdot e^{i\theta},\quad 0\le\theta\le\pi$$

and $|g(z)|\le M$.

$$\int_{C_1} f(z)\,dz = \int_0^{\pi} \frac{b_1}{r\cdot e^{i\theta}}\cdot ir\cdot e^{i\theta}\cdot d\theta + \int_{C_2} g(z)\,dz = b_1\pi i + \int_{C_1} g(z)\,dz$$



as $r\to 0$, $M\pi r \to 0$.

$$\lim_{r\to 0}\int_{C_1} f(z)\,dz = \pi i\, b_1 = \operatorname*{Res}_{z=a} f(z)\,(\pi i)$$

So, p.v. $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f(z) + \pi i \sum \operatorname{Res} f(z)$

on upper half-plane; on real axis, simple pole

$$|f(z)| < \frac{k}{|z|^r}$$

MOOKEUK

Ex 4.

$$\text{pr.v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)}$$

$z=1$: $\operatorname*{Res}_{z=1} f(z) = \left.\frac{1}{(z-2)(z^2+1)}\right|_{z=1} = -\frac{1}{2}$

$z=2$: $\operatorname*{Res}_{z=2} f(z) = \left.\frac{1}{(z-1)(z^2+1)}\right|_{z=2} = \frac{1}{5}$

$z=i$: $\operatorname*{Res}_{z=i} f(z) = \left.\frac{1}{(z^2-3z+2)(z+i)}\right|_{z=i} = \frac{1}{6+2i} = \frac{3-i}{20}$

$$\text{pr.v.} \int_{-\infty}^{\infty} \frac{dx}{(x^2-3x+2)(x^2+1)} = 2\pi i\left(\frac{3-i}{20}\right) + \pi i\left(-\frac{1}{2}+\frac{1}{5}\right)$$

$$= \frac{\pi}{10}$$

MOOKEUK