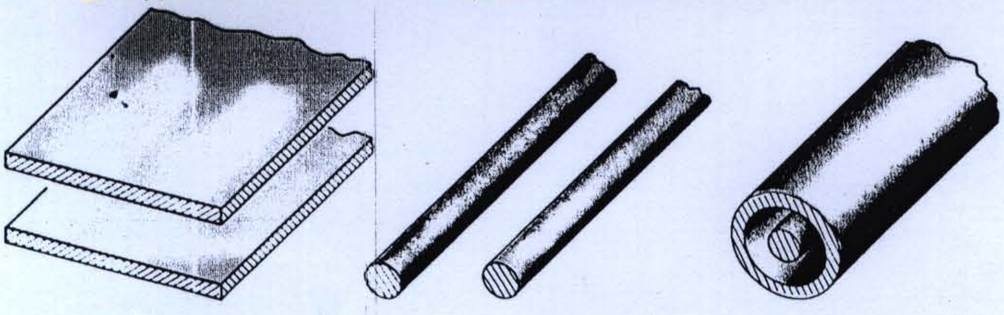
Chap. 9 Theory and Applications of Transmission Lines

9–1 Introduction

For efficient point-to-point transmission of power and information the source energy must be directed or guided. In this chapter we study transverse electromagnetic (TEM) waves guided by transmission lines. The TEM mode of guided waves is one in which **E** and **H** are perpendicular to each other and both are transverse to the direction of propagation along the guiding line. We discussed the propagation of



(a) Parallel-plate transmission line. (b) Two-wire transmission line. (c) Coaxial transmission line.

FIGURE 9–1
Common types of transmission lines.

9–2 Transverse Electromagnetic Wave along a Parallel-Plate Transmission Line (lossless dielectric)

Let us consider a y-polarized TEM wave propagating in the $+z$-direction along a uniform parallel-plate transmission line. Figure 9–2 shows the cross-sectional dimensions of such a line and the chosen coordinate system. For time-harmonic fields the wave equation to be satisfied in the sourceless dielectric region becomes the homogeneous Helmholtz's equation, Eq. (8–46). In the present case the appropriate phasor solution for the wave propagating in the $+z$-direction is

$$\mathbf{E} = \mathbf{a}_y E_y = \mathbf{a}_y E_0 e^{-\gamma z}. \tag{9–1a}$$

The associated **H** field is, from Eq. (8–31),

$H(\vec{r}) = \frac{1}{\eta}\vec{a}_n \times E(\vec{r}) \Rightarrow$ (8-29)

$$\mathbf{H} = \mathbf{a}_x H_x = -\mathbf{a}_x \frac{E_0}{\eta} e^{-\gamma z}, \tag{9–1b}$$

where γ and η are the propagation constant and the intrinsic impedance, respectively, of the dielectric medium. Fringe fields at the edges of the plates are neglected. Assuming perfectly conducting plates and a lossless dielectric, we have, from Chapter 8,

$$\gamma = j\beta = j\omega\sqrt{\mu\epsilon} \tag{9–2}$$

and

$$\eta = \sqrt{\frac{\mu}{\epsilon}}. \tag{9–3}$$

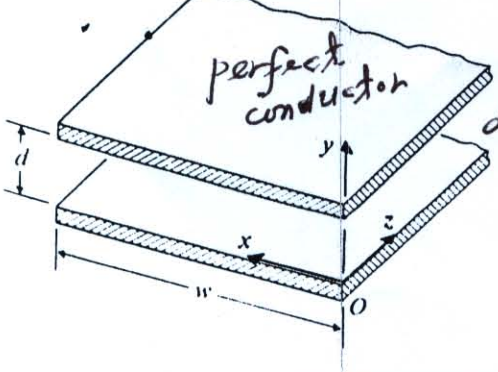


FIGURE 9–2
Parallel-plate transmission line.

$$\begin{cases} \vec{E} = \vec{a}_y E_0 e^{-\gamma z} & (9\text{-}1a) \\ H = -\vec{a}_x E_0 e^{-\gamma z}/\eta & (9\text{-}1b) \end{cases}$$

$\gamma = j\beta = j\omega\sqrt{\mu\varepsilon}$ (9-2)

$\eta = \sqrt{\mu/\varepsilon}$ (9-3)

(7-68)

$$\begin{pmatrix} E_{1t} = 0 \\ \vec{a}_{n2} \times \vec{H}_1 = \vec{J}_s \\ \vec{a}_{n2} \cdot \vec{D}_1 = \rho_s \\ B_{1n} = 0 \end{pmatrix}$$

The boundary conditions to be satisfied at the interfaces of the dielectric and the perfectly conducting planes are, from Eqs. (7–68a, b, c, and d), as follows:

At both $y = 0$ and $y = d$:

$$E_t = 0 \tag{9–4}$$

and

$$H_n = 0, \tag{9–5}$$

which are obviously satisfied because $E_x = E_z = 0$ and $H_y = 0$.

At $y = 0$ (lower plate), $\mathbf{a}_n = \mathbf{a}_y$:

$$\mathbf{a}_y \cdot \mathbf{D} = \rho_{s\ell} \quad \text{or} \quad \rho_{s\ell} = \epsilon E_y = \epsilon E_0 e^{-j\beta z}; \tag{9–6a}$$

$$\mathbf{a}_y \times \mathbf{H} = \mathbf{J}_{s\ell} \quad \text{or} \quad \mathbf{J}_{s\ell} = -\mathbf{a}_z H_x = \mathbf{a}_z \frac{E_0}{\eta} e^{-j\beta z}. \tag{9–7a}$$

At $y = d$ (upper plate), $\mathbf{a}_n = -\mathbf{a}_y$:

$$-\mathbf{a}_y \cdot \mathbf{D} = \rho_{su} \quad \text{or} \quad \rho_{su} = -\epsilon E_y = -\epsilon E_0 e^{-j\beta z}; \tag{9–6b}$$

$$-\mathbf{a}_y \times \mathbf{H} = \mathbf{J}_{su} \quad \text{or} \quad \mathbf{J}_{su} = \mathbf{a}_z H_x = -\mathbf{a}_z \frac{E_0}{\eta} e^{-j\beta z}. \tag{9–7b}$$

Equations (9–6) and (9–7) indicate that surface charges and surface currents on the conducting planes vary sinusoidally with z, as do E_y and H_x. This is illustrated schematically in Fig. 9–3.

Field phasors **E** and **H** in Eqs. (9–1a) and (9–1b) satisfy the two Maxwell's curl equations:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{9–8}$$

and

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}. \tag{9–9}$$

Since $\mathbf{E} = \mathbf{a}_y E_y$ and $\mathbf{H} = \mathbf{a}_x H_x$, Eqs. (9–8) and (9–9) become

Note: $\begin{pmatrix} E_y = E_0 e^{-\gamma z} \\ H_x = -\frac{E_0}{\eta} e^{-\gamma z} \end{pmatrix}$ satisfy

$$\begin{cases} \dfrac{dE_y}{dz} = j\omega\mu H_x & (9\text{–}10) \\[2ex] \dfrac{dH_x}{dz} = j\omega\varepsilon E_y & (9\text{-}11) \end{cases}$$

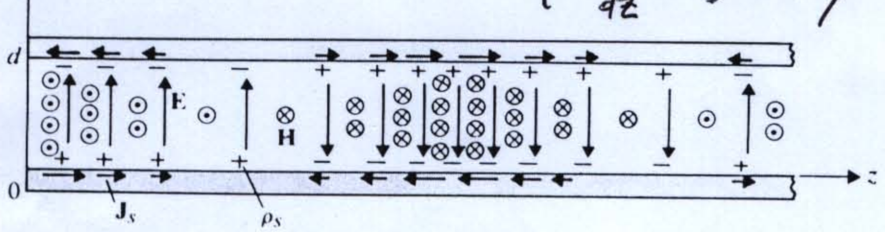


FIGURE 9–3
Field, charge, and current distributions along a parallel-plate transmission line.

$$\nabla \times E = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \underbrace{E_0 e^{-\gamma z}}_{=E_y} & 0 \end{vmatrix} = -\vec{a}_x \frac{\partial E_y}{\partial z} = \left(\vec{a}_x E_0 \gamma e^{-\gamma z}\right)$$

and

$$\frac{dH_x}{dz} = j\omega\epsilon E_y. \tag{9–11}$$

Ordinary derivatives appear above because phasors E_y and H_x are functions of z only.
Integrating Eq. (9–10) over y from 0 to d, we have

$$\frac{d}{dz}\int_0^d E_y\,dy = j\omega\mu \int_0^d H_x\,dy$$

$E_y = E_0 e^{-j\beta z}$
$J_{su}(z) = -\frac{E_0}{\eta} e^{-j\beta z}$
$\beta = \omega\sqrt{\mu\varepsilon}$
$\eta = \sqrt{\mu/\varepsilon}$
can be directly obtained $\Rightarrow$

or

$$-\frac{dV(z)}{dz} = j\omega\mu J_{su}(z)d = j\omega\left(\mu\frac{d}{w}\right)[J_{su}(z)w] = j\omega L I(z), \tag{9–12}$$

where

$$V(z) = -\int_0^d E_y\,dy = -E_y(z)d \quad (*1)$$

is the potential difference or voltage between the upper and lower plates,

$$I(z) = J_{su}(z)w \quad (*2)$$

is the total current flowing in the $+z$ direction in the upper plate (w = plate width), and

$$L = \frac{\Lambda}{I} = \frac{\mu H_x \times d \times 1m}{J_{su}w} = \frac{\mu H_x d}{H_x w} \quad (9\text{-}7b)$$

$$\boxed{L = \mu\frac{d}{w} \quad \text{(H/m)}} \tag{9–13}$$

is the inductance per unit length of the parallel-plate transmission line. The dependence of phasors $V(z)$ and $I(z)$ on z is noted explicitly in Eq. (9–12) for emphasis.
Similarly, we integrate Eq. (9–11) over x from 0 to w to obtain

$$\frac{d}{dz}\int_0^w H_x\,dx = j\omega\epsilon \int_0^w E_y\,dx$$

$E_y = E_0 e^{-j\beta z}$
$J_{su}(z) = -\frac{E_0}{\eta} e^{-j\beta z}$
$\beta = \omega\sqrt{\mu\varepsilon}$
$\eta = \sqrt{\mu/\varepsilon}$
directly follows $\Rightarrow$

or

$$-\frac{dI(z)}{dz} = -j\omega\epsilon E_y(z)w = j\omega\left(\epsilon\frac{w}{d}\right)[-E_y(z)d] = j\omega C V(z), \tag{9–14}$$

where

$$\boxed{C = \epsilon\frac{w}{d} \quad \text{(F/m)}} \tag{9–15}$$

is the capacitance per unit length of the parallel-plate transmission line.
Equations (9–12) and (9–14) constitute a pair of ***time-harmonic transmission-line equations*** for phasors $V(z)$ and $I(z)$. They may be combined to yield second-order

differential equations for $V(z)$ and for $I(z)$:

$$\frac{d^2V(z)}{dz^2} = -\omega^2 LCV(z), \tag{9-16a}$$

$$\frac{d^2I(z)}{dz^2} = -\omega^2 LCI(z). \tag{9-16b}$$

The solutions of Eqs. (9–16a) and (9–16b) are, for waves propagating in the $+z$-direction,

$$V(z) = V_0 e^{-j\beta z} \tag{9-17a}$$

and

$$I(z) = I_0 e^{-j\beta z}, \tag{9-17b}$$

where the phase constant

$$\boxed{\beta = \omega\sqrt{LC} = \omega\sqrt{\mu\epsilon} \qquad \text{(rad/m)}} \tag{9-18}$$

is the same as that given in Eq. (9–2). The relation between V_0 and I_0 can be found by using either Eq. (9–12) or Eq. (9–14):

$$\boxed{Z_0 = \frac{V(z)}{I(z)} = \frac{V_0}{I_0} = \sqrt{\frac{L}{C}} \qquad (\Omega),} \tag{9-19}$$

which becomes, in view of the results of Eqs. (9–13) and (9–15),

$$\boxed{Z_0 = \frac{d}{w}\sqrt{\frac{\mu}{\epsilon}} = \frac{d}{w}\eta \qquad (\Omega).} \tag{9-20}$$

The quantity Z_0 is the impedance at any location that looks toward an infinitely long (no reflections) transmission line. It is called the ***characteristic impedance*** of the line. The ratio of $V(z)$ and $I(z)$ at any point on a finite line of any length terminated in Z_0 is Z_0.† For a parallel-plate transmission line with perfectly conducting plates of width w and separated by a lossless dielectric slab of thickness d, the characteristic impedance Z_0 is (d/w) times the intrinsic impedance η of the dielectric medium.

The velocity of propagation along the line is

$$\boxed{u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\epsilon}} \qquad \text{(m/s),}} \tag{9-21}$$

which is the same as the phase velocity of a TEM plane wave in the dielectric medium.

† This statement will be proved in Section 9–4 (see Eq. 9–107).

§ 9-2.1 Lossy Parallel-plate Transmission Lines

The conductance per unit length across the two plates

$$G = \frac{\sigma}{\varepsilon} C \quad \text{by (5-81)}$$

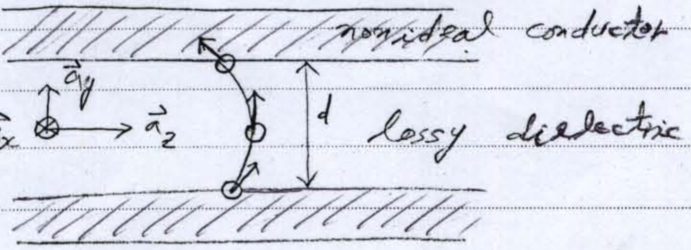
$$\doteq \sigma \frac{W}{d} \quad \text{by (9-15)}$$

The time-harmonic transmission line equations

(Approximation)

<Note> Assumption: (A.1) $|E_z| \ll |E_y|$, (A.2) $\varepsilon_c \simeq \frac{\sigma_c}{j\omega}$ (good conductor)

$\sigma_c < \infty \Rightarrow E_z \neq 0$



complex number

(1) $\nabla^2 E + k^2 E = 0$, $k^2 \triangleq \omega^2 \mu \varepsilon$

(2) $\nabla^2 E_c + k_c^2 E_c = 0$, $k_c^2 \triangleq \omega^2 \mu_c \varepsilon_c$

Let $\begin{cases} E_z = E_z(y) e^{-\gamma z} \vec{a}_z, & \vec{E}_x = 0, \\ E_{cz} = E_{zc}(y) e^{-\gamma z} \vec{a}_z, & \vec{E}_{cx} = 0 \end{cases}$

$$\Rightarrow \begin{cases} E_z''(y) + \bar{k}^2 E_z(y) = 0, & \bar{k}^2 \triangleq \gamma^2 + k^2 \\ E_{zc}''(y) + \bar{k}_c^2 E_{zc}(y) = 0, & \bar{k}_c^2 \triangleq \gamma^2 + k_c^2 \end{cases}$$

$$\Rightarrow \begin{cases} E_z = (C \sin \bar{k} y + D \cos \bar{k} y) e^{-\gamma z} \vec{a}_z \\ E_{zc} = (C_c e^{-j\bar{k}_c y} + D_c e^{j \bar{k}_c y}) e^{-\gamma z} \vec{a}_z \end{cases}$$

$$\Rightarrow \begin{cases} E_z = C e^{-\gamma z} \sin \bar{k} y \, \vec{a}_z & \text{(from odd symmetry)} \\ E_{zc} = E_{c0} e^{-j\bar{k}_c (y - \frac{d}{2})} e^{-\gamma z} \vec{a}_z & \text{(the field must decay)} \\ C = E_{c0} / \sin \frac{\bar{k} d}{2} & \text{(from boundary conditions)} \end{cases}$$

$H_x(y)$

From (7-94a) and (9-109) with $\begin{cases} H_x = H_0(y)\, e^{-\gamma z}\, \vec{a}_x \\ E_y = E_0(y)\, e^{-\gamma z}\, \vec{a}_y \end{cases}$ ($E_0(y) = E_y(y)$)

$$\begin{cases} \dfrac{\partial E_z}{\partial y} + \gamma E_y = -j\omega\mu H_x \quad (*2) \\ -\gamma H_x - \dfrac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad (*1) \end{cases}$$

H_z indep. of x

$$\Rightarrow \begin{cases} H_x = \dfrac{1}{\bar{k}^2}\left(j\omega\varepsilon \dfrac{\partial E_z}{\partial y} - \gamma \cancel{\dfrac{\partial H_z}{\partial x}}\right) = \left(\dfrac{j\omega\varepsilon}{\bar{k}^2}\right)\dfrac{\partial E_z}{\partial y} = \dfrac{j\omega\varepsilon C}{\bar{k}} e^{-\gamma z}\cos\bar{k}y \\ \text{Similary, we have} \\ H_{xc} = \dfrac{\omega\varepsilon_c E_{co}}{\bar{k}_c} e^{-j\bar{k}_c(y-\frac{d}{2})} e^{-\gamma z} \end{cases}$$

$$(A.1) \Rightarrow \vec{E} \simeq \vec{E_y} \Rightarrow \bar{k} \simeq 0 \Rightarrow \gamma \simeq j\omega\sqrt{\mu\varepsilon}$$

$$\Rightarrow j\bar{k}_c \simeq j\omega\sqrt{\mu_c\varepsilon_c}\left(1 - \frac{\mu\varepsilon}{\mu_c\varepsilon_c}\right)^{1/2} \simeq j\omega\sqrt{\mu_c\varepsilon_c} \simeq (1+j)\sqrt{\pi f \mu_c \sigma_c} \text{ (by (8-53))}$$

$\sqrt{\pi f \mu_c \sigma_c} \triangleq \dfrac{1}{\delta_c}$

$$\Rightarrow \vec{E}_{zc} = E_{co}\, e^{-\frac{(1+j)}{\delta_c}(y-\frac{d}{2})} e^{-\gamma z}\, \vec{a}_z$$

$$\Rightarrow \vec{J}_c = \sigma_c \vec{E}_{zc} = \sigma_c E_{co}\, e^{-(1+j)(y-\frac{d}{2})/\delta_c} e^{-\gamma z}\, \vec{a}_z$$

$$\Rightarrow I = W\int_{\frac{d}{2}}^{\infty} J_c\, dy = \frac{W\sigma_c\delta_c E_{co}}{(1+j)} e^{-\gamma z} = \left(\frac{W E_{co}}{\eta_c}\right) e^{-\gamma z} \quad \left(\text{by (8-54)}\right) \quad \eta_c = \frac{1+j}{\sigma_c\delta_c}$$

$$\frac{E_z}{H_x} = \frac{E_{zc}}{H_{xc}} \text{ at } y = \frac{d}{2} \Rightarrow \tan\left(\frac{\bar{k}d}{2}\right) = j\,\frac{\bar{k}_c}{\bar{k}}\,\frac{\varepsilon}{\varepsilon_c}$$

$$\Rightarrow \frac{\bar{k}d}{2} \simeq j\,\frac{\bar{k}_c}{\bar{k}}\,\frac{\varepsilon}{\varepsilon_c} \text{ (since } \bar{k} \simeq 0) \Rightarrow \bar{k}^2 = \left(\frac{1+j}{\delta_c}\right)\left(\frac{2}{d}\right)\left(\frac{\varepsilon}{\varepsilon_c}\right) \simeq j\omega\eta_c\left(\frac{2\varepsilon}{d}\right)$$

$$\Rightarrow H_x = \left(\frac{j\omega\varepsilon}{\bar{k}}\right)\left(\frac{E_{co}}{\sin\frac{\bar{k}d}{2}}\right)\cos\bar{k}y\, e^{-\gamma z}\,\vec{a}_x \simeq \left(\frac{j\omega\varepsilon}{\bar{k}^2}\right)\left(\frac{2E_{co}}{d}\right) e^{-\gamma z}\,\vec{a}_x$$
$$= \frac{E_{co}}{\eta_c} e^{-\gamma z}\,\vec{a}_x = \frac{I}{W}\,\vec{a}_x$$

$$-j\omega\mu H = \nabla\times E \Rightarrow -j\omega\mu H_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}$$

$$\Rightarrow -j\omega\mu\left(\frac{I}{W}\right) = C\bar{k}\cos\bar{k}y\, e^{-\gamma z} - \frac{\partial E_y}{\partial z}$$
$$\simeq \frac{2E_{co}}{d} e^{-\gamma z} - \frac{\partial E_y}{\partial z} \qquad \text{since } \bar{k}y \simeq 0$$
$$= \left(\frac{2\eta_c I}{Wd}\right) - \frac{\partial E_y}{\partial z}$$

$$\Rightarrow \frac{1}{W}\left(j\omega\mu + \frac{2\eta_c}{d}\right) I(z) = \frac{\partial E_y}{\partial z}$$

$$\Rightarrow \frac{d}{W}\left(j\omega\mu + \frac{2\eta_c}{d}\right) I(z) = -\frac{d}{dz}V(z) \quad \left(\text{by taking } \int_{-\frac{d}{2}}^{\frac{d}{2}} dy \text{, fixed } z\right)$$

$$\Rightarrow \boxed{\left[R + j\omega(L + L_i)\right] I(z) = -\frac{d}{dz}V(z)} \quad \text{where } L_i = \frac{R}{\omega}$$

(9-31)

$R \triangleq \left(\frac{2}{W}\right)R_s$, and $R_s \triangleq \sqrt{\dfrac{\pi f \mu_c}{\sigma_c}}$ $\left(= 1/(\sigma_c\delta_c)\right)$

internal series inductance (negligible at high freq.)

Similarly, from (9-109), we have

$$\frac{\partial H_x}{\partial z} = -(\sigma + j\omega\varepsilon) E_y \quad \text{instead of (9-11)}$$

$$\Rightarrow \boxed{-\frac{dI(z)}{dz} = \left[\sigma \frac{W}{d} + j\omega\left(\varepsilon \frac{W}{d}\right)\right] \int_{-\frac{d}{2}}^{\frac{d}{2}} E_y \, dy = (\underbrace{G}_{\text{conductance}} + j\omega C)\, V(z)}$$

(9-33)'

Justification of (9-26a)

$$\left.\begin{aligned} E_{zc}\left(\tfrac{d}{2}\right) &= E_{co}\, e^{-\gamma z} \simeq \eta_c H_x \\ J_{su} &\simeq \frac{I}{W} = H_x \end{aligned}\right\} \Rightarrow \underbrace{Z_s}_{\text{surface impedance}} \triangleq \frac{E_z}{J_s} = \frac{E_{zc}(\frac{d}{2})}{J_{su}} = \eta_c$$

since the current flows in a very thin surface layer

Justification of (A.1)

By (*1), P. (129), ((*2)는 Ez 근사식의 미분이 필요하므로 부정확)

$$-\gamma H_x - \cancel{\frac{\partial H_z}{\partial x}} = j\omega\varepsilon E_y$$

Since H_z is independent of x, $E_y = \dfrac{-\gamma}{j\omega\varepsilon} H_x \underset{\bar{K} \simeq 0}{\simeq} \dfrac{-j\omega\sqrt{\mu\varepsilon}}{j\omega\varepsilon} H_x$

$$\Rightarrow E_y \simeq -\eta H_x$$

$$\Rightarrow \left|\frac{E_{zc}(\frac{d}{2})}{E_y}\right| = \left|\frac{\eta_c H_x}{-\eta H_x}\right| = \left|\frac{\eta_c}{\eta}\right| = \left|\frac{(1+j)\sqrt{\frac{\omega\mu_c}{2\sigma_c}}}{\sqrt{\mu/\varepsilon}}\right|$$

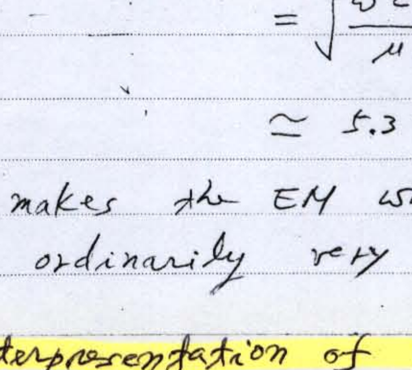
$$= \sqrt{\frac{\omega\varepsilon\mu_c}{\mu\sigma_c}}$$

$\simeq 5.3 \times 10^{-5}$ for copper plate in air

$\Rightarrow |\vec{E}_z| \neq 0$ makes the EM wave strictly not TEM but $\vec{E}_z$ is ordinarily very small

Physical Interpretation of $R = \left(\frac{2}{W}\right) R_s$

If $\delta_c \ll 1$,



$$R = \frac{1}{\sigma_c} \frac{1+1}{W\delta_c} = \left(\frac{2}{W}\right) R_s$$

Transmission-Line Equation of lossy parallel-plate TL

$$\left.\begin{matrix}(9\text{-}31)' \\ (9\text{-}33)'\end{matrix}\right\} \Rightarrow \begin{cases} R\, i(z,t) + (L+L_i)\dfrac{\partial i(z,t)}{\partial t} = -\dfrac{\partial v(z,t)}{\partial z} & (9\text{-}31) \\ -\dfrac{\partial i(z,t)}{\partial z} = G\, v(z,t) + C\dfrac{\partial v(z,t)}{\partial t} & (9\text{-}33) \end{cases}$$

9–2.2 MICROSTRIP LINES

The development of solid-state microwave devices and systems has led to the widespread use of a form of parallel-plate transmission lines called microstrip lines or simply ***striplines***. A stripline usually consists of a dielectric substrate sitting on a grounded conducting plane with a thin narrow metal strip on top of the substrate, as shown in Fig. 9–4(a). Since the advent of printed-circuit techniques, striplines can be easily fabricated and integrated with other circuit components. However, because the results that we have derived in this section were based on the assumption of two wide conducting plates (with negligible fringing effect) of equal width, they are not expected to apply here exactly.

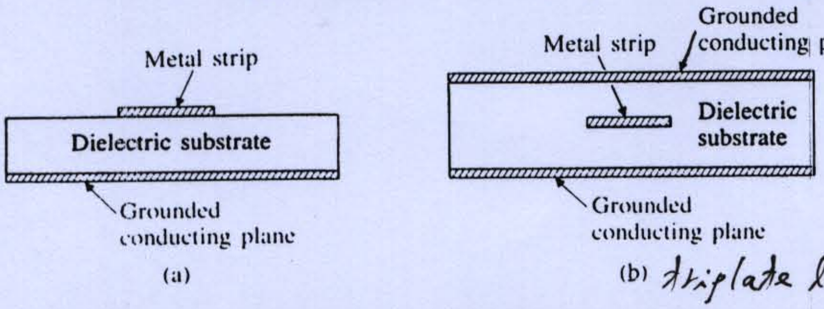


FIGURE 9–4
Two types of microstrip lines.

9–3 General Transmission-Line Equations (stationary TL)

We will now derive the equations that govern general two-conductor uniform transmission lines that include parallel-plate, two-wire, and coaxial lines. Transmission lines differ from ordinary electric networks in one essential feature. Whereas the physical dimensions of electric networks are very much smaller than the operating wavelength, transmission lines are usually a considerable fraction of a wavelength and may even be many wavelengths long. The circuit elements in an ordinary electric network can be considered discrete and as such may be described by lumped parameters. It is assumed that currents flowing in lumped-circuit elements do not vary spatially over the elements, and that no standing waves exist. A transmission line, on the other hand, is a distributed-parameter network and must be described by circuit parameters that are distributed throughout its length. Except under matched conditions, standing waves exist in a transmission line.

Consider a differential length Δz of a transmission line that is described by the following four parameters:

- R, resistance per unit length (both conductors), in Ω/m.
- L, inductance per unit length (both conductors), in H/m.
- G, conductance per unit length, in S/m.
- C, capacitance per unit length, in F/m.

Assume :

Note that R and L are series elements and G and C are shunt elements. Figure 9–5 shows the equivalent electric circuit of such a line segment. The quantities $v(z, t)$ and $v(z + \Delta z, t)$ denote the instantaneous voltages at z and $z + \Delta z$, respectively. Similarly, $i(z, t)$ and $i(z + \Delta z, t)$ denote the instantaneous currents at z and $z + \Delta z$, respectively. Applying Kirchhoff's voltage law, we obtain

$$v(z, t) - R\,\Delta z\, i(z, t) - L\,\Delta z\,\frac{\partial i(z, t)}{\partial t} - v(z + \Delta z, t) = 0, \qquad (9\text{–}30)$$

Assume :

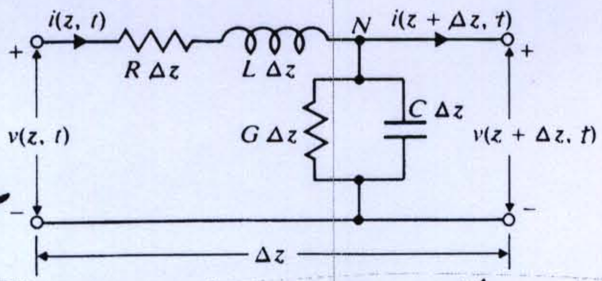


FIGURE 9–5
Equivalent circuit of a differential length Δz of a two-conductor transmission line.

Applying Kirchhoff's voltage law,

$$v(z,t) - R\Delta z\, i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z, t) = 0 \quad (9\text{–}30)$$

which leads to

$$-\frac{v(z+\Delta z, t) - v(z,t)}{\Delta z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t}. \quad (9\text{–}30a)$$

In the limit as $\Delta z \to 0$, Eq. (9–30a) becomes

$$\boxed{-\frac{\partial v(z,t)}{\partial z} = Ri(z,t) + L\frac{\partial i(z,t)}{\partial t}.} \quad \text{PDE} \quad (9\text{–}31)$$

Similarly, applying Kirchhoff's current law to the node N in Fig. 9–5, we have

$$i(z,t) - G\,\Delta z v(z+\Delta z, t) - C\,\Delta z\frac{\partial v(z+\Delta z, t)}{\partial t} - i(z+\Delta z, t) = 0. \quad (9\text{–}32)$$

On dividing by Δz and letting Δz approach zero, Eq. (9–32) becomes

$$\boxed{-\frac{\partial i(z,t)}{\partial z} = Gv(z,t) + C\frac{\partial v(z,t)}{\partial t}.} \quad \text{PDE} \quad (9\text{–}33)$$

Equations (9–31) and (9–33) are a pair of first-order partial differential equations in $v(z, t)$ and $i(z, t)$. They are the ***general transmission-line equations.***[†]

For harmonic time dependence the use of phasors simplifies the transmission-line equations to ordinary differential equations. For a cosine reference we write

$$v(z,t) = \mathcal{R}e[V(z)e^{j\omega t}], \quad (9\text{–}34a)$$

$$i(z,t) = \mathcal{R}e[I(z)e^{j\omega t}], \quad (9\text{–}34b)$$

where $V(z)$ and $I(z)$ are functions of the space coordinate z only and both may be complex. Substitution of Eqs. (9–34a) and (9–34b) in Eqs. (9–31) and (9–33) yields

[†] Sometimes referred to as the ***telegraphist's equations*** or ***telegrapher's equations***.

Kirchhoff's Voltage Law : $\sum_k V_k = \sum_i R_i I_i + \sum_j L_j \frac{dI_j}{dt}$ (from (6-41), (6-130), and (9-54a))

Kirchhoff's Current Law : $\sum_k I_k = -\sum_i G_i V_i - \sum_j C_j \frac{dV_j}{dt}$ (from equation of continuity)

hold even for time-varying EM fields

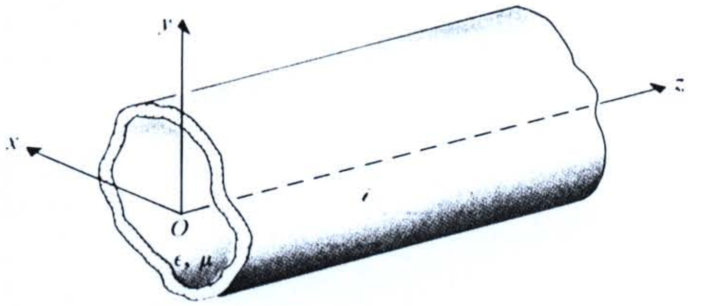


FIGURE 10-1
A uniform waveguide with an arbitrary cross section.

factor

$$e^{-\gamma z}e^{j\omega t} = e^{(j\omega t - \gamma z)} = e^{-\alpha z}e^{j(\omega t - \beta z)}. \tag{10-1}$$

As an example, for a cosine reference we may write the instantaneous expression for the **E** field in Cartesian coordinates as

$$\mathbf{E}(x, y, z; t) = \mathscr{R}_e[\mathbf{E}^0(x, y)e^{(j\omega t - \gamma z)}], \tag{10-2}$$

where $\mathbf{E}^0(x, y)$ is a two-dimensional vector phasor that depends only on the cross-sectional coordinates. The instantaneous expression for the **H** field can be written in a similar way. Hence, in using a phasor representation in equations relating field quantities we may replace partial derivatives with respect to t and z simply by products with $(j\omega)$ and $(-\gamma)$, respectively; the common factor $e^{(j\omega t - \gamma z)}$ can be dropped.

We consider a straight waveguide in the form of a dielectric-filled metal tube having an arbitrary cross section and lying along the z-axis, as shown in Fig. 10–1. According to Eqs. (7–105) and (7–106), the electric and magnetic field intensities in the charge-free dielectric region inside satisfy the following homogeneous vector Helmholtz's equations:

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \tag{10-3}$$

and

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = 0, \tag{10-4}$$

where **E** and **H** are three-dimensional vector phasors, and k is the wavenumber:

$$k = \omega\sqrt{\mu\epsilon}. \tag{10-5}$$

The three-dimensional Laplacian operator ∇^2 may be broken into two parts: $\nabla^2_{u_1u_2}$ for the cross-sectional coordinates and ∇^2_z for the longitudinal coordinate. For waveguides with a rectangular cross section we use Cartesian coordinates:

$$\begin{aligned}\nabla^2\mathbf{E} &= (\nabla^2_{xy} + \nabla^2_z)\mathbf{E} = \left(\nabla^2_{xy} + \frac{\partial^2}{\partial z^2}\right)\mathbf{E} \\ &= \nabla^2_{xy}\mathbf{E} + \gamma^2\mathbf{E}.\end{aligned} \tag{10-6}$$

Combination of Eqs. (10–3) and (10–6) gives

$$\nabla^2_{xy}\mathbf{E} + (\gamma^2 + k^2)\mathbf{E} = 0. \tag{10-7}$$

Similarly, from Eq. (10–4) we have

$$\nabla_{xy}^2 \mathbf{H} + (\gamma^2 + k^2)\mathbf{H} = 0. \tag{10–8}$$

We note that each of Eqs. (10–7) and (10–8) is really three second-order partial differential equations, one for each component of **E** and **H**. The exact solution of these component equations depends on the cross-sectional geometry and the boundary conditions that a particular field component must satisfy at conductor-dielectric interfaces. We note further that by writing $\nabla_{r\phi}^2$ for the transversal operator ∇_{xy}^2, Eqs. (10–7) and (10–8) become the governing equations for waveguides with a circular cross section.

Of course, the various components of **E** and **H** are not all independent, and it is not necessary to solve all six second-order partial differential equations for the six components of **E** and **H**. Let us examine the interrelationships among the six components in Cartesian coordinates by expanding the two source-free curl equations, Eqs. (7–104a) and (7–104b):

From $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$:		From $\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$:	
$\dfrac{\partial E_z^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0$	(10–9a)	$\dfrac{\partial H_z^0}{\partial y} + \gamma H_y^0 = j\omega\epsilon E_x^0$	(10–10a)
$-\gamma E_x^0 - \dfrac{\partial E_z^0}{\partial x} = -j\omega\mu H_y^0$	(10–9b)	$-\gamma H_x^0 - \dfrac{\partial H_z^0}{\partial x} = j\omega\epsilon E_y^0$	(10–10b)
$\dfrac{\partial E_y^0}{\partial x} - \dfrac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0$	(10–9c)	$\dfrac{\partial H_y^0}{\partial x} - \dfrac{\partial H_x^0}{\partial y} = j\omega\epsilon E_z^0$	(10–10c)

Note that partial derivatives with respect to z have been replaced by multiplications by $(-\gamma)$. All the component field quantities in the equations above are phasors that depend only on x and y, the common $e^{-\gamma z}$ factor for z-dependence having been omitted. By manipulating these equations we can express the transverse field components H_x^0, H_y^0, and E_x^0, and E_y^0 in terms of the two longitudinal components E_z^0 and H_z^0. For instance, Eqs. (10–9a) and (10–10b) can be combined to eliminate E_y^0 and obtain H_x^0 in terms of E_z^0 and H_z^0. We have

$$H_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y}\right), \tag{10–11}$$

$$H_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x}\right), \tag{10–12}$$

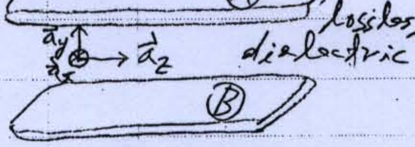
$$E_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y}\right), \tag{10–13}$$

$$E_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x}\right), \tag{10–14}$$

Justification of general transmission-line equations in (9-31) and (9-33) based on Maxwell's equations.

Assume : (1) two lossless conductors Ⓐ and Ⓑ of any general shape

(2) TEM wave $\begin{cases} \vec{E} = E_x^o(x,y)\, e^{-\gamma z}\, \vec{a}_x + E_y^o(x,y)\, e^{-\gamma z}\, \vec{a}_y \\ H = H_x^o(x,y)\, e^{-\gamma z}\, \vec{a}_x + H_y^o(x,y)\, e^{-\gamma z}\, \vec{a}_y \end{cases}$



From (10-11) – (10-14),

$$E_z = H_z = 0 \underset{(10\text{-}11)\sim(10\text{-}14)}{\Longrightarrow} \gamma^2 + k^2 = 0$$

(TEM wave) (otherwise, all other components must also be zero)

$$\Rightarrow \gamma = j\omega\sqrt{\mu\varepsilon}$$

$$\Rightarrow \nabla^2 E = \nabla_{xy}^2 E + \underbrace{\frac{\partial^2 E}{\partial z^2}}_{=\gamma^2 E} = -k^2 E \qquad (10\text{-}7)$$

$$\Rightarrow \begin{cases} \nabla_{xy}^2 E = 0 \\ \nabla_{xy}^2 H = 0 \end{cases} \Rightarrow \begin{cases} \nabla_{xy}^2 E^o = 0 \\ \nabla_{xy}^2 H^o = 0 \end{cases} \qquad (*1)$$

On the other hand, from (10-9) and (10-10),

$$\begin{cases} H_y^o = \dfrac{\gamma}{j\omega\mu} E_x^o = \dfrac{E_x^o}{\eta} \\ H_x^o = -\dfrac{\gamma}{j\omega\mu} E_y^o = -\dfrac{E_y^o}{\eta} \end{cases} \Rightarrow E^o \cdot H^o = 0 \qquad (*2)$$

$\Rightarrow$ E has a normal component only at the surface of the conductor

This with (*2) $\Rightarrow$ H has a tangent component only at the surface of the conductor

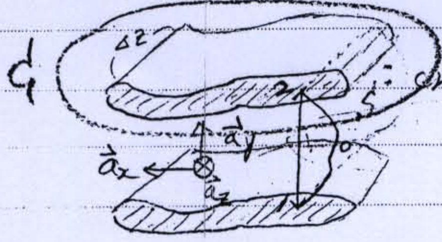
(Recall: $A(\vec{r}) = \frac{\mu}{4\pi}\int_{V'} \frac{J(\vec{r}')e^{-jk|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}dv'$)

(1) The E field distribution in the transverse plane is exactly a static distribution

(2) The line integral of E between conductors is the same for all paths lying in a given transverse plane

$$\left(E = -\nabla V - j\omega A \Rightarrow \int E\cdot d\vec{r} = -V - j\omega \int A\cdot d\vec{r} \right)$$

(3) The magnetic field pattern in the transverse plane then corresponds exactly to that arising from static currents flowing entirely on the surfaces of the perfect conductors



: cross section of two conductor transmission line at z

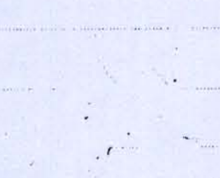
$$V(z) \overset{(by\ (2))}{=} -\int_1^2 E\cdot d\vec{r} = -\int_1^2 E_y\, dy$$

$$\Rightarrow \frac{\partial V}{\partial z} = -\int_1^2 \frac{\partial E_y}{\partial z}\, dy$$

$$\Rightarrow \frac{\partial V}{\partial z} = -j\omega\mu \int_1^2 H_x\, dy$$

$$= -j\omega\mu \left\{ \lim_{\Delta z \to 0} \left[\int_z^{z+\Delta z}\int_1^2 H_x\, dy\, dz \right] / \Delta z \right.$$

$$= -j\omega \lim_{\Delta z \to 0} \frac{1}{\Delta z}\int_{S'} B\cdot d\vec{s} \overset{(\because \Delta z \simeq 0)}{=} -j\omega \lim_{\Delta z \to 0} [(\Delta z L) I]/\Delta z = -j\omega L I(z) \quad (9\text{-}35a)'$$



since

$$\nabla \times E = -j\omega\mu H \Rightarrow \begin{cases} \frac{\partial E_y}{\partial z} = j\omega\mu H_x \\ \frac{\partial E_x}{\partial z} = -j\omega\mu H_y \end{cases}$$

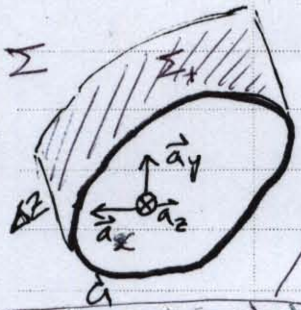
$$(E_z = 0)$$

since $E_z = 0 \Rightarrow \int E \cdot d\vec{S} = 0$

$$I(z) = \oint_c \vec{H} \cdot d\vec{r} = \oint (H_x dx + H_y dy)$$

$$\Rightarrow \frac{\partial I}{\partial z} = \oint_c \left(\frac{\partial H_x}{\partial z} dx + \frac{\partial H_y}{\partial z} dy\right)$$

$$= -\varepsilon j\omega \oint_c (E_x dy - E_y dx) = -\varepsilon j\omega \left\{ \lim_{\Delta z \to 0} \left[\int_z^{z+\Delta z} \oint_C (E_x dy - E_y dx) dz \right] / \Delta z \right\}$$



$$= -\varepsilon j\omega \lim_{\Delta z \to 0} \left[\oint_\Sigma \vec{E} \cdot d\vec{\Sigma} / \Delta z \right] = -j\omega \lim_{\Delta z \to 0} \left[\int_{Surface} \rho_s \cdot d\vec{S} / \Delta z \right]$$

by Divergence Theorem

$(\because \Delta z \simeq 0)$

$$= -j\omega \left[\lim_{\Delta z \to 0} \frac{(\Delta z C_l) V}{\Delta z} \right] = -j\omega C V(z) \qquad (9\text{-}35b)'$$

$\vec{r} = x\vec{a}_x + f(x)\vec{a}_y + z\vec{a}_z$ on Σ_+ where $u = x, v = z$

$\int E \cdot d\Sigma = \int E \cdot (\vec{r}_u \times \vec{r}_v) du dv$

and $E_z = 0$

since

$$\nabla \times H = j\omega\varepsilon E \Rightarrow \begin{cases} \frac{\partial H_y}{\partial z} = -j\omega\varepsilon E_x \\ \frac{\partial H_x}{\partial z} = j\omega\varepsilon E_y \end{cases}$$

$(H_z = 0)$

Conclusion

(2) The well-known method of analysis based on low-frequency circuit notions in §9-3 gives the correct answer, since it is actually equivalent to an analysis starting from Maxwell's equations – despite the use of static L's and C's from a problem, certainly not static

(1) Lossy transmission line 경우도 (9-35)가 성립함은 Maxwell's equation으로 부터 보일수 있으나, 그 과정이 매우 복잡하다

(3) Discuss the contents in §9-3.1

the following ordinary differential equations for phasors $V(z)$ and $I(z)$:

$$\boxed{\begin{aligned} -\frac{dV(z)}{dz} &= (R + j\omega L)I(z), \\ -\frac{dI(z)}{dz} &= (G + j\omega C)V(z). \end{aligned}}$$ (9–35a) (9–35b)

Equations (9–35a) and (9–35b) are ***time-harmonic transmission-line equations***, which reduce to Eqs. (9–12) and (9–14) under lossless conditions ($R = 0$, $G = 0$).

9–3.1 WAVE CHARACTERISTICS ON AN INFINITE TRANSMISSION LINE

The coupled time-harmonic transmission-line equations, Eqs. (9–35a) and (9–35b), can be combined to solve for $V(z)$ and $I(z)$. We obtain

$$\boxed{\frac{d^2V(z)}{dz^2} = \gamma^2 V(z)} \tag{9–36a}$$

and

$$\boxed{\frac{d^2I(z)}{dz^2} = \gamma^2 I(z),} \tag{9–36b}$$

where

$$\boxed{\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} \quad (m^{-1})} \tag{9–37}$$

is the ***propagation constant*** whose real and imaginary parts, α and β, are the ***attenuation constant*** (Np/m) and ***phase constant*** (rad/m) of the line, respectively. The nomenclature here is similar to that for plane-wave propagation in lossy media as defined in Section 8–3. These quantities are not really constants because, in general, they depend on ω in a complicated way.

The solutions of Eqs. (9–36a) and (9–36b) are

$$\begin{aligned} V(z) &= V^+(z) + V^-(z) \\ &= V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}, \end{aligned} \tag{9–38a}$$

$$\begin{aligned} I(z) &= I^+(z) + I^-(z) \\ &= I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}, \end{aligned} \tag{9–38b}$$

where the plus and minus superscripts denote waves traveling in the $+z$- and $-z$-directions, respectively. Wave amplitudes (V_0^+, I_0^+) and (V_0^-, I_0^-) are related by Eqs. (9–35a) and (9–35b), and it is easy to verify (Problem P.9–5) that

(9-38) → (9-35a) ⇒

$$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R + j\omega L}{\gamma}. \tag{9–39}$$

For an infinite line (actually a semi-infinite line with the source at the left end) the terms containing the $e^{\gamma z}$ factor must vanish. There are no reflected waves; only the waves traveling in the $+z$-direction exist. We have

$$V(z) = V^+(z) = V_0^+ e^{-\gamma z}, \tag{9-40a}$$

$$I(z) = I^+(z) = I_0^+ e^{-\gamma z}. \tag{9-40b}$$

The ratio of the voltage and the current at any z for an infinitely long line is independent of z and is called the ***characteristic impedance*** of the line.

$$\boxed{Z_0 = \frac{R + j\omega L}{\gamma} = \frac{\gamma}{G + j\omega C} = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \quad (\Omega).} \tag{9-41}$$

Note that γ and Z_0 are characteristic properties of a transmission line whether or not the line is infinitely long. They depend on R, L, G, C, and ω—not on the length of the line. An infinite line simply implies that there are no reflected waves.

There is a close analogy between the general governing equations and the wave characteristics of a transmission line and those of uniform plane waves in a lossy medium. This analogy will be discussed in the following example.

EXAMPLE 9–2 Demonstrate the analogy between the wave characteristics on a transmission line and uniform plane waves in a lossy medium.

Solution In a lossy medium with a complex permittivity $\epsilon_c = \epsilon' - j\epsilon''$ and a complex permeability $\mu = \mu' - j\mu''$ the Maxwell's curl equations (7–104a) and (7–104b) become

$$\nabla \times \mathbf{E} = -j\omega(\mu' - j\mu'')\mathbf{H}, \tag{9-42a}$$

$$\nabla \times \mathbf{H} = j\omega(\epsilon' - j\epsilon'')\mathbf{E}. \tag{9-42b}$$

If we assume a uniform plane wave characterized by an E_x that varies only with z, Eq. (9–42a) reduces to (see Eq. 8–12b)

$$\nabla \times E = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x^+(z) & 0 & 0 \end{vmatrix}$$

$$-\frac{dE_x(z)}{dz} = j\omega(\mu' - j\mu'')H_y = (\omega\mu'' + j\omega\mu')H_y. \tag{9-43a}$$

Similarly, we obtain from Eq. (9–42b) the following relation:

$$-\frac{dH_y(z)}{dz} = (\omega\epsilon'' + j\omega\epsilon')E_x. \tag{9-43b}$$

Comparing Eqs. (9–43a) and (9–43b) with Eqs. (9–35a) and (9–35b), respectively, we recognize immediately the analogy of the governing equations for E_x and H_y of a uniform plane wave and those for V and I on a transmission line.

$$\begin{cases} -\dfrac{dV(z)}{dz} = (R + j\omega L)\, I(z) & (9\text{-}35a) \\[2ex] -\dfrac{dV(z)}{dz} = (G + j\omega C)\, V(z) & (9\text{-}35b) \end{cases}$$

Equations (9–43a) and (9–43b) can be combined to give

$$\frac{d^2V(z)}{dz^2} = \gamma^2 V(z) \quad (9\text{-}36a)$$

$$\frac{d^2E_x(z)}{dz^2} = \gamma^2 E_x(z) \tag{9–44a}$$

and

$$\frac{d^2I(z)}{dz^2} = \gamma^2 I(z) \quad (9\text{-}36b)$$

$$\frac{d^2H_y(z)}{dz^2} = \gamma^2 H_y(z), \tag{9–44b}$$

which are entirely similar to Eqs. (9–36a) and (9–36b). The propagation constant of the uniform plane wave is

$$\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)} \quad (9\text{-}37)$$

$$\gamma = \alpha + j\beta = \sqrt{(\omega\mu'' + j\omega\mu')(\omega\epsilon'' + j\omega\epsilon')}, \tag{9–45}$$

which should be compared with Eq. (9–37) for the transmission line. The intrinsic impedance of the lossy medium (the wave impedance of the plane wave traveling in the $+z$-direction) is (see Eq. 8–30)

$$Z_0 = \frac{V(z)}{I(z)} = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad (9\text{-}41)$$

$$\eta_c = \sqrt{\frac{\mu'' + j\mu'}{\epsilon'' + j\epsilon'}}, \tag{9–46}$$

which is analogous to the expression for the characteristic impedance of a transmission line in Eq. (9–41).

Because of the above analogies, many of the results obtained for normal incidence of uniform plane waves can be adapted to transmission-line problems, and vice versa. ■

The general expressions for the characteristic impedance in Eq. (9–41) and the propagation constant in Eq. (9–37) are relatively complicated. The following three limiting cases have special significance.

1. *Lossless Line* ($R = 0$, $G = 0$).

a) Propagation constant:

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}; \tag{9–47}$$

$$\alpha = 0, \tag{9–48}$$

$$\beta = \omega\sqrt{LC} \qquad \text{(a linear function of } \omega). \tag{9–49}$$

b) Phase velocity:

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \qquad \text{(constant)}. \tag{9–50}$$

c) Characteristic impedance:

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}}; \tag{9–51}$$

$$R_0 = \sqrt{\frac{L}{C}} \qquad \text{(constant)}, \tag{9–52}$$

$$X_0 = 0. \tag{9–53}$$

Note:

$$LC = \left(\mu\frac{d}{W}\right)\left(\epsilon\frac{W}{d}\right) = \mu\epsilon$$

$$L/C = \left(\frac{\mu}{\epsilon}\right)\left(\frac{d}{W}\right)^2 \neq \frac{\mu}{\epsilon}$$

2. *Low-Loss Line* ($R \ll \omega L$, $G \ll \omega C$). The low-loss conditions are more easily satisfied at very high frequencies.

   a) Propagation constant:

$$\gamma = \alpha + j\beta = j\omega\sqrt{LC}\left(1 + \frac{R}{j\omega L}\right)^{1/2}\left(1 + \frac{G}{j\omega C}\right)^{1/2}$$

$$\cong j\omega\sqrt{LC}\left(1 + \frac{R}{2j\omega L}\right)\left(1 + \frac{G}{2j\omega C}\right) \tag{9-54}$$

$$\cong j\omega\sqrt{LC}\left[1 + \frac{1}{2j\omega}\left(\frac{R}{L} + \frac{G}{C}\right)\right];$$

$$\alpha \cong \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right), \tag{9-55}$$

$$\beta \cong \omega\sqrt{LC} \qquad \text{(approximately a linear function of } \omega\text{).} \tag{9-56}$$

   b) Phase velocity:

$$u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{LC}} \qquad \text{(approximately constant).} \tag{9-57}$$

   c) Characteristic impedance:

$Z_0 = \sqrt{\frac{L}{C}}$ (9-51)

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{L}{C}}\left(1 + \frac{R}{j\omega L}\right)^{1/2}\left(1 + \frac{G}{j\omega C}\right)^{-1/2}$$

$$\cong \sqrt{\frac{L}{C}}\left[1 + \frac{1}{2j\omega}\left(\frac{R}{L} - \frac{G}{C}\right)\right]; \tag{9-58}$$

$$R_0 \cong \sqrt{\frac{L}{C}}, \tag{9-59}$$

$$X_0 \cong -\sqrt{\frac{L}{C}}\,\frac{1}{2\omega}\left(\frac{R}{L} - \frac{G}{C}\right) \cong 0. \tag{9-60}$$

3. *Distortionless Line* ($R/L = G/C$). If the condition

$$\frac{R}{L} = \frac{G}{C} \tag{9-61}$$

is satisfied, the expressions for both γ and Z_0 simplify.

   a) Propagation constant:

$$\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)\left(\frac{RC}{L} + j\omega C\right)}$$

$$= \sqrt{\frac{C}{L}}\,(R + j\omega L); \tag{9-62}$$

$$\alpha = R\sqrt{\frac{C}{L}}, \tag{9-63}$$

$$\beta = \omega\sqrt{LC} \qquad \text{(a linear function of } \omega\text{).} \tag{9-64}$$

b) Phase velocity:

$$u_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \quad \text{(constant).} \tag{9–65}$$

c) Characteristic impedance:

$$Z_0 = R_0 + jX_0 = \sqrt{\frac{R + j\omega L}{(RC/L) + j\omega C}} = \sqrt{\frac{L}{C}}; \tag{9–66}$$

$$R_0 = \sqrt{\frac{L}{C}} \quad \text{(constant),} \tag{9–67}$$

$$X_0 = 0. \tag{9–68}$$

Thus, except for a nonvanishing attenuation constant, the characteristics of a distortionless line are the same as those of a lossless line—namely, a constant phase velocity ($u_p = 1/\sqrt{LC}$) and a constant real characteristic impedance ($Z_0 = R_0 = \sqrt{L/C}$).

A constant phase velocity is a direct consequence of the linear dependence of the phase constant β on ω. Since a signal usually consists of a band of frequencies, it is essential that the different frequency components travel along a transmission line at the same velocity in order to avoid distortion. This condition is satisfied by a lossless line and is approximated by a line with very low losses. For a lossy line, wave amplitudes will be attentuated, and distortion will result when different frequency components attenuate differently, even when they travel with the same velocity. The condition specified in Eq. (9–61) leads to both a constant α and a constant u_p—thus the name ***distortionless line***.

$\frac{R}{L} = \frac{G}{C}$ (9–61)

The phase constant of a lossy transmission line is determined by expanding the expression for γ in Eq. (9–37). In general, the phase constant is not a linear function of ω; thus it will lead to a u_p, which depends on frequency. As the different frequency components of a signal propagate along the line with different velocities, the signal suffers ***dispersion***. A general, lossy, transmission line is therefore *dispersive*, as is a lossy dielectric.

EXAMPLE 9–3 It is found that the attenuation on a 50 (Ω) distortionless transmission line is 0.01 (dB/m). The line has a capacitance of 0.1 (nF/m).

a) Find the resistance, inductance, and conductance per meter of the line.

b) Find the velocity of wave propagation.

c) Determine the percentage to which the amplitude of a voltage traveling wave decreases in 1 (km) and in 5 (km).

Solution

a) For a distortionless line,

$$\frac{R}{L} = \frac{G}{C}.$$

$\alpha = 0.01$ dB/m

$= (?)$ Np/m — see the footnote in p. 368

$20 \log_{10} e^{\alpha(\text{Np/m})} = 0.01$ dB/m

$\Rightarrow \alpha(\text{Np/m}) = 0.01 / \left(20 \log_{10} e\right)$

8.69

9–3.2 TRANSMISSION-LINE PARAMETERS — lossless or low-loss conductors

The electrical properties of a transmission line at a given frequency are completely characterized by its four distributed parameters R, L, G, and C. These parameters for a parallel-plate transmission line are listed in Table 9–1. We will now obtain them for two-wire and coaxial transmission lines.

Our basic premise is that the conductivity of the conductors in a transmission line is usually so high that the effect of the series resistance on the computation of the propagation constant is negligible, the implication being that the waves on the line are approximately TEM. We may write, in dropping R from Eq. (9–37),

$$\gamma = \alpha + j\beta = \underbrace{\sqrt{(R+j\omega L)(G+j\omega C)}}_{(9\text{-}37)} \Rightarrow \gamma \cong j\omega\sqrt{LC}\left(1+\frac{G}{j\omega C}\right)^{1/2}. \quad \text{if } R \simeq 0 \tag{9–69}$$

From Eq. (8–44) we know that the propagation constant for a TEM wave in a medium with constitutive parameters (μ, ϵ, σ) is

(free space)

$$\gamma = j\omega\sqrt{\mu\epsilon}\left(1+\frac{\sigma}{j\omega\epsilon}\right)^{1/2}. \quad \text{(holds approximately if } R \simeq 0\text{)} \tag{9–70}$$

But

$$\frac{G}{C} = \frac{\sigma}{\epsilon} \tag{9–71}$$

in accordance with Eq. (5–81); hence comparison of Eqs. (9–69) and (9–70) yields

$$\boxed{LC = \mu\epsilon.} \quad \text{Approximation in case of } R \simeq 0 \tag{9–72}$$

Equation (9–72) is a very useful relation, because if L is known for a line with a given medium, C can be determined, and vice versa. Knowing C, we can find G from Eq. (9–71). Series resistance R is determined by introducing a small axial E_z as a slight perturbation of the TEM wave and by finding the ohmic power dissipated in a unit length of the line, as was done in Subsection 9–2.1.

Equation (9–72), of course, also holds for a lossless line. ***The velocity of wave propagation on a lossless transmission line, $u_p = 1/\sqrt{LC}$, therefore, is equal to the velocity of propagation, $1/\sqrt{\mu\epsilon}$, of unguided plane wave in the dielectric of the line.*** This fact has been pointed out in connection with Eq. (9–21) for parallel-plate lines.

1. *Two-wire transmission line.* The capacitance per unit length of a two-wire transmission line, whose wires have a radius a and are separated by a distance D, has been found in Eq. (4–47). We have

$$\boxed{C = \frac{\pi\epsilon}{\cosh^{-1}(D/2a)} \qquad \text{(F/m).}} \tag{9–73}^{\dagger}$$

From Eqs. (9–72) and (9–71) we obtain

$$L' \simeq \frac{\mu_0}{\pi}\ln\frac{D}{a} \simeq \boxed{L = \frac{\mu}{\pi}\cosh^{-1}\left(\frac{D}{2a}\right) \qquad \text{(H/m)}} \quad \text{external inductance} \tag{9–74}^{\dagger}$$

$\text{if } (D/2a)^2 \gg 1$

$= \frac{\mu_0}{\pi}\left(\frac{1}{4} + \frac{D}{a}\right)$ (8-145)

and

$$\boxed{G = \frac{\pi\sigma}{\cosh^{-1}(D/2a)} \qquad \text{(S/m).}} \tag{9–75}^{\dagger}$$

---

† $\cosh^{-1}(D/2a) \cong \ln(D/a)$ if $(D/2a)^2 \gg 1$.

To determine R, we go back to Eq. (9–28) and express the ohmic power dissipated per unit length of both wires in terms of p_σ. Assuming the current J_s (A/m) to flow in a very thin surface layer, the current in each wire is $I = 2\pi a J_s$, and

$$P_\sigma = 2\pi a p_\sigma = \frac{1}{2} I^2 \left(\frac{R_s}{2\pi a}\right) \quad \text{(W/m)}. \tag{9–76}$$

Hence the series resistance per unit length for both wires is

$$\boxed{R = 2\left(\frac{R_s}{2\pi a}\right) = \frac{1}{\pi a}\sqrt{\frac{\pi f \mu_c}{\sigma_c}} \quad (\Omega/\text{m}).} \tag{9–77}$$

In deriving Eqs. (9–76) and (9–77), we have assumed the surface current J_s to be uniform over the circumference of both wires. This is an approximation, inasmuch as the proximity of the two wires tends to make the surface current nonuniform.

2. *Coaxial transmission line.* The external inductance per unit length of a coaxial transmission line with a center conductor of radius a and an outer conductor of inner radius b has been found in Eq. (6–140):

$$\underset{(6\text{-}140)}{\frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\ln\frac{b}{a}} = L' \underset{\text{if } b \gg a \text{ or } \delta_c \ll 1}{\simeq} \boxed{L = \frac{\mu}{2\pi}\ln\frac{b}{a} \quad \text{(H/m).}} \quad \text{external inductance} \tag{9–78}$$

From Eq. (9–72) we obtain

$$\boxed{C = \frac{2\pi\epsilon}{\ln(b/a)} \quad \text{(F/m),}} \tag{9–79}$$

and from Eq. (9–71),

σ here is σeff in (7-112)

$$\boxed{G = \frac{2\pi\sigma}{\ln(b/a)} \quad \text{(S/m),}} \tag{9–80}$$

where σ is the equivalent conductivity of the lossy dielectric. If one prefers, σ could be replaced by $\omega\epsilon''$ as in Eq. (7–112).

To determine R, we again return to Eq. (9–27), where J_{si} on the surface of the center conductor is different from J_{so} on the inner surface of the outer conductor. We must have

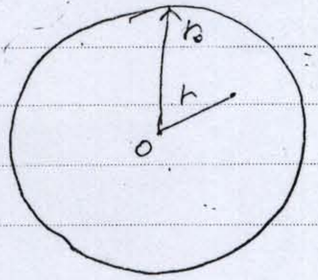
$$I = 2\pi a J_{si} = 2\pi b J_{so}. \tag{9–81}$$

The power dissipated in a unit length of the center and outer conductors are, respectively,

$$P_{\sigma i} = 2\pi a p_{\sigma i} = \frac{1}{2} I^2 \left(\frac{R_s}{2\pi a}\right), \tag{9–82}$$

Determination of R

Current distribution in a wire of circular cross section (§5.16, §5.18, RAMO)



$$\vec{J}_c(r) = \vec{a}_z \left[J_c(r_0)\, J_0(Tr) / J_0(Tr_0) \right] e^{-\gamma z}$$

where $T \triangleq j^{-\frac{1}{2}} \sqrt{2}/\delta_c$

J_0 : zero-order Bessel func.

$$\Rightarrow \frac{|\vec{J}_c(r)|}{|\vec{J}_c(r_0)|} = e^{-(r_0 - r)/\delta_c} \quad \text{if } r_0/\delta_c \gg 1$$

$$\oint \vec{H} \cdot d\vec{r} = I \Rightarrow H_\phi \Big|_{r=r_0} = \frac{I}{2\pi r_0}$$

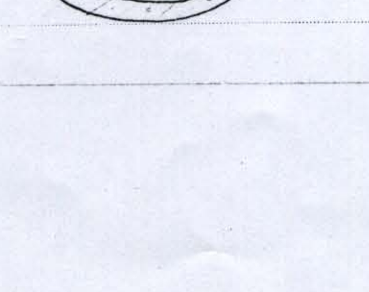
$$\nabla \times \vec{E} = -j\omega\mu_c \vec{H} \Rightarrow H_\phi = \frac{1}{j\omega\mu_c} \frac{dE_z}{dr}$$

$$E_z = \frac{J_z}{\sigma_c} = \frac{J_c(r_0)}{\sigma_c} \frac{J_0(Tr)}{J_0(Tr_0)} e^{-\gamma z}$$

$$\rightarrow \quad I = -\left[\frac{2\pi r_0 J_c(r_0)}{T} \right] \frac{J_0'(Tr_0)}{J_0(Tr_0)} e^{-\gamma z}$$

$$\Rightarrow Z_i \text{ (internal impedence per unit length)} = \frac{E_z \big|_{r=r_0}}{I}$$

$$= -\frac{T J_0(Tr_0)}{2\pi r_0 \sigma_c J_0'(Tr_0)}$$

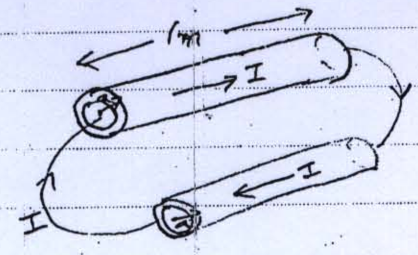


$$\boxed{Z_i \simeq \frac{(1+j)}{2\pi r_0} R_s \quad \text{if } r_0/\delta_c \gg 1}$$

$$\Uparrow$$

$$\boxed{R = \frac{1}{\sigma_c} \frac{1}{2\pi r_0 \delta_c} = \frac{R_s}{2\pi r_0}}$$

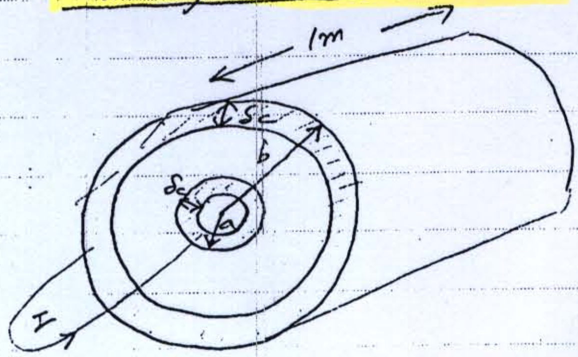
Case of two-wire line



$$R = \frac{R_s}{2\pi a} + \frac{R_s}{2\pi a} = \frac{R_s}{\pi a}$$

if $a/\delta_c \gg 1$

Case of coaxial line



$$R = R_{in} + R_{out}$$

$$= \frac{R_s}{2\pi a} + \frac{R_s}{2\pi b}$$

$$= \frac{R_s}{2\pi}\left(\frac{1}{a} + \frac{1}{b}\right)$$

if $a/\delta_c \gg 1$

TABLE 9–2
Distributed Parameters of Two-Wire and Coaxial Transmission Lines

Parameter	Two-Wire Line	Coaxial Line	Unit
R	$\dfrac{R_s}{\pi a}$	$\dfrac{R_s}{2\pi}\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$	Ω/m
L	$\dfrac{\mu}{\pi}\cosh^{-1}\left(\dfrac{D}{2a}\right)$	$\dfrac{\mu}{2\pi}\ln\dfrac{b}{a}$	H/m
G	$\dfrac{\pi\sigma}{\cosh^{-1}(D/2a)}$	$\dfrac{2\pi\sigma}{\ln(b/a)}$	S/m
C	$\dfrac{\pi\epsilon}{\cosh^{-1}(D/2a)}$	$\dfrac{2\pi\epsilon}{\ln(b/a)}$	F/m

Note: $R_s = \sqrt{\pi f \mu_c/\sigma_c}$; $\cosh^{-1}(D/2a) \cong \ln(D/a)$ if $(D/2a)^2 \gg 1$. Internal inductance is not included.

Either

(9.31) & (9-33) $\Rightarrow$

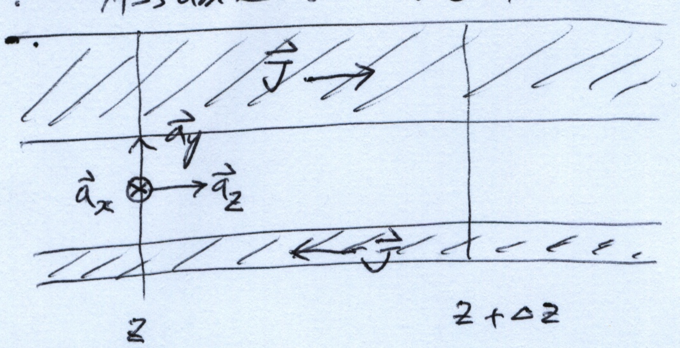
or

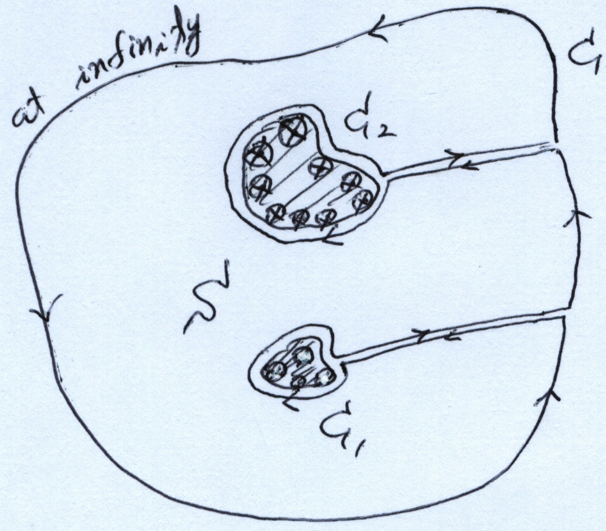
$$\boxed{-\frac{\partial}{\partial z}(VI) = \frac{\partial}{\partial t}\left(\frac{1}{2}CV^2 + \frac{1}{2}LI^2\right) + I^2R + GV^2}$$

Poynting's Theorem

<Fact1> Instantaneous Input Power at z is $V(z,t)\,I(z,t)$

Proof : Assume : TEM wave





cross section of a general two-conductor uniform transmission line at z

$$-\int_{S}(\vec E\times\vec H)\cdot d\vec S = \int_{S}\left(\nabla V+\frac{\partial \vec A}{\partial t}\right)\times\vec H\cdot dS(-\vec a_z)$$

$$= -\int_{S}(\nabla V\times\vec H)\cdot dS\,\vec a_z \quad \left(\text{since } \vec A=\frac{\mu}{4\pi}\int_{V'}\frac{\vec J(r',t-R/u)}{R}\,dv' \text{ and hence } (\vec A\times\vec H)\perp\vec a_z\right)$$

$$= -\int_{S}(\nabla\times V\vec H)\cdot dS\,\vec a_z+\int_{S}V(\nabla\times\vec H)\cdot dS\,\vec a_z$$

$$= -\oint_{C}V\vec H\cdot d\vec r+\int_{S}V(\underbrace{\cancel{\vec J}}_{0}+\underbrace{\frac{\partial\vec E}{\partial t})\cdot dS\,\vec a_z}_{=0 \text{ since } \vec E\parallel\vec a_y}$$

$$= -\oint_{C_1}V\vec H\cdot d\vec r-\oint_{C_2}V\vec H\cdot d\vec r \quad \text{since } V=0 \text{ at infinity}$$

$$= -V(1)\,I(z,t)+V(2)\,I(z,t) \quad \text{since } V=\text{constant on conductor surface}$$

$$= V(z,t)\,I(z,t)$$

<Fact2> Time-Average Input Power at z is $\frac{1}{2}\mathrm{Re}[V(z)I^*(z)]$ when $V(z,t)=V(z)e^{j\omega t}$ and $I(z,t)=I(z)e^{j\omega t}$

9-3.3 ATTENUATION CONSTANT FROM POWER RELATIONS

The attenuation constant of a traveling wave on a transmission line is the real part of the propagation constant; it can be determined from the basic definition in Eq. (9-37):

$$\alpha = \mathcal{R}e(\gamma) = \mathcal{R}e\left[\sqrt{(R + j\omega L)(G + j\omega C)}\right]. \tag{9-85}$$

The attenuation constant can also be found from a power relationship. The phasor voltage and phasor current distributions on an infinitely long transmission line (no reflections) may be written as (Eqs. (9-40a) and (9-40b) with the plus superscript dropped for simplicity):

$$V(z) = V_0 e^{-(\alpha + j\beta)z}, \tag{9-86a}$$

$$I(z) = \frac{V_0}{Z_0} e^{-(\alpha + j\beta)z}. \tag{9-86b}$$

The time-average power propagated along the line at any z is

$$\begin{aligned} P(z) &= \tfrac{1}{2}\mathcal{R}e[V(z)I^*(z)] \quad \text{(why?)} \\ &= \frac{V_0^2}{2|Z_0|^2} R_0 e^{-2\alpha z}. \end{aligned} \tag{9-87}$$

The law of conservation of energy requires that the rate of decrease of $P(z)$ with distance along the line equals the time-average power loss P_L per unit length. Thus,

$$\left.\begin{aligned} -\frac{dV}{dz} &= (R + j\omega L)I \\ -\frac{dI}{dz} &= (G + j\omega C)V \quad (9\text{-}35a,b) \\ \mathcal{R}e\, A &= \tfrac{1}{2}(A + A^*) \end{aligned}\right\} \xrightarrow[(9\text{-}87)]{} (*) \quad \boxed{-\frac{\partial P(z)}{\partial z} = P_L(z)} \quad \text{where } P_L(z) \text{ is given by (9-89)}$$

$$= 2\alpha P(z), \quad \text{from (9-87)}$$

from which we obtain the following formula:

$$\boxed{\alpha = \frac{P_L(z)}{2P(z)} \quad \text{(Np/m)}.} \tag{9-88}$$

<Note> (*) can be shown directly using Poynting's Theorem in Chapter 8

EXAMPLE 9-4

a) Use Eq. (9-88) to find the attenuation constant of a lossy transmission line with distributed parameters R, L, G, and C.

b) Specialize the result in part (a) to obtain the attenuation constants of a low-loss line and of a distortionless line.

Solution

a) For a lossy transmission line the time-average power loss per unit length is

$$\begin{aligned} P_L(z) &= \tfrac{1}{2}\left[|I(z)|^2 R + |V(z)|^2 G\right] \\ &= \frac{V_0^2}{2|Z_0|^2}(R + G|Z_0|^2)e^{-2\alpha z}. \end{aligned} \tag{9-89}$$

Substitution of Eqs. (9-87) and (9-89) in Eq. (9-88) gives

$$\boxed{\alpha = \frac{1}{2R_0}(R + G|Z_0|^2) \quad \text{(Np/m)}.} \tag{9-90}$$

b) For a low-loss line, $Z_0 \cong R_0 = \sqrt{L/C}$, Eq. (9-90) becomes

$$\alpha \cong \frac{1}{2}\left(\frac{R}{R_0} + GR_0\right)$$

9-4 Wave characteristics on Finite Transmission Lines

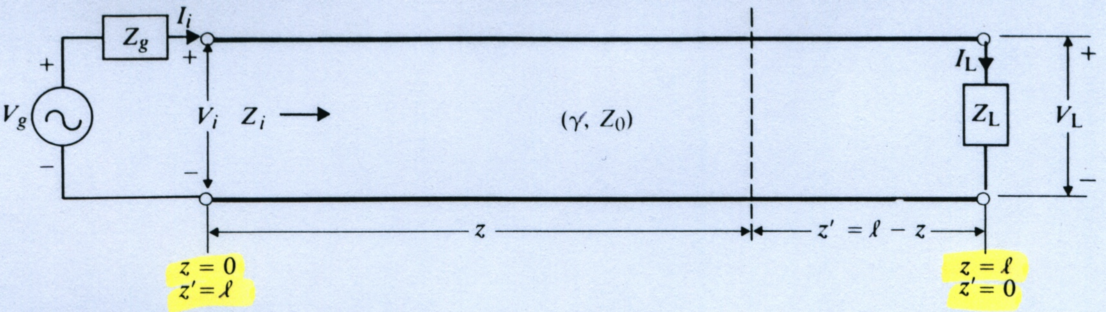


FIGURE 9–6
Finite transmission line terminated with load impedance Z_L.

Given the characteristic γ and Z_0 of the line and its length ℓ, there are four unknowns V_0^+, V_0^-, I_0^+, and I_0^- in Eqs. (9–93a) and (9–93b). These four unknowns are not all independent because they are constrained by the relations at $z = 0$ and at $z = \ell$. Both $V(z)$ and $I(z)$ can be expressed either in terms of V_i and I_i at the input end (Problem P.9–12), or in terms of the conditions at the load end. Consider the latter case.

Let $z = \ell$ in Eqs. (9–93a) and (9–93b). We have

$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z}$ (9-93a)(9-38a)

$$V_L = V_0^+ e^{-\gamma\ell} + V_0^- e^{\gamma\ell}, \tag{9–96a}$$

$I(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z}$ (9-93b)(9-38b)

$$I_L = \frac{V_0^+}{Z_0} e^{-\gamma\ell} - \frac{V_0^-}{Z_0} e^{\gamma\ell}. \tag{9–96b}$$

$\frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-}$ (9-94) ((9-39)) $= Z_0 = \frac{R + j\omega L}{\gamma}$ (9-41) characteristic impedance

Solving Eqs. (9–96a) and (9–96b) for V_0^+ and V_0^-, we have

$$V_0^+ = \tfrac{1}{2}(V_L + I_L Z_0)e^{\gamma\ell}, \tag{9–97a}$$

$$V_0^- = \tfrac{1}{2}(V_L - I_L Z_0)e^{-\gamma\ell}. \tag{9–97b}$$

Substituting Eq. (9–95) in Eqs. (9–97a) and (9–97b), and using the results in Eqs. (9–93a) and (9–93b), we obtain

$$V(z) = \frac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma(\ell - z)} + (Z_L - Z_0)e^{-\gamma(\ell - z)}\right], \tag{9–98a}$$

$$I(z) = \frac{I_L}{2Z_0}\left[(Z_L + Z_0)e^{\gamma(\ell - z)} - (Z_L - Z_0)e^{-\gamma(\ell - z)}\right]. \tag{9–98b}$$

Since ℓ and z appear together in the combination $(\ell - z)$, it is expedient to introduce a new variable $z' = \ell - z$, which is the distance measured backward from the load. Equations (9–98a) and (9–98b) then become

$$V(z') = \frac{I_L}{2}\left[(Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'}\right], \tag{9–99a}$$

$$I(z') = \frac{I_L}{2Z_0}\left[(Z_L + Z_0)e^{\gamma z'} - (Z_L - Z_0)e^{-\gamma z'}\right]. \tag{9–99b}$$

We note here that although the same symbols V and I are used in Eqs. (9–99a) and (9–99b) as in Eqs. (9–98a) and (9–98b), the dependence of $V(z')$ and $I(z')$ on z' is different from the dependence of $V(z)$ and $I(z)$ on z.

The use of hyperbolic functions simplifies the equations above. Recalling the relations

$$e^{\gamma z'} + e^{-\gamma z'} = 2\cosh \gamma z' \qquad \text{and} \qquad e^{\gamma z'} - e^{-\gamma z'} = 2\sinh \gamma z',$$

we may write Eqs. (9–99a) and (9–99b) as

(9-99a) $\frac{I_L}{2}\left[(Z_L+Z_0)e^{\gamma z'} + (Z_L-Z_0)e^{-\gamma z'}\right] =$

$$\boxed{V(z') = I_L(Z_L \cosh \gamma z' + Z_0 \sinh \gamma z'),} \tag{9–100a}$$

(9-99b) $\frac{I_L}{2Z_0}\left[(Z_L+Z_0)e^{\gamma z'} - (Z_L-Z_0)e^{-\gamma z'}\right] =$

$$\boxed{I(z') = \frac{I_L}{Z_0}(Z_L \sinh \gamma z' + Z_0 \cosh \gamma z'),} \tag{9–100b}$$

which can be used to find the voltage and current at any point along a transmission line in terms of I_L, Z_L, γ, and Z_0.

The ratio $V(z')/I(z')$ is the impedance when we look toward the load end of the line at a distance z' from the load.

$$Z(z') = \frac{V(z')}{I(z')} = Z_0 \frac{Z_L \cosh \gamma z' + Z_0 \sinh \gamma z'}{Z_L \sinh \gamma z' + Z_0 \cosh \gamma z'} \tag{9–101}$$

or

$$\boxed{Z(z') = Z_0 \frac{Z_L + Z_0 \tanh \gamma z'}{Z_0 + Z_L \tanh \gamma z'} \quad (\Omega).} \tag{9–102}$$

At the source end of the line, $z' = \ell$, the generator looking into the line sees an ***input impedance*** Z_i.

$$\boxed{Z_i = (Z)_{\substack{z=0 \\ z'=\ell}} = Z_0 \frac{Z_L + Z_0 \tanh \gamma\ell}{Z_0 + Z_L \tanh \gamma\ell} \quad (\Omega).} \tag{9–103}$$

As far as the conditions at the generator are concerned, the terminated finite transmission line can be replaced by Z_i, as shown in Fig. 9–7. The input voltage V_i and input current I_i in Fig. 9–6 are found easily from the equivalent circuit in Fig. 9–7.

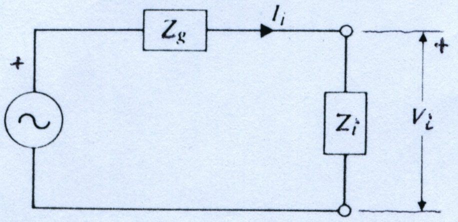


FIGURE 9–7
Equivalent circuit for finite transmission line in Figure 9–6 at generator end.

They are

$$V_i = \frac{Z_i}{Z_g + Z_i} V_g, \tag{9-104a}$$

$$I_i = \frac{V_g}{Z_g + Z_i}. \tag{9-104b}$$

Of course, the voltage and current at any other location on line cannot be determined by using the equivalent circuit in Fig. 9–7.

The average power delivered by the generator to the input terminals of the line is

$$(P_{av})_i = \tfrac{1}{2}\mathcal{R}e[V_i I_i^*]_{z=0,\, z'=\ell}. \tag{9-105}$$

The average power delivered to the load is

$$(P_{av})_L = \tfrac{1}{2}\mathcal{R}e[V_L I_L^*]_{z=\ell,\, z'=0}$$
$$= \frac{1}{2}\left|\frac{V_L}{Z_L}\right|^2 R_L = \frac{1}{2}|I_L|^2 R_L. \tag{9-106}$$

For a lossless line, conservation of power requires that $(P_{av})_i = (P_{av})_L$.

A particularly important special case is when a line is terminated with its characteristic impedance—that is, when $Z_L = Z_0$. The input impedance, Z_i in Eq. (9–103), is seen to be equal to Z_0. As a matter of fact, the impedance of the line looking toward the load at any distance z' from the load is, from Eq. (9–102),

$$(9\text{-}102)\quad Z_0\left(\frac{Z_L + Z_0 \tanh \gamma z'}{Z_0 + Z_L \tanh \gamma z'}\right) = Z(z') \Rightarrow Z(z') = Z_0 \qquad (\text{for } Z_L = Z_0). \tag{9-107}$$

The voltage and current equations in Eqs. (9–98a) and (9–98b) reduce to

$$(9\text{-}98a)\ \frac{I_L}{2}\left[(Z_L+Z_0)e^{\gamma(\ell-z)} + (Z_L-Z_0)e^{-\gamma(\ell-z)}\right] = V(z) = (I_L Z_0 e^{\gamma\ell})e^{-\gamma z} = V_i e^{-\gamma z}, \quad \text{if } Z_L = Z_0 \tag{9-108a}$$

$$(9\text{-}98b)\ \frac{I_L}{2Z_0}\left[(Z_L+Z_0)e^{\gamma(\ell-z)} - (Z_L-Z_0)e^{-\gamma(\ell-z)}\right] = I(z) = (I_L e^{\gamma\ell})e^{-\gamma z} = I_i e^{-\gamma z}. \tag{9-108b}$$

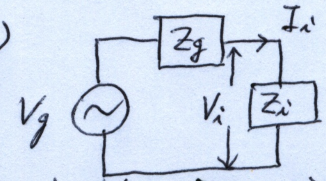
Equations (9–108a) and (9–108b) correspond to the pair of voltage and current equations—Eqs. (9–40a) and (9–40b)—representing waves traveling in $+z$-direction, and there are no reflected waves. Hence, ***when a finite transmission line is terminated with its own characteristic impedance (when a finite transmission line is matched), the voltage and current distributions on the line are exactly the same as though the line has been extended to infinity.***

no reflection if $Z_L = Z_0$
$$\begin{cases} V(z) = V_0^+ e^{-\gamma z}, & (9\text{-}40a) \\ I(z) = I_0^+ e^{-\gamma z}, \quad V_0^+/I_0^+ = Z_0 & (9\text{-}40b) \end{cases}$$

EXAMPLE 9–5 A signal generator having an internal resistance 1 (Ω) and an open-circuit voltage $v_g(t) = 0.3 \cos 2\pi 10^8 t$ (V) is connected to a 50 (Ω) lossless transmission line. The line is 4 (m) long, and the velocity of wave propagation on the line is 2.5×10^8 (m/s). For a matched load, find (a) the instantaneous expressions for the voltage and current at an arbitrary location on the line, (b) the instantaneous expressions for the voltage and current at the load, and (c) the average power transmitted to the load.

(cf.) Matched Condition in (Fig. 9-7)

$(P_{av})_i = \frac{1}{2}\mathcal{R}e[V_i I_i^*]$ (by (9-89))



$\underset{Z_i}{\operatorname{Arg\,max}}\,(P_{av})_i = Z_g^*$ (shown in problem P.9-11)

Solution

a) In order to find the voltage and current at an arbitrary location on the line, it is first necessary to obtain those at the input end ($z = 0, z' = \ell$). The given quantities are as follows:

$$V_g = 0.3\underline{/0^\circ} \quad \text{(V)}, \qquad \text{a phasor with a cosine reference,}$$
$$Z_g = R_g = 1 \quad (\Omega),$$
$$Z_0 = R_0 = 50 \quad (\Omega),$$
$$\omega = 2\pi \times 10^8 \quad \text{(rad/s)},$$
$$u_p = 2.5 \times 10^8 \quad \text{(m/s)},$$
$$\ell = 4 \quad \text{(m)}.$$

Since the line is terminated with a matched load, $Z_i = Z_0 = 50$ (Ω). The voltage and current at the input terminals can be evaluated from the equivalent circuit in Fig. 9–7. From Eqs. (9–104a) and (9–104b) we have

$$V_i = \frac{50}{1 + 50} \times 0.3\underline{/0^\circ} = 0.294\underline{/0^\circ} \quad \text{(V)},$$
$$I_i = \frac{0.3\underline{/0^\circ}}{1 + 50} = 0.0059\underline{/0^\circ} \quad \text{(A)}.$$

Since only forward-traveling waves exist on a matched line, we use Eqs. (9–86a) and (9–86b) for the voltage and current, respectively, at an arbitrary location. For the given line, $\alpha = 0$ and

$$\beta = \frac{\omega}{u_p} = \frac{2\pi \times 10^8}{2.5 \times 10^8} = 0.8\pi \quad \text{(rad/m)}.$$

Thus,

$$V(z) = 0.294e^{-j0.8\pi z} \quad \text{(V)},$$
$$I(z) = 0.0059e^{-j0.8\pi z} \quad \text{(A)}.$$

These are phasors. The corresponding instantaneous expressions are, from Eqs. (9–34a) and (9–34b),

$$\begin{aligned} v(z, t) &= \mathcal{R}e\left[0.294e^{j(2\pi 10^8 t - 0.8\pi z)}\right] \\ &= 0.294 \cos(2\pi 10^8 t - 0.8\pi z) \quad \text{(V)}, \\ i(z, t) &= \mathcal{R}e\left[0.0059e^{j(2\pi 10^8 t - 0.8\pi z)}\right] \\ &= 0.0059 \cos(2\pi 10^8 t - 0.8\pi z) \quad \text{(A)}. \end{aligned}$$

b) At the load, $z = \ell = 4$ (m),

$$v(4, t) = 0.294 \cos(2\pi 10^8 t - 3.2\pi) \quad \text{(V)},$$
$$i(4, t) = 0.0059 \cos(2\pi 10^8 t - 3.2\pi) \quad \text{(A)}.$$

c) The average power transmitted to the load on a lossless line is equal to that at the input terminals.

$$\begin{aligned} (P_{av})_L = (P_{av})_i &= \tfrac{1}{2}\mathcal{R}e\left[V(z)I^*(z)\right] \\ &= \tfrac{1}{2}(0.294 \times 0.0059) = 8.7 \times 10^{-4} \text{ (W)} = 0.87 \quad \text{(mW)}. \end{aligned}$$

9–4.1 TRANSMISSION LINES AS CIRCUIT ELEMENTS

Not only can transmission lines be used as wave-guiding structures for transferring power and information from one point to another, but at ultrahigh frequencies—UHF: frequency from 300 (MHz) to 3 (GHz), wavelength from 1 (m) to 0.1 (m)—they may serve as circuit elements. At these frequencies, ordinary lumped-circuit elements are difficult to make, and stray fields become important. Sections of transmission lines can be designed to give an inductive or capacitive impedance and are used to match an arbitrary load to the internal impedance of a generator for maximum power transfer. The required length of such lines as circuit elements becomes practical in the UHF range. At frequencies much lower than 300 (MHz) the required lines tend to be too long, whereas at frequencies higher than 3 (GHz) the physical dimensions become inconveniently small, and it would be advantageous to use waveguide components.

In most cases, transmission-line segments can be considered lossless: $\gamma = j\beta$, $Z_0 = R_0$, and $\tanh \gamma\ell = \tanh(j\beta\ell) = j\tan\beta\ell$. The formula in Eq. (9–103) for the input impedance Z_i of a lossless line of length ℓ terminated in Z_L becomes

$$\boxed{Z_i = R_0 \frac{Z_L + jR_0 \tan\beta\ell}{R_0 + jZ_L \tan\beta\ell} \quad (\Omega).} \tag{9–109}$$

(Lossless line)

Comparison of Eq. (9–109) with Eq. (8–171) again confirms the similarity between normal incidence of a uniform plane wave on a plane interface and wave propagation along a terminated transmission line.

We now consider several important special cases.

1. *Open-circuit termination* ($Z_L \to \infty$). We have, from Eq. (9–109),

$$Z_{io} = jX_{io} = -\frac{jR_0}{\tan\beta\ell} = -jR_0 \cot\beta\ell. \tag{9–110}$$

Equation (9–110) shows that the input impedance of an open-circuited lossless line is purely reactive. The line can, however, be either capacitive or inductive because the function $\cot\beta\ell$ can be either positive or negative, depending on the value of $\beta\ell\ (=2\pi\ell/\lambda)$. Figure 9–8 is a plot of $X_{io} = -R_0 \cot\beta\ell$ versus ℓ. We see that X_{io} can assume all values from $-\infty$ to $+\infty$.

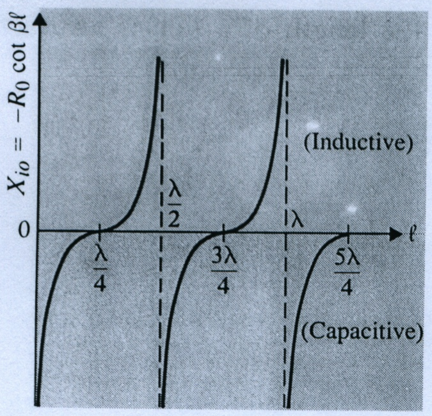
When the length of an open-circuited line is very short in comparison with a wavelength, $\beta\ell \ll 1$, we can obtain a very simple formula for its capacitive reactance by noting that $\tan\beta\ell \cong \beta\ell$. From Eq. (9–110) we have

$$\left\{\begin{array}{l} R_0 = \sqrt{L/C} \\ \beta = \omega\sqrt{LC} \end{array}\right.$$

$$Z_{io} = jX_{io} \cong -j\frac{R_0}{\beta\ell} = -j\frac{\sqrt{L/C}}{\omega\sqrt{LC}\ell} = -j\frac{1}{\omega C\ell}, \tag{9–111}$$

which is the impedance of a capacitance of $C\ell$ farads.

In practice, it is not possible to obtain an infinite load impedance at the end of a transmission line, especially at high frequencies, because of coupling to nearby objects and because of radiation from the open end.



$Z_{is} = \infty$ if $\ell = \frac{\lambda}{2} m$, $m = 0, \pm 1, \pm 2, \ldots$

$(Z_{is} = \infty)$

FIGURE 9–8
Input reactance of open-circuited transmission line. (lossless)

2. *Short-circuit termination* ($Z_L = 0$). In this case, Eq. (9–109) reduces to

$$Z_{is} = jX_{is} = jR_0 \tan \beta\ell. \tag{9–112}$$

Since $\tan \beta\ell$ can range from $-\infty$ to $+\infty$, the input impedance of a short-circuited lossless line can also be either purely inductive or purely capacitive, depending on the value of $\beta\ell$. Figure 9–9 is a graph of X_{is} versus ℓ. We note that Eq. (9–112) has exactly the same form as that—Eq. (8–172)—of the wave impedance of the total field at a distance ℓ from a perfectly conducting plane boundary.

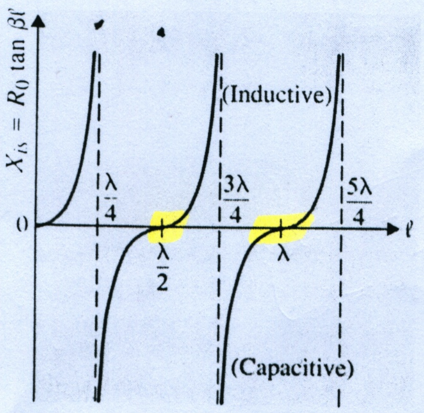
Comparing Figs. 9–8 and 9–9, we see that in the range where X_{io} is capacitive X_{is} is inductive, and vice versa. The input reactances of open-circuited and short-circuited lossless transmission lines are the same if their lengths differ by an odd multiple of $\lambda/4$.

When the length of a short-circuited line is very short in comparison with a wavelength, $\beta\ell \ll 1$, Eq. (9–112) becomes approximately

$$Z_{is} = jX_{is} \cong jR_0\beta\ell = j\sqrt{\frac{L}{C}}\,\omega\sqrt{LC}\,\ell = j\omega L\ell, \tag{9–113}$$

inductive reactance

which is the impedance of an inductance of $L\ell$ henries.



$Z_{is} = 0$ if $\ell = \frac{\lambda}{2} m$,

$m = 0, \pm 1, \pm 2, \ldots$

$(Z_{is} = 0)$

FIGURE 9–9
Input reactance of short-circuited transmission line. (lossless)

3. *Quarter-wave section* ($\ell = \lambda/4$, $\beta\ell = \pi/2$). When the length of a line is an odd multiple of $\lambda/4$, $\ell = (2n-1)\lambda/4$, ($n = 1, 2, 3, \ldots$),

$$\beta\ell = \frac{2\pi}{\lambda}(2n-1)\frac{\lambda}{4} = (2n-1)\frac{\pi}{2},$$

$$\tan\beta\ell = \tan\left[(2n-1)\frac{\pi}{2}\right] \to \pm\infty,$$

and Eq. (9–109) becomes

$$\boxed{Z_i = \frac{R_0^2}{Z_L} \quad \text{(Quarter-wave line).}} \tag{9–114}$$

Hence, ***a quarter-wave lossless line transforms the load impedance to the input terminals as its inverse multiplied by the square of the characteristic resistance.*** It acts as an impedance inverter and is often referred to as a ***quarter-wave transformer.*** An open-circuited, quarter-wave line appears as a short circuit at the input terminals, and a short-circuited quarter-wave line appears as an open circuit. Actually, if the series resistance of the line itself is not neglected, the input impedance of a short-circuited, quarter-wave line is an impedance of a very high value similar to that of a parallel resonant circuit. It is interesting to compare Eq. (9–114) with the formula for quarter-wave impedance transformation with multiple dielectrics, Eq. (8–182a).

4. *Half-wave section* ($\ell = \lambda/2$, $\beta\ell = \pi$). When the length of a line is an integral multiple of $\lambda/2$, $\ell = n\lambda/2$ ($n = 1, 2, 3, \ldots$),

$$\beta\ell = \frac{2\pi}{\lambda}\left(\frac{n\lambda}{2}\right) = n\pi,$$

$$\tan\beta\ell = 0,$$

and Eq. (9–109) reduces to

$$\boxed{Z_i = Z_L \quad \text{(Half-wave line).}} \tag{9–115}$$

Equation (9–115) states that ***a half-wave lossless line transfers the load impedance to the input terminals without change.*** From Eq. (9–103) we observe that a half-wave line with loss does not have this property unless $Z_L = Z_0$.

By measuring the input impedance of a line section under open- and short-circuit conditions, we can determine the characteristic impedance and the propagation constant of the line. The following expressions follow directly from Eq. (9–103).

Open-circuited line, $Z_L \to \infty$: $\quad Z_{io} = Z_0 \coth \gamma\ell$. (9–116)

Short-circuited line, $Z_L = 0$: $\quad Z_{is} = Z_0 \tanh \gamma\ell$. (9–117)

$$Z_i = Z_0 \frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \quad (\Omega) \quad (9\text{-}103)$$

From Eqs. (9–116) and (9–117) we have

$$\boxed{Z_0 = \sqrt{Z_{io}Z_{is}} \qquad (\Omega)} \tag{9–118}$$

and

$$\boxed{\gamma = \frac{1}{\ell}\tanh^{-1}\sqrt{\frac{Z_{is}}{Z_{io}}} \qquad (\text{m}^{-1}).} \tag{9–119}$$

Equations (9–118) and (9–119) apply whether or not the line is lossy.

EXAMPLE 9–6 The open-circuit and short-circuit impedances measured at the input terminals of a lossless transmission line of length 1.5 (m), which is less than a quarter wavelength, are $-j54.6$ (Ω) and $j103$ (Ω), respectively. (a) Find Z_0 and γ of the line. (b) Without changing the operating frequency, find the input impedance of a short-circuited line that is twice the given length. (c) How long should the short-circuited line be in order for it to appear as an open circuit at the input terminals?

Solution The given quantities are

$$Z_{io} = -j54.6, \qquad Z_{is} = j103, \qquad \ell = 1.5.$$

a) Using Eqs. (9–118) and (9–119), we find

$$Z_0 = \sqrt{-j54.6(j103)} = 75 \quad (\Omega)$$

$$\gamma = \frac{1}{1.5}\tanh^{-1}\sqrt{\frac{j103}{-j54.6}} = \frac{j}{1.5}\tan^{-1} 1.373 = j0.628 \quad (\text{rad/m}).$$

b) For a short-circuited line twice as long, $\ell = 3.0$ (m),

$$\gamma\ell = j0.628 \times 3.0 = j1.884 \quad (\text{rad}).$$

The input impedance is, from Eq. (9–117),

$$\begin{aligned} Z_{is} &= 75\tanh(j1.884) = j75\tan 108° \\ &= j75(-3.08) = -j231 \quad (\Omega). \end{aligned}$$

Note that Z_{is} for the 3 (m) line is now a capacitive reactance, whereas that for the 1.5 (m) line in part (a) is an inductive reactance. We may conclude from Fig. 9–9 that 1.5 (m) $< \lambda/4 <$ 3.0 (m).

c) In order for a short-circuited line to appear as an open circuit at the input terminals, it should be an odd multiple of a quarter-wavelength long:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.628} = 10 \quad (\text{m}).$$

Hence the required line length is

$$\begin{aligned} \ell &= \frac{\lambda}{4} + (n-1)\frac{\lambda}{2} \\ &= 2.5 + 5(n-1) \quad (\text{m}), \qquad n = 1, 2, 3, \ldots. \end{aligned}$$

So far in this subsection we have considered only open- and short-circuited lossless lines as circuit elements. We have seen in Figs. 9–8 and 9–9 that, depending on the length of the line, the input impedance of an open- or short-circuited lossless line can be either purely inductive or purely capacitive. Let us now examine the input impedance of a lossy line with a short-circuit termination. When the line length is a multiple of $\lambda/2$, the input impedance will not vanish as in Fig. 9–9. Instead, we have, from Eq. (9–117),

$$\begin{aligned} Z_{is} = Z_0 \tanh \gamma\ell &= Z_0 \frac{\sinh(\alpha + j\beta)\ell}{\cosh(\alpha + j\beta)\ell} \\ &= Z_0 \frac{\sinh \alpha\ell \cos \beta\ell + j \cosh \alpha\ell \sin \beta\ell}{\cosh \alpha\ell \cos \beta\ell + j \sinh \alpha\ell \sin \beta\ell}. \end{aligned} \tag{9–120}$$

Assume: $\alpha\ell \ll 1$

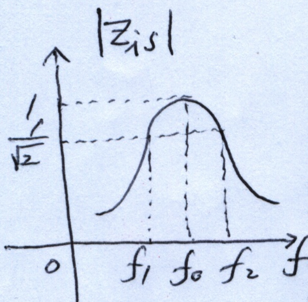
For $\ell = n\lambda/2$, $\beta\ell = n\pi$, $\sin \beta\ell = 0$, Eq. (9–120) reduces to

$$Z_{is} = Z_0 \tanh \alpha\ell \cong Z_0(\alpha\ell), \quad \left(\begin{array}{c}\text{Min. } |Z_{is}| \\ \text{at } \sin\beta\ell = 0\end{array}\right) \tag{9–121}$$

where we have assumed a low-loss line: $\alpha\ell \ll 1$ and $\tanh \alpha\ell \cong \alpha\ell$. The quantity Z_{is} in Eq. (9–121) is small but not zero. At $\ell = n\lambda/2$ we have the condition of a series-resonant circuit.

When the length of a shorted lossy line is an odd multiple of $\lambda/4$, the input impedance will not go to infinity as indicated in Fig. 9–9. For $\ell = n\lambda/4$, $\beta\ell = n\pi/2$ (n = odd), $\cos \beta\ell = 0$, and Eq. (9–120) becomes

$$Z_{is} = \frac{Z_0}{\tanh \alpha\ell} \cong \frac{Z_0}{\alpha\ell}, \quad \left(\begin{array}{c}\text{Max. } |Z_{is}| \\ \text{at } \cos\beta\ell = 0\end{array}\right) \tag{9–122}$$



which is large but not infinite. We have the condition of a parallel-resonant circuit. It is a frequency-selective circuit, and we can determine the ***quality factor***, or Q, of such a circuit by first finding its ***half-power bandwidth***, or simply the ***bandwidth***. The bandwidth of a parallel-resonant circuit is the frequency range $\Delta f = f_2 - f_1$ around the resonant frequency f_0, where $f_2 = f_0 + \Delta f/2$ and $f_1 = f_0 - \Delta f/2$ are half-power frequencies at which the voltage across the parallel circuit is $1/\sqrt{2}$ or 70.7% of its maximum value at f_0 (assuming a constant-current source). Hence the associated power, which is proportional to $|Z_{is}|^2$ and is maximum at f_0, is one-half of its value at f_1 and f_2.

$Q \triangleq \dfrac{f_0}{f_2 - f_1}$

Let $f = f_0 + \delta f$, where δf is a small frequency shift from the resonant frequency. We have

$$\begin{aligned} \beta\ell &= \frac{2\pi f}{u_p}\ell = \frac{2\pi(f_0 + \delta f)}{u_p}\ell \\ &= \frac{n\pi}{2} + \frac{n\pi}{2}\left(\frac{\delta f}{f_0}\right), \qquad n = \text{odd}, \end{aligned} \tag{9–123}$$

$$\cos \beta\ell = -\sin\left[\frac{n\pi}{2}\left(\frac{\delta f}{f_0}\right)\right] \cong -\frac{n\pi}{2}\left(\frac{\delta f}{f_0}\right), \tag{9–124}$$

$$\sin \beta\ell = \cos\left[\frac{n\pi}{2}\left(\frac{\delta f}{f_0}\right)\right] \cong 1, \tag{9–125}$$

$\boxed{\text{Proof of (9-122)}}$

Max. $|Z_{is}|$ at $f = f_0$

when $\ell = \left(\frac{\lambda_0}{4}\right) n$ (n = odd) ($\Rightarrow \cos\beta_0\ell = 0$)

where $\beta = \dfrac{2\pi}{\lambda} = \dfrac{2\pi f}{u_p}$

where we have assumed $(n\pi/2)(\delta f/f_0) \ll 1$. Substituting Eqs. (9–123), (9–124), and (9–125) in Eq. (9–120), noting that $\alpha \ell \ll 1$, and retaining only small terms of the first order, we obtain

j ≈

$$Z_{is} = Z_0 \frac{\sinh \alpha l \cos \beta l + j \cosh \alpha l \sin \beta l}{\cosh \alpha l \cos \beta l + j \sinh \alpha l \sin \beta l} \simeq Z_{is} = \frac{Z_0}{\alpha \ell + j\dfrac{n\pi}{2}\left(\dfrac{\delta f}{f_0}\right)} \tag{9–126}$$

$\approx -\frac{n\pi}{2}\frac{\delta f}{f_0} + j\alpha l$

and

$$|Z_{is}|^2 = \frac{|Z_0|^2}{(\alpha\ell)^2 + \left[\dfrac{n\pi}{2}\left(\dfrac{\delta f}{f_0}\right)\right]^2}. \tag{9–127}$$

At $f = f_0$, $\delta f = 0$, $|Z_{is}|^2$ is a maximum and equals $|Z_{is}|^2_{max} = |Z_0|^2/(\alpha\ell)^2$. Thus,

$$\frac{|Z_{is}|^2}{|Z_{is}|^2_{max}} = \frac{1}{1 + \left[\dfrac{n\pi}{2\alpha\ell}\left(\dfrac{\delta f}{f_0}\right)\right]^2}. \tag{9–128}$$

When $\delta f = \pm \Delta f/2$, we have the half-power frequencies f_2 and f_1, at which the ratio in Eq. (9–128) equals $\frac{1}{2}$, or

$$l = \left(\frac{\lambda_0}{4}\right)n,\ \beta_0 = \frac{2\pi}{\lambda_0} \Rightarrow \frac{n\pi}{2\alpha\ell}\left(\frac{\Delta f}{2f_0}\right) = \frac{\overset{=\beta_0}{\beta}}{2\alpha}\left(\frac{\Delta f}{f_0}\right) = 1, \qquad n = \text{odd}. \tag{9–129}$$

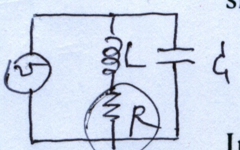
Therefore, the Q of the parallel-resonant circuit (a shorted lossy line having a length equal to an odd multiple of $\lambda/4$) is

$$\boxed{Q = \frac{f_0}{\Delta f} = \frac{\beta}{2\alpha}.} \tag{9–130}$$

Using the expressions of α and β for a low-loss line in Eqs. (9–55) and (9–56), we obtain

$$Q = \frac{\omega L}{R + GL/C} = \frac{1}{[(R/\omega L) + (G/\omega C)]}. \tag{9–131}$$

For a well-insulated line, $GL/C \ll R$, and Eq. (9–131) reduces to the familiar expression for the Q of a parallel-resonant circuit:



$$Q = \frac{\omega L}{R}. \tag{9–132}$$

Series resistance in L

In a similar manner an analysis can be made for the resonant behavior of an open-circuited low-loss transmission line whose length is an odd multiple of $\lambda/4$ (series resonance) or a multiple of $\lambda/2$ (parallel resonance). (See Problem P.9–21.)

(The situation is reversed)

EXAMPLE 9–7 The measured attenuation of an air-dielectric coaxial transmission line at 400 (MHz) is 0.01 (dB/m). Determine the Q and the half-power bandwidth of a quarter-wavelength section of the line with a short-circuit termination.

Solution At $f = 4 \times 10^8$ (Hz),

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{4 \times 10^8} = 0.75 \quad \text{(m)},$$

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.75} = 8.38 \quad \text{(rad/m)},$$

$$\alpha = 0.01 \text{ (dB/m)} = \frac{0.01}{8.69} \quad \text{(Np/m)}.$$

Therefore,

$$Q = \frac{\beta}{2\alpha} = \frac{8.38 \times 8.69}{2 \times 0.01} = 3641,$$

which is much higher than the Q obtainable from any lumped-element parallel-resonant circuit at 400 (MHz). The half-power bandwidth is

$$\Delta f = \frac{f_0}{Q} = \frac{4 \times 10^8}{3641} = 0.11 \times 10^6 \quad \text{(Hz)}$$
$$= 0.11 \quad \text{(MHz)}, \quad \text{or } 110 \quad \text{(kHz)}.$$

9–4.2 LINES WITH RESISTIVE TERMINATION

When a transmission line is terminated in a load impedance Z_L different from the characteristic impedance Z_0, both an incident wave (from the generator) and a reflected wave (from the load) exist. Equation (9–99a) gives the phasor expression for the voltage at any distance $z' = \ell - z$ from the load end. Note that in Eq. (9–99a), the term with $e^{\gamma z'}$ represents the incident voltage wave and the term with $e^{-\gamma z'}$ represents the reflected voltage wave. We may write

$$V(z') = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}\left[1 + \frac{Z_L - Z_0}{Z_L + Z_0} e^{-2\gamma z'}\right]$$
$$= \frac{I_L}{2}(Z_L + Z_0)e^{\gamma z'}[1 + \Gamma e^{-2\gamma z'}], \qquad (9\text{–}133a)$$

where

$$\boxed{\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta_\Gamma} \quad \text{(Dimensionless)}} \qquad (9\text{–}134)$$

is the ratio of the complex amplitudes of the reflected and incident voltage waves at the load ($z' = 0$) and is called the ***voltage reflection coefficient*** of the load impedance Z_L. It is of the same form as the definition of the reflection coefficient in Eq. (8–140) for a plane wave incident normally on a plane interface between two dielectric media. It is, in general, a complex quantity with a magnitude $|\Gamma| \leq 1$. The current equation

$$V(z') = \frac{1}{2} I_L [\underbrace{(Z_L + Z_0) e^{+\gamma z'}}_{\text{incident}} + \underbrace{(Z_L - Z_0) e^{-\gamma z'}}_{\text{reflected}}] \qquad (9\text{–}99a)$$

corresponding to $V(z')$ in Eq. (9–133a) is, from Eq. (9–99b),

$$I(z') = \frac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma z'}[1 - \Gamma e^{-2\gamma z'}]. \tag{9–133b}$$

The current reflection coefficient defined as the ratio of the complex amplitudes of the reflected and incident current waves, I_0^-/I_0^+, is different from the voltage reflection coefficient. As a matter of fact, the former is the negative of the latter, inasmuch as $I_0^-/I_0^+ = -V_0^-/V_0^+$, as is evident from Eq. (9–94). In what follows we shall refer only to the voltage reflection coefficient.

For a *lossless* transmission line, $\gamma = j\beta$, Eqs. (9–133a) and (9–133b) become

$Z_0 = R_0 = \sqrt{\frac{L}{C}}$

$$\begin{aligned} V(z') &= \frac{I_L}{2}(Z_L + R_0)e^{j\beta z'}[1 + \Gamma e^{-j2\beta z'}] \\ &= \frac{I_L}{2}(Z_L + R_0)e^{j\beta z'}[1 + |\Gamma|e^{j(\theta_\Gamma - 2\beta z')}] \end{aligned} \tag{9–135a}$$

and

$$I(z') = \frac{I_L}{2R_0}(Z_L + R_0)e^{j\beta z'}[1 - |\Gamma|e^{j(\theta_\Gamma - 2\beta z')}]. \tag{9–135b}$$

The voltage and current phasors on a lossless line are more easily visualized from Eqs. (9–100a) and (9–100b) by setting $\gamma = j\beta$ and $V_L = I_L Z_L$. Noting that $\cosh j\theta = \cos\theta$, and $\sinh j\theta = j\sin\theta$, we obtain

$$\boxed{\begin{aligned} V(z') &= V_L\cos\beta z' + jI_L R_0 \sin\beta z', \\ I(z') &= I_L \cos\beta z' + j\frac{V_L}{R_0}\sin\beta z'. \end{aligned}} \qquad \begin{aligned} &(9\text{–}136a) \\ &(9\text{–}136b) \end{aligned}$$

(Lossless line)

If the terminating impedance is purely resistive, $Z_L = R_L$, $V_L = I_L R_L$, the voltage and current magnitudes are given by

$$|V(z')| = V_L\sqrt{\cos^2\beta z' + (R_0/R_L)^2\sin^2\beta z'}, \tag{9–137a}$$

$$|I(z')| = I_L\sqrt{\cos^2\beta z' + (R_L/R_0)^2\sin^2\beta z'}, \tag{9–137b}$$

where $R_0 = \sqrt{L/C}$. Plots of $|V(z')|$ and $|I(z')|$ as functions of z' are standing waves with their maxima and minima occurring at fixed locations along the line.

Analogously to the plane-wave case in Eq. (8–147), we define the ratio of the maximum to the minimum voltages along a finite, terminated line as the ***standing-wave ratio*** **(SWR)**, S:

$$\boxed{S = \frac{|V_{max}|}{|V_{min}|} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \quad \text{(Dimensionless).}} \tag{9–138}$$

lossless line

$\Gamma = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

$S = \dfrac{1+|\Gamma|}{1-|\Gamma|}$

The inverse relation of Eq. (9–138) is

$$\boxed{|\Gamma| = \frac{S-1}{S+1} \qquad \text{(Dimensionless).}} \tag{9–139}$$

It is clear from Eqs. (9–138) and (9–139) that on a lossless transmission line

$\Gamma = 0, \quad S = 1 \quad$ when $Z_L = Z_0$ (Matched load);

$\Gamma = -1, \quad S \to \infty \quad$ when $Z_L = 0$ (Short circuit);

$\Gamma = +1, \quad S \to \infty \quad$ when $Z_L \to \infty$ (Open circuit).

Because of the wide range of S, it is customary to express it on a logarithmic scale: $20 \log_{10} S$ in (dB). Standing-wave ratio S defined in terms of $|I_{max}|/|I_{min}|$ results in the same expression as that defined in terms of $|V_{max}|/|V_{min}|$ in Eq. (9–138). A high standing-wave ratio on a line is undesirable because it results in a large power loss.

$P_{avg} = \frac{1}{2} Re[V I^*] = (1-|\Gamma|^2) \times |I_L(Z_L+R_0)|^2/2R_0$ from (9-135)

Examination of Eqs. (9–135a) and (9–135b) reveals that $|V_{max}|$ and $|I_{min}|$ occur together when

$$\theta_\Gamma - 2\beta z'_M = -2n\pi, \qquad n = 0, 1, 2, \ldots. \tag{9–140}$$

On the other hand, $|V_{min}|$ and $|I_{max}|$ occur together when

$$\theta_\Gamma - 2\beta z'_m = -(2n+1)\pi, \qquad n = 0, 1, 2, \ldots. \tag{9–141}$$

For resistive terminations on a lossless line, $Z_L = R_L$, $Z_0 = R_0$, and Eq. (9–134) simplifies to

$$\Gamma = \frac{R_L - R_0}{R_L + R_0} \qquad \text{(Resistive load).} \tag{9–142}$$

The voltage reflection coefficient is therefore purely real. Two cases are possible.

1. $R_L > R_0$: In this case, Γ is positive real and $\theta_\Gamma = 0$.

2. $R_L < R_0$: Γ negative and $\theta_\Gamma = -\pi$

2.

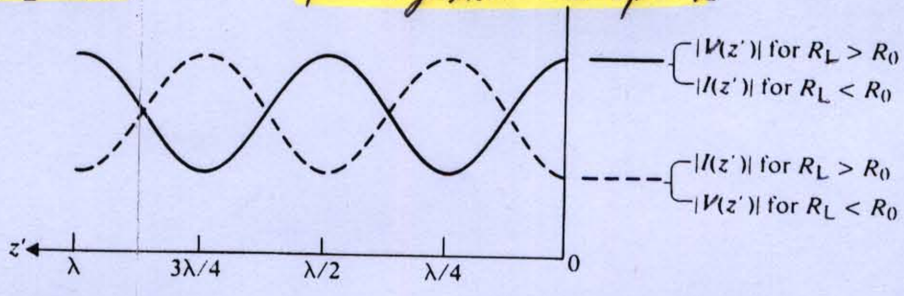


FIGURE 9–10
Voltage and current standing waves on resistance-terminated lossless lines.

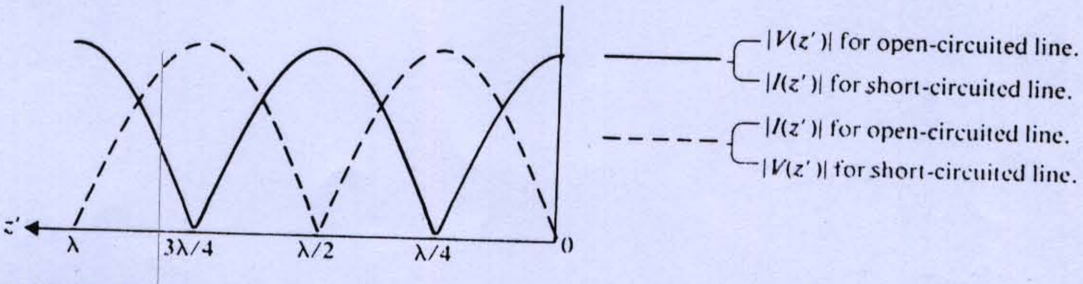


FIGURE 9–11
Voltage and current standing waves on open- and short-circuited lossless lines.

EXAMPLE 9–8 The standing-wave ratio S on a transmission line is an easily measurable quantity. (a) Show how the value of a terminating resistance on a lossless line of known characteristic impedance R_0 can be determined by measuring S. (b) What is the impedance of the line looking toward the load at a distance equal to one quarter of the operating wavelength?

Solution

a) Since the terminating impedance is purely resistive, $Z_L = R_L$, we can determine whether R_L is greater than R_0 (if there are voltage maxima at $z' = 0, \lambda/2, \lambda$, etc.) or whether R_L is less than R_0 (if there are voltage minima at $z' = 0, \lambda/2, \lambda$, etc.). This can be easily ascertained by measurements.

First, if $R_L > R_0$, $\theta_\Gamma = 0$. Both $|V_{max}|$ and $|I_{min}|$ occur at $\beta z' = 0$; and $|V_{min}|$ and $|I_{max}|$ occur at $\beta z' = \pi/2$. We have, from Eqs. (9–136a) and (9–136b),

$$\left(\begin{aligned} V(z') &= V_L \cos\beta z' + j I_L R_0 \sin\beta z' \\ I(z') &= I_L \cos\beta z' + j \frac{V_L}{R_0} \sin\beta z' \end{aligned}\right)$$

(9-136 a,b)

$$|V_{max}| = V_L, \qquad |V_{min}| = V_L \frac{R_0}{R_L};$$

$$|I_{min}| = I_L, \qquad |I_{max}| = I_L \frac{R_L}{R_0}.$$

Thus,

$$\frac{|V_{max}|}{|V_{min}|} = \frac{|I_{max}|}{|I_{min}|} = S = \frac{R_L}{R_0}$$

or

$$R_L = SR_0. \tag{9–145}$$

Second, if $R_L < R_0$, $\theta_\Gamma = -\pi$. Both $|V_{min}|$ and $|I_{max}|$ occur at $\beta z' = 0$; and $|V_{max}|$ and $|I_{min}|$ occur at $\beta z' = \pi/2$. We have

$$|V_{min}| = V_L, \qquad |V_{max}| = V_L \frac{R_0}{R_L};$$

$$|I_{max}| = I_L, \qquad |I_{min}| = I_L \frac{R_L}{R_0}.$$

Therefore,

$$\frac{|V_{max}|}{|V_{min}|} = \frac{|I_{max}|}{|I_{min}|} = S = \frac{R_0}{R_L}$$

or

$$R_L = \frac{R_0}{S}. \tag{9–146}$$

b) The operating wavelength, λ, can be determined from twice the distance between two neighboring voltage (or current) maxima or minima. At $z' = \lambda/4$, $\beta z' = \pi/2$, $\cos\beta z' = 0$, and $\sin\beta z' = 1$. Equations (9–136a) and (9–136b) become

$$V(\lambda/4) = jI_L R_0,$$

$$I(\lambda/4) = j\frac{V_L}{R_0}.$$

(*Question:* What is the significance of the j in these equations?) The ratio of $V(\lambda/4)$ to $I(\lambda/4)$ is the input impedance of a quarter-wavelength, resistively terminated, lossless line.

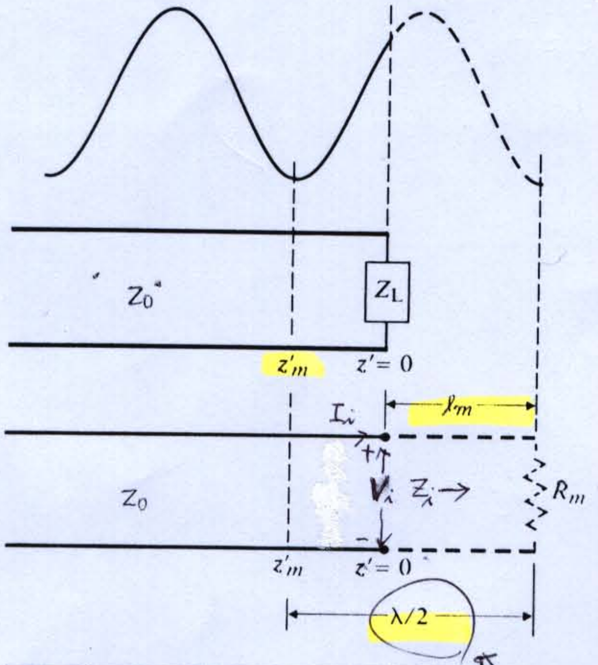
$$Z_i(z' = \lambda/4) = R_i = \frac{V(\lambda/4)}{I(\lambda/4)}$$
$$= \frac{R_0^2}{R_L}.$$

This result is anticipated because of the impedance-transformation property of a quarter-wave line given in Eq. (9–114). ■

9–4.3 LINES WITH ARBITRARY TERMINATION (lossless)

In the preceding subsection we noted that the standing wave on a *resistively terminated* lossless transmission line is such that a voltage maximum (a current minimum) occurs at the termination where $z' = 0$ if $R_L > R_0$, and a voltage minimum (a current maximum) occurs there if $R_L < R_0$. What will happen if the terminating impedance is not a pure resistance? It is intuitively correct to expect that a voltage maximum or minimum will not occur at the termination and that both will be shifted away from the termination. In this subsection we will show that information on the direction and amount of this shift can be used to determine the terminating impedance.

Let the terminating (or load) impedance be $Z_L = R_L + jX_L$, and assume the voltage standing wave on the line to look like that depicted in Fig. 9–12. We note that neither a voltage maximum nor a voltage minimum appears at the load at $z' = 0$. If we let the standing wave continue, say, by an extra distance ℓ_m, it will reach a minimum. The voltage minimum is where it should be if the original terminating impedance Z_L is replaced by a line section of length ℓ_m terminated by a pure resistance $R_m < R_0$



$$Z_i = R_0 \frac{Z_L + jR_0 \tan\beta\ell}{R_0 + jZ_L \tan\beta\ell} \ (\Omega) \ (9\text{-}109)$$
$$\parallel$$
$$Z_L \quad \text{if } Z_L = R_m \text{ and } \ell = \ell_m$$

FIGURE 9–12
Voltage standing wave on a line terminated by an arbitrary impedance, and equivalent line section with pure resistive load.

$z'_m + \ell_m = \lambda/2$ from (9-141) $\theta_\Gamma - 2\beta z'_m = -(2n+1)\pi$
where $\lambda = 2\pi/\beta$

$R_m < R_0$, as shown in the figure. The voltage distribution on the line to the left of the actual termination (where $z' > 0$) is not changed by this replacement.

The fact that any complex impedance can be obtained as the input impedance of a section of lossless line terminated in a resistive load can be seen from Eq. (9–109). Using R_m for Z_L and ℓ_m for ℓ, we have

$$Z_L \text{ in Fig. 9-12} = R_i + jX_i = R_0 \frac{R_m + jR_0 \tan \beta\ell_m}{R_0 + jR_m \tan \beta\ell_m}. \tag{9–147}$$

The real and imaginary parts of Eq. (9–147) form two equations, from which the two unknowns, R_m and ℓ_m, can be solved (see Problem P.9–28).

The load impedance Z_L can be determined experimentally by measuring the standing-wave ratio S and the distance z'_m in Fig. 9–12. (Remember that $z'_m + \ell_m = \lambda/2$.) The procedure is as follows:

1. Find $|\Gamma|$ from S. Use $|\Gamma| = \dfrac{S-1}{S+1}$ from Eq. (9–139).

2. Find θ_Γ from z'_m. Use $\theta_\Gamma = 2\beta z'_m - \pi$ for $n = 0$ from Eq. (9–141). $|V_{min}|$ occurs when $\theta_\Gamma - 2\beta z'_m = -(2n+1)\pi$,

3. Find Z_L, which is the ratio of Eqs. (9–135a) and (9–135b) at $z' = 0$:

$$\Gamma = \frac{Z_L - Z_o}{Z_L + Z_o} = |\Gamma| e^{j\theta_\Gamma} \; (9\text{-}134) \Rightarrow Z_L = R_L + jX_L = R_0 \frac{1 + |\Gamma|e^{j\theta_\Gamma}}{1 - |\Gamma|e^{j\theta_\Gamma}}. \tag{9–148}$$

$Z_o = R_o$

The value of R_m that, if terminated on a line of length ℓ_m, will yield an input impedance Z_L can be found easily from Eq. (9–147). Since $R_m < R_0$, $R_m = R_0/S$. from (9-146)

The procedure leading to Eq. (9–148) is used to determine Z_L from a measurement of S and of z'_m, the distance from the termination to the first voltage minimum. Of course, the distance from the termination to a voltage maximum, z'_M, could be used instead of z'_m. In that case, Eq. (9–140) should be used to find θ_Γ in Step 2 above.

EXAMPLE 9–9 The standing-wave ratio on a lossless 50 (Ω) transmission line terminated in an unknown load impedance is found to be 3.0. The distance between successive voltage minima is 20 (cm), and the first minimum is located at 5 (cm) from the load. Determine (a) the reflection coefficient Γ, and (b) the load impedance Z_L. In addition, find (c) the equivalent length and terminating resistance of a line such that the input impedance is equal to Z_L.

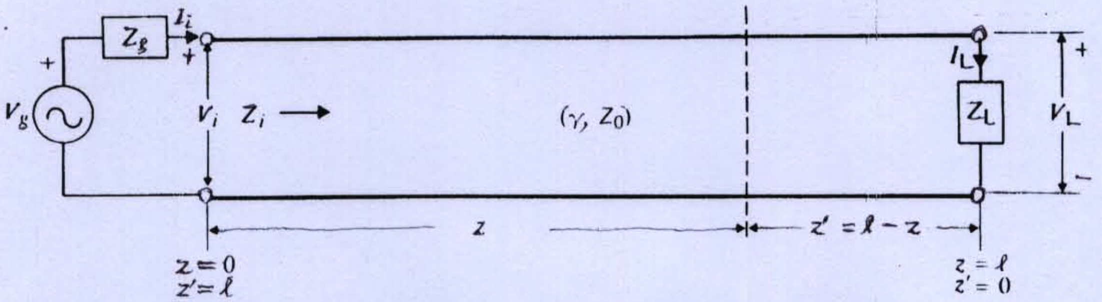
Solution

a) The distance between successive voltage minima is half a wavelength.

$$\lambda = 2 \times 0.2 = 0.4 \quad \text{(m)}, \qquad \beta = \frac{2\pi}{\lambda} = \frac{2\pi}{0.4} = 5\pi \quad \text{(rad/m)}.$$

Step 1: We find the magnitude of the reflection coefficient, $|\Gamma|$, from the standing-wave ratio $S = 3$.

$$|\Gamma| = \frac{S-1}{S+1} = \frac{3-1}{3+1} = 0.5.$$



(9-133a)
(9-133b) $\Leftarrow$

$$V(z') = \frac{I_L}{2}[(Z_L + Z_0)e^{\gamma z'} + (Z_L - Z_0)e^{-\gamma z'}], \tag{9-99a}$$

$$I(z') = \frac{I_L}{2Z_0}[(Z_L + Z_0)e^{\gamma z'} - (Z_L - Z_0)e^{-\gamma z'}]. \tag{9-99b}$$

$\Gamma \triangleq \frac{Z_L - Z_0}{Z_L + Z_0}$

c) Now we find R_m and ℓ_m in Fig. 9-12. We may use Eq. (9-147),

$$30 - j40 = 50\left(\frac{R_m + j50 \tan \beta \ell_m}{50 + jR_m \tan \beta \ell_m}\right),$$

and solve the simultaneous equations obtained from the real and imaginary parts for R_m and $\beta \ell_m$. Actually, we know $z'_m + \ell_m = \lambda/2$ and $R_m = R_0/S$. Hence,†

$$\ell_m = \frac{\lambda}{2} - z'_m = 0.2 - 0.05 = 0.15 \quad \text{(m)}$$

and

$$R_m = \frac{50}{3} = 16.7 \quad (\Omega).$$

9-4.4 TRANSMISSION-LINE CIRCUITS

Our discussions on the properties of transmission lines so far have been restricted primarily to the effects of the load on the input impedance and on the characteristics of voltage and current waves. No attention has been paid to the generator at the "other end," which is the source of the waves. Just as the constraint (the boundary condition), $V_L = I_L Z_L$, which the voltage V_L and the current I_L must satisfy at the load end ($z = \ell$, $z' = 0$), a constraint exists at the generator end where $z = 0$ and $z' = \ell$. Let a voltage generator V_g with an internal impedance Z_g represent the source connected to a finite transmission line of length ℓ that is terminated in a load impedance Z_L, as shown in Fig. 9-6. The additional constraint at $z = 0$ will enable the voltage and current anywhere on the line to be expressed in terms of the source characteristics (V_g, Z_g), the line characteristics (γ, Z_0, ℓ), and the load impedance (Z_L).

The constraint at $z = 0$ is

$$V_i = V_g - I_i Z_g. \tag{9-149}$$

But, from Eqs. (9-133a) and (9-133b),

at $z' = \ell$

$$V_i = \frac{I_L}{2}(Z_L + Z_0)e^{\gamma \ell}[1 + \Gamma e^{-2\gamma \ell}] \tag{9-150a}$$

and

$$I_i = \frac{I_L}{2Z_0}(Z_L + Z_0)e^{\gamma \ell}[1 - \Gamma e^{-2\gamma \ell}]. \tag{9-150b}$$

† Another set of solutions to part (c) is $\ell'_m = \ell_m - \lambda/4 = 0.05$ (m) and $R'_m = SR_0 = 150$ (Ω). Do you see why?

$R_m > R_0 \Rightarrow \ell'_m + z'_m = \lambda/4$

$\theta_\Gamma = 0 \Rightarrow R_m = R_0 \frac{1 + |\Gamma|}{1 - |\Gamma|} = R_0 S$

(cf.) Voltage Reflection Coefficient of Load Impedance
$\Gamma \triangleq (Z_L - Z_0)/(Z_L + Z_0)$

Substitution of Eqs. (9–150a) and (9–150b) in Eq. (9–149) enables us to find

$$\frac{I_L}{2}(Z_L + Z_0)e^{\gamma\ell} = \frac{Z_0 V_g}{Z_0 + Z_g}\frac{1}{[1 - \Gamma_g \Gamma e^{-2\gamma\ell}]}, \tag{9–151}$$

where

$$\Gamma_g = \frac{Z_g - Z_0}{Z_g + Z_0} \tag{9–152}$$

is the ***voltage reflection coefficient*** at the generator end. Using Eq. (9–151) in Eqs. (9–133a) and (9–133b), we obtain

These (steady-state) harmonic solution satisfies the boundary conditions at both generator and load ends

$$\boxed{V(z') = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z}\left(\frac{1 + \Gamma e^{-2\gamma z'}}{1 - \Gamma_g \Gamma e^{-2\gamma\ell}}\right).} \tag{9–153a}$$

Similarly,

$$\boxed{I(z') = \frac{V_g}{Z_0 + Z_g} e^{-\gamma z}\left(\frac{1 - \Gamma e^{-2\gamma z'}}{1 - \Gamma_g \Gamma e^{-2\gamma\ell}}\right).} \tag{9–153b}$$

Equations (9–153a) and (9–153b) are analytical phasor expressions for the voltage and current at any point on a finite line fed by a sinusoidal voltage source V_g. These are rather complicated expressions, but their significance can be interpreted in the following way. Let us concentrate our attention on the voltage equation (9–153a); obviously, the interpretation of the current equation (9–153b) is quite similar. We expand Eq. (9–153a) as follows:

$$\begin{aligned} V(z') &= \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z}(1 + \Gamma e^{-2\gamma z'})(1 - \Gamma_g \Gamma e^{-2\gamma\ell})^{-1} \\ &= \frac{Z_0 V_g}{Z_0 + Z_g} e^{-\gamma z}(1 + \Gamma e^{-2\gamma z'})(1 + \Gamma_g \Gamma e^{-2\gamma\ell} + \Gamma_g^2 \Gamma^2 e^{-4\gamma\ell} + \cdots) \\ &= \frac{Z_0 V_g}{Z_0 + Z_g}[e^{-\gamma z} + (\Gamma e^{-\gamma\ell})e^{-\gamma z'} + \Gamma_g(\Gamma e^{-2\gamma\ell})e^{-\gamma z} + \cdots] \\ &= V_1^+ + V_1^- + V_2^+ + V_2^- + \cdots, \end{aligned} \tag{9–154}$$

where

$$V_1^+ = \frac{V_g Z_0}{Z_0 + Z_g} e^{-\gamma z} = V_M e^{-\gamma z}, \tag{9–154a}$$

$$V_1^- = \Gamma(V_M e^{-\gamma\ell})e^{-\gamma z'}, \tag{9–154b}$$

$$V_2^+ = \Gamma_g(\Gamma V_M e^{-2\gamma\ell})e^{-\gamma z}. \tag{9–154c}$$

$$\vdots$$

The quantity

$$V_M = \frac{Z_0 V_g}{Z_0 + Z_g} \tag{9–155}$$

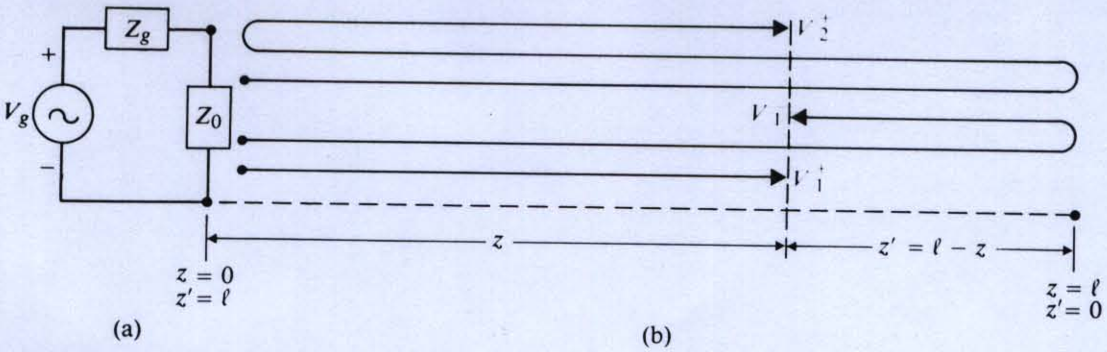


FIGURE 9–13
A transmission-line circuit and traveling waves.

is the complex amplitude of the voltage wave initially sent down the transmission line from the generator. It is obtained directly from the simple circuit shown in Fig. 9–13(a). The phasor V_1^+ in Eq. (9–154a) represents the initial wave traveling in the $+z$-direction. Before this wave reaches the load impedance Z_L, it sees Z_0 of the line as if the line were infinitely long.

When the first wave $V_1^+ = V_M e^{-\gamma z}$ reaches Z_L at $z = \ell$, it is reflected because of mismatch, resulting in a wave V_1^- with a complex amplitude $\Gamma(V_M e^{-\gamma\ell})$ traveling in the $-z$-direction. As the wave V_1^- returns to the generator at $z = 0$, it is again reflected for $Z_g \neq Z_0$, giving rise to a second wave V_2^+ with a complex amplitude $\Gamma_g(\Gamma V_M e^{-2\gamma\ell})$ traveling in $+z$-direction. This process continues indefinitely with reflections at both ends, and the resulting standing wave $V(z')$ is the sum of all the waves traveling in both directions. This is illustrated schematically in Fig. 9–13(b). In practice, $\gamma = \alpha + j\beta$ has a real part, and the attenuation effect of $e^{-\alpha\ell}$ diminishes the amplitude of a reflected wave each time the wave transverses the length of the line.

When the line is terminated with a matched load, $Z_L = Z_0$, $\Gamma = 0$, only V_1^+ exists, and it stops at the matched load with no reflections. If $Z_L \neq Z_0$ but $Z_g = Z_0$ (if the internal impedance of the generator is matched to the line), then $\Gamma \neq 0$ and $\Gamma_g = 0$. As a consequence, both V_1^+ and V_1^- exist, and V_2^+, V_2^- and all higher-order reflections vanish.

EXAMPLE 9–10 A 100 (MHz) generator with $V_g = 10\underline{/0^\circ}$ (V) and internal resistance 50 (Ω) is connected to a lossless 50 (Ω) air line that is 3.6 (m) long and terminated in a $25 + j25$ (Ω) load. Find (a) $V(z)$ at a location z from the generator, (b) V_i at the input terminals and V_L at the load, (c) the voltage standing-wave ratio on the line, and (d) the average power delivered to the load.

Solution Referring to Fig. 9–6, the given quantities are

$$V_g = 10\underline{/0^\circ} \quad (\text{V}), \qquad Z_g = 50 \quad (\Omega), \qquad f = 10^8 \quad (\text{Hz}),$$
$$R_0 = 50 \quad (\Omega), \qquad Z_L = 25 + j25 = 35.36\underline{/45^\circ} \quad (\Omega), \qquad \ell = 3.6 \quad (\text{m}).$$

Thus,

$$\beta = \frac{\omega}{c} = \frac{2\pi 10^8}{3 \times 10^8} = \frac{2\pi}{3} \quad \text{(rad/m)}, \qquad \beta\ell = 2.4\pi \quad \text{(rad)},$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{(25 + j25) - 50}{(25 + j25) + 50} = \frac{-25 + j25}{75 + j25} = \frac{35.36\underline{/135^\circ}}{79.1\underline{/18.4^\circ}}$$
$$= 0.447\underline{/116.6^\circ} = 0.447\underline{/0.648\pi},$$

$$\Gamma_g = 0.$$

a) From Eq. (9–153a) we have

$$V(z) = \frac{Z_0 V_g}{Z_0 + Z_g} e^{-j\beta z}[1 + \Gamma e^{-j2\beta(\ell - z)}]$$
$$= \frac{50(10)}{100} e^{-j2\pi z/3}[1 + 0.447 e^{j(0.648 - 4.8)\pi} e^{j4\pi z/3}]$$
$$= 5[e^{-j2\pi z/3} + 0.447 e^{j(2z/3 - 0.152)\pi}] \quad \text{(V)}.$$

We see that, because $\Gamma_g = 0$, $V(z)$ is the superposition of only two traveling waves, V_1^+ and V_1^-, as defined in Eq. (9–154).

b) At the input terminals,

$$V_i = V(0) = 5(1 + 0.447 e^{-j0.152\pi})$$
$$= 5(1.396 - j0.207)$$
$$= 7.06\underline{/-8.43^\circ} \quad \text{(V)}.$$

At the load,

$$V_L = V(3.6) = 5[e^{-j0.4\pi} + 0.447 e^{j0.248\pi}]$$
$$= 5(0.627 - j0.637) = 4.47\underline{/-45.5^\circ} \quad \text{(V)}.$$

c) The voltage standing-wave ratio (VSWR) is

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + 0.447}{1 - 0.447} = 2.62.$$

d) The average power delivered to the load is

$$P_{av} = \frac{1}{2}\left|\frac{V_L}{Z_L}\right|^2 R_L = \frac{1}{2}\left(\frac{4.47}{35.36}\right)^2 \times 25 = 0.200 \quad \text{(W)}.$$

It is interesting to compare this result with the case of a matched load when $Z_L = Z_0 = 50 + j0$ (Ω). In that case, $\Gamma = 0$,

$$|V_L| = |V_i| = \frac{V_g}{2} = 5 \quad \text{(V)},$$

and a maximum average power is delivered to the load:

$$\text{Maximum } P_{av} = \frac{V_L^2}{2R_L} = \frac{5^2}{2 \times 50} = 0.25 \quad \text{(W)},$$

§ 9-5 Transients on Transmission Lines

(lossless)

$$-\frac{\partial v(z,t)}{\partial z} = R\,\cancelto{0}{i(z,t)} + L\,\frac{\partial i(z,t)}{\partial t} \qquad (9\text{-}31)$$

$$-\frac{\partial i(z,t)}{\partial z} = G\,\cancelto{0}{v(z,t)} + C\,\frac{\partial v(z,t)}{\partial t} \qquad (9\text{-}33)$$

From (9-31) with $R=0$,

$$\frac{\partial^2 v(z,t)}{\partial^2 z} = -L\,\frac{\partial^2 i(z,t)}{\partial z\,\partial t} \qquad (9\text{-}31)'$$

From (9-33) with $G=0$,

$$-\frac{\partial^2 i(z,t)}{\partial t\,\partial z} = C\,\frac{\partial^2 v(z,t)}{\partial^2 t} \qquad (9\text{-}33)'$$

$$(9\text{-}31)' \text{ and } (9\text{-}33)' \Rightarrow \frac{\partial^2 v(z,t)}{\partial^2 z} = LC\,\frac{\partial^2 v(z,t)}{\partial t^2} \qquad (*1)$$

$\Rightarrow$ d'Alembert's solution:

$$(*2)\quad \begin{cases} v(z,t) = f(t - z\sqrt{LC}) + g(t + z\sqrt{LC}) \\ i(z,t) = \{f(t - z\sqrt{LC}) - g(t + z\sqrt{LC})\} \Big/ \overset{R_0}{\overset{\parallel}{\sqrt{L/C}}} \end{cases}$$

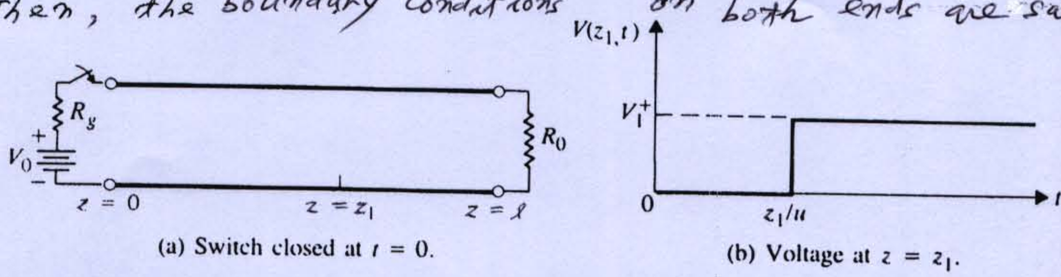
(Case I) $Z_L = R_0$ (Fig. 9-14):

Let $v(z,t) = V_1^+\, s(t - z\sqrt{LC})$

$i(z,t) = V_1^+\, s(t - z\sqrt{LC}) / R_0$

where $V_1^+ \triangleq R_0 V_0 / (R_0 + R_g)$

Then, the boundary conditions on both ends are satisfied.



(a) Switch closed at $t = 0$.

(b) Voltage at $z = z_1$.

FIGURE 9-14
A d-c source applied to a line terminated in characteristic resistance R_0 through a series resistance R_g.

(*2) at $t = 2T$, $z = 0$
$(t_2 = 0)$

$$V_0 - R_g\, i(z, t_2) = v(z, t_2) = V_1^+ + V_2^+ s(t_2 - z\sqrt{LC}) + V_1^- s(t_1 - z\sqrt{LC})$$

$$R_g\, i(z,t_2) = R_g\{[V_1^+ + V_2^+ s(t_2 - z\sqrt{LC})] - V_1^- s(t_1 - z\sqrt{LC})\}/R_0$$

$$\Rightarrow V_0 - R_g (V_1^+ + V_2^+ - V_1^-)/R_0 = V_1^+ + V_2^+ + V_1^-$$

$$V_0 = (R_0 + R_g) V_1^+ / R_0$$

$\Rightarrow$ (9-161) and (9-162)

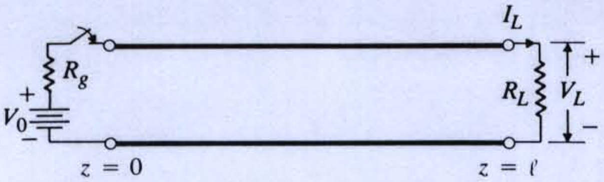


FIGURE 9–15
A d-c source applied to a terminated lossless line at $t = 0$ (general case).

has the same shape with a magnitude I_1^+ given in Eq. (9–157). When the voltage and current waves reach the termination at $z = \ell$, there are no reflected waves because $\Gamma = 0$. A steady state is established, and the entire line is charged to a voltage equal to V_1^+.

(Case II)
$Z_L = R_L \neq R_0$

If both the series resistance R_g and the load resistance R_L are not equal to R_0, as in Fig. 9–15, the situation is more complicated. When the switch is closed at $t = 0$, the d-c source sends a voltage wave of magnitude

$0 \le t < T$: $v(z,t) = V_1^+ s(t - z\sqrt{LC})$ where

$$V_1^+ = \frac{R_0}{R_0 + R_g} V_0 \tag{9–158}$$

in the $+z$-direction with a velocity $u = 1/\sqrt{LC}$ as before because the V_1^+ wave has no knowledge of the length of the line or the nature of the load at the other end; it proceeds as if the line were infinitely long. At $t = T = \ell/u$ this wave reaches the load end $z = \ell$. Since $R_L \neq R_0$, a reflected wave will travel in the $-z$-direction with a magnitude

(*1)

$T \le t < 2T$: $v(z', t_1) = V_1^+ s(t_1 + z'\sqrt{LC}) + V_1^- s(t_1 - z'\sqrt{LC})$
$(0 \le t_1 < T)$
$i(z', t_1) = \{V_1^+ s(t_1 + z'\sqrt{LC}) - V_1^- s(t_1 - z'\sqrt{LC})\}/R_0$

$$V_1^- = \Gamma_L V_1^+, \tag{9–159}$$

$$\Gamma_L = \frac{R_L - R_0}{R_L + R_0} \tag{9–160}$$

$\Leftarrow$ (*1) (or from (9-154b))

where
$v(0,0) = V_L = R_L I_L$
$i(0,0) = I_L$
$\Rightarrow V_1^+ + V_1^- = R_L I_L$
$V_1^+ - V_1^- = R_0 I_L$
$\Rightarrow$ (9-159), (9-160)

is the reflection coefficient of the load resistance R_L. This reflected wave arrives at the input end at $t = 2T$, where it is reflected by $R_g \neq R_0$. A new voltage wave having a magnitude V_2^+ then travels down the line, where

$$V_2^+ = \Gamma_g V_1^- = \Gamma_g \Gamma_L V_1^+. \tag{9–161}$$

$$\Gamma_g = \frac{R_g - R_0}{R_g + R_0} \tag{9–162}$$

$\Leftarrow$ (*2) (or from (9-154c))

is the reflection coefficient of the series resistance R_g. This process will go on indefinitely with waves traveling back and forth, being reflected at each end at $t = nT$ ($n = 1, 2, 3, \ldots$).

Two points are worth noting here. First, some of the reflected waves traveling in either direction may have a negative amplitude, since Γ_L or Γ_g (or both) may be negative. Second, except for an open circuit or a short circuit, Γ_L and Γ_g are less than unity. Thus the magnitude of the successive reflected waves becomes smaller

d'Alembert's solution:

(*2)'
$$v(z', t') = f(t' + z'\sqrt{LC}) + g(t' - z'\sqrt{LC})$$
$$i(z', t') = \{f(t' + z'\sqrt{LC}) - g(t' - z'\sqrt{LC})\}/\sqrt{L/C} \quad (= R_0)$$

and smaller, leading to a convergent process. The progression of the transient voltage waves on the lossless line in Fig. 9–15 for $R_L = 3R_0$ ($\Gamma_L = \frac{1}{2}$) and $R_g = 2R_0$ ($\Gamma_g = \frac{1}{3}$) is illustrated in Figs. 9–16(a), 9–16(b), and 9–16(c) for three different time intervals. The corresponding current waves are given in Figs. 9–16(d), 9–16(e), and 9–16(f), which are self-explanatory. The voltage and current at any particular location on the line in any particular time interval are just the *algebraic sums* $(V_1^+ + V_1^- + V_2^+ + V_2^- + \ldots)$ and $(I_1^+ + I_1^- + I_2^+ + I_2^- + \ldots)$, respectively.

It is interesting to check the ultimate value of the voltage across the load, $V_L = V(\ell)$, as t increases indefinitely. We have

$$\begin{aligned} V_L &= V_1^+ + V_1^- + V_2^+ + V_2^- + V_3^+ + V_3^- + \cdots \\ &= V_1^+(1 + \Gamma_L + \Gamma_g\Gamma_L + \Gamma_g\Gamma_L^2 + \Gamma_g^2\Gamma_L^2 + \Gamma_g^2\Gamma_L^3 + \cdots) \\ &= V_1^+[(1 + \Gamma_g\Gamma_L + \Gamma_g^2\Gamma_L^2 + \cdots) + \Gamma_L(1 + \Gamma_g\Gamma_L + \Gamma_g^2\Gamma_L^2 + \cdots)] \\ &= V_1^+\left[\left(\frac{1}{1-\Gamma_g\Gamma_L}\right) + \left(\frac{\Gamma_L}{1-\Gamma_g\Gamma_L}\right)\right] \\ &= V_1^+\left(\frac{1+\Gamma_L}{1-\Gamma_g\Gamma_L}\right). = \left(\frac{R_L}{R_L+R_g}\right)V_0 \end{aligned} \tag{9–163}$$

For the present case, $V_1^+ = V_0/3$, $\Gamma_L = \frac{1}{2}$, and $\Gamma_g = 1/3$, Eq. (9–163) gives

$$V_L = \tfrac{9}{5}V_1^+ = \tfrac{3}{5}V_0 \tag{9–163a}$$

Similarly, $I_L = \left(\frac{1-\Gamma_L}{1-\Gamma_g\Gamma_L}\right)\frac{V_1^+}{R_0} = \frac{V_L}{R_L}$

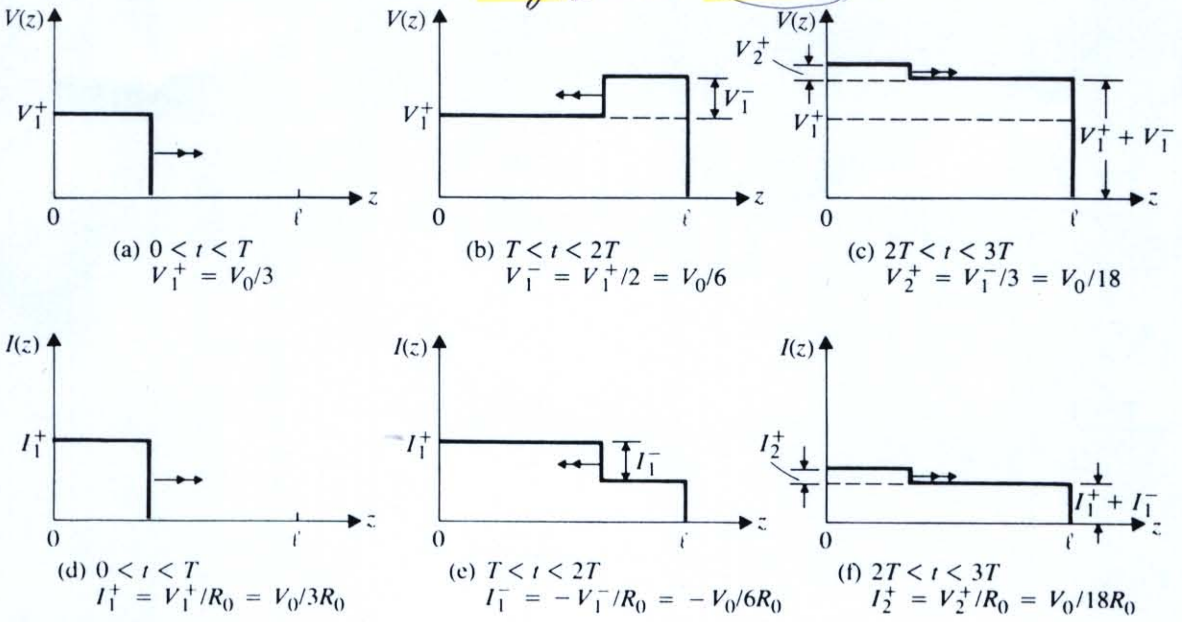
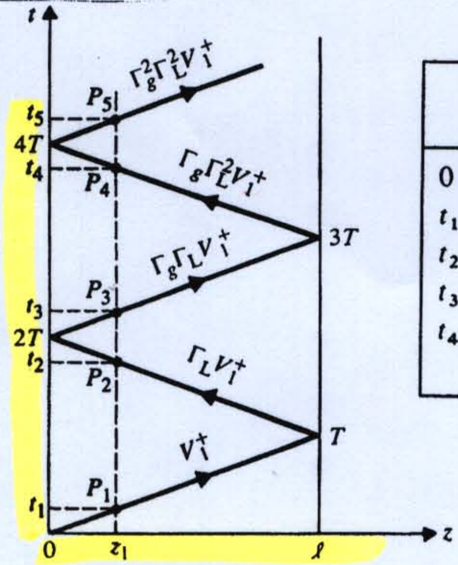


FIGURE 9–16
Transient voltage and current waves on transmission line in Fig. 9–15 for $R_L = 3R_0$ and $R_g = 2R_0$.

Note : $\frac{V_1^+}{I_1^+} = R_0$, $\frac{V_1^-}{I_1^-} = -R_0$, $\frac{V_2^+}{I_2^+} = R_0$

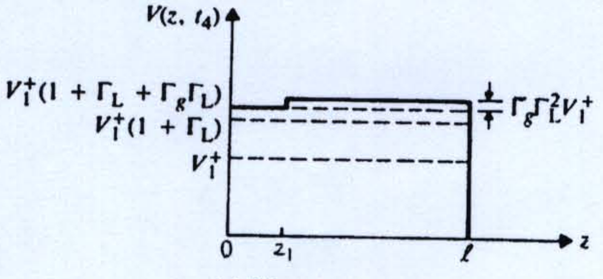


Time Range	Voltage	Voltage Discontinuity
$0 \le t < t_1$ ($t_1 = z_1/u$)	0	0
$t_1 \le t < t_2$ ($t_2 = 2T - t_1$)	V_1^+	V_1^+ at t_1
$t_2 \le t < t_3$ ($t_3 = 2T + t_1$)	$V_1^+(1 + \Gamma_L)$	$\Gamma_L V_1^+$ at t_2
$t_3 \le t < t_4$ ($t_4 = 4T - t_1$)	$V_1^+(1 + \Gamma_L + \Gamma_g\Gamma_L)$	$\Gamma_g\Gamma_L V_1^+$ at t_3
$t_4 \le t < t_5$ ($t_5 = 4T + t_1$)	$V_1^+(1 + \Gamma_L + \Gamma_g\Gamma_L + \Gamma_g\Gamma_L^2)$	$\Gamma_g\Gamma_L^2 V_1^+$ at t_4
⋮	⋮	⋮

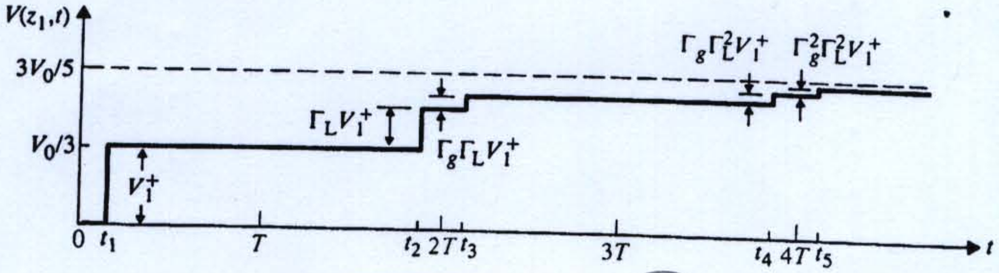
FIGURE 9-17
Voltage reflection diagram for transmission-line circuit in Fig. 9-15.

$$V_1^+ = \frac{R_0}{R_0 + R_g} V_0$$

$$\Gamma_g = \frac{R_g - R_0}{R_g + R_0}, \quad \Gamma_L = \frac{R_L - R_0}{R_L + R_0}$$

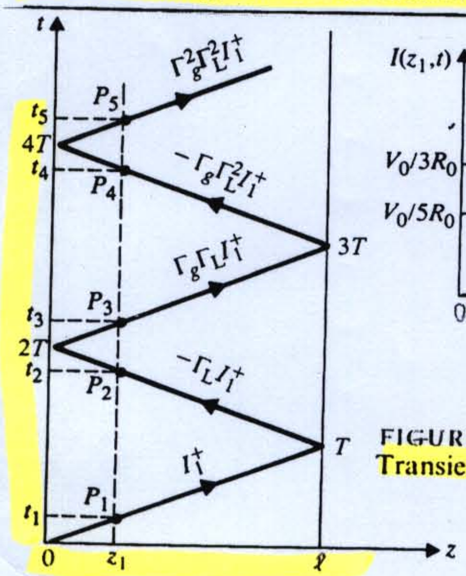


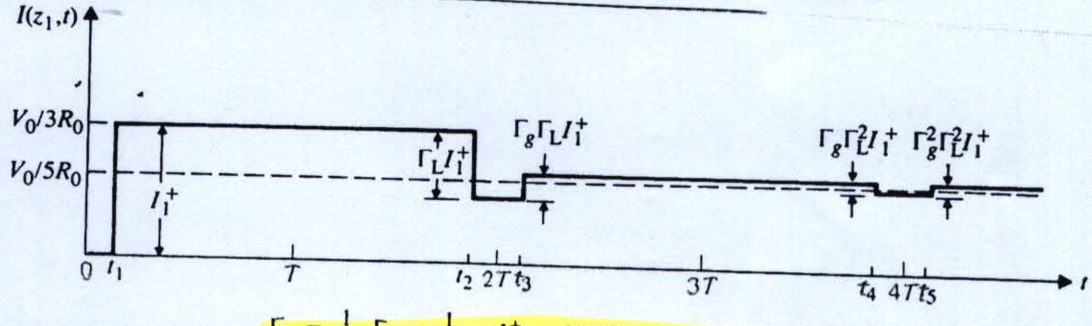
(a) $V(z, t_4)$ versus z;
$\Gamma_L = \frac{1}{2}$, $\Gamma_g = \frac{1}{3}$, $V_1^+ = V_0/3$.



(b) $V(z_1, t)$ versus t; $V(z_1, \infty) = 3V_0/5$. $= V(\ell, \infty) = \dfrac{R_L}{R_g + R_L} V_0$

FIGURE 9-18
Transient voltage on lossless transmission line for $R_L = 3R_0$ and $R_g = 2R_0$.





$\Gamma_L = \frac{1}{2}$, $\Gamma_g = \frac{1}{3}$, $I_1^+ = V_0/3R_0$, $I(z_1, \infty) = V_0/5R_0$.

$= I(\ell, \infty) = \dfrac{V(\ell, \infty)}{R_L}$

FIGURE 9-20
Transient current on lossless transmission line for $R_L = 3R_0$ and $R_g = 2R_0$.

FIGURE 9-19
Current reflection diagram for transmission-line circuit in Fig. 9-15.

EXAMPLE 9–11 A rectangular pulse of an amplitude 15 (V) and a duration 1 (μs) is applied through a series resistance of 25 (Ω) to the input terminals of a 50 (Ω) lossless coaxial transmission line. The line is 400 (m) long and is short-circuited at the far end. Determine the voltage response at the midpoint of the line as a fuction of time up to 8 (μs). The dielectric constant of the insulating material in the cable is 2.25.

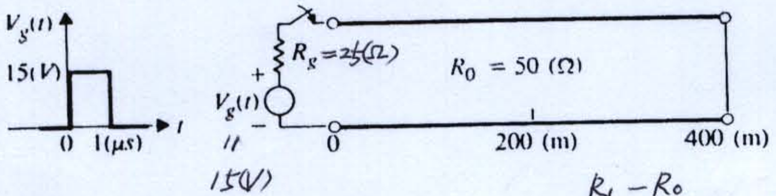


FIGURE 9–22
A pulse applied to a short-circuited line.

$$\frac{R_L - R_o}{R_L + R_o} = \Gamma_L = -1, \qquad \Gamma_g = \frac{25-50}{25+50} = -\frac{1}{3},$$

$$v_g(t) = \overset{15}{\cancel{10}}[U(t) - U(t-10^{-6})],$$

$$u = \frac{c}{\sqrt{\epsilon_r}} = \frac{3 \times 10^8}{\sqrt{2.25}} = 2 \times 10^8 \quad \text{(m/s)},$$

$$T = \frac{\ell}{u} = \frac{400}{2 \times 10^8} = 2 \times 10^{-6}\ \text{(s)} = 2 \quad (\mu\text{s})$$

$$V_1^+ = \frac{15R_0}{R_0 + R_g} = \frac{15 \times 50}{50 + 25} = 10 \quad \text{(V)}.$$

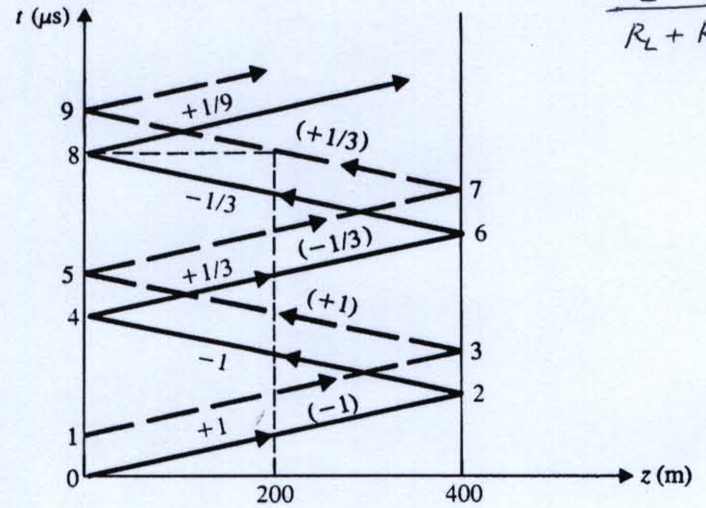


FIGURE 9–23
Voltage reflection diagram for Example 9–11.

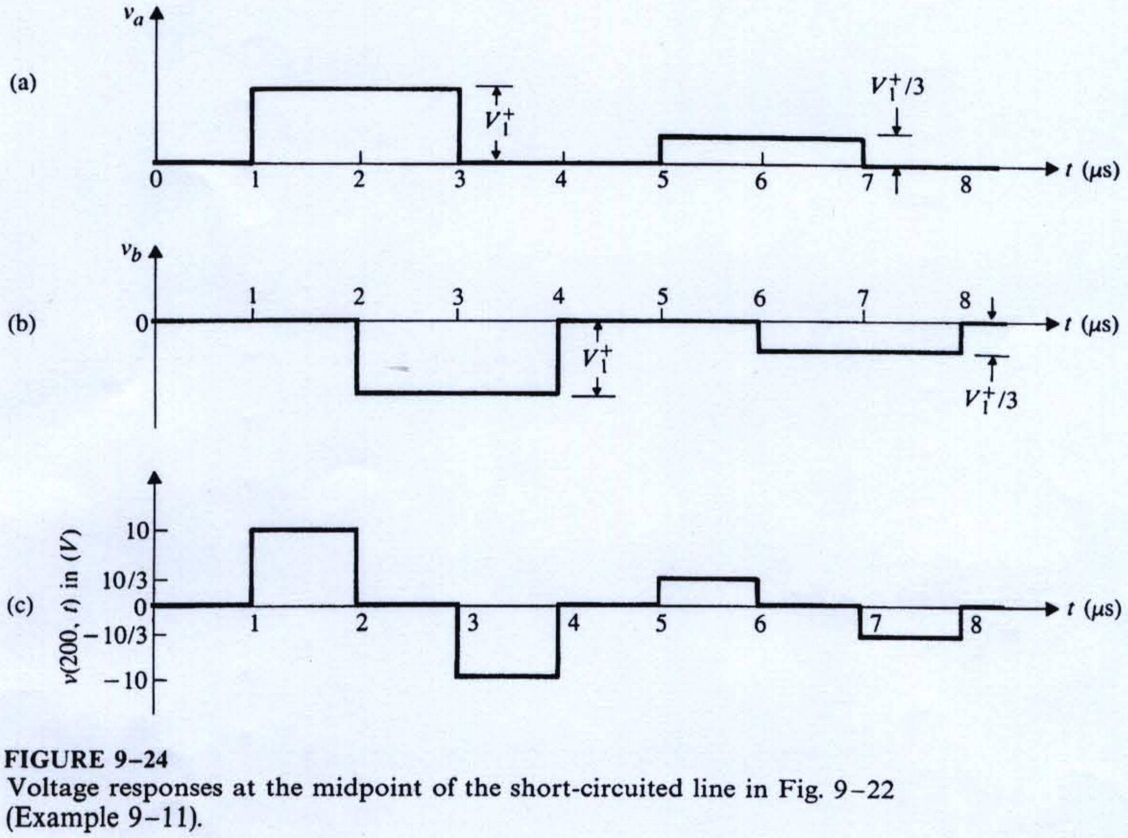


FIGURE 9–24
Voltage responses at the midpoint of the short-circuited line in Fig. 9–22 (Example 9–11).

9-5.3 INITIALLY CHARGED LINE

In our discussion of transients on transmission lines we have assumed that the lines themselves have no initial voltages or currents when an external source is applied. Actually, any disturbance or change in a transmission-line circuit will start transients along the line even without an external source if initial voltages and/or currents exist. We examine in this subsection a situation involving an initially charged line and develop a method of analysis.

Consider the following example.

EXAMPLE 9–12 A lossless, air-dielectric, open-circuited transmission line of characteristic resistance R_0 and length ℓ is initially charged to a voltage V_0. At $t = 0$ the line is connected to a resistance R. Determine the voltage across and the current in R as functions of time. Assume that $R = R_0$.

When the switch is closed, a voltage wave of amplitude V_1^+ will be sent down the line in the $+z$-direction, where

$$V_1^+ = -\frac{R_0}{R + R_0} V_0 = -\frac{V_0}{2}.$$

At $t = \ell/c$, the V_1^+ wave reaches the open end, having reduced the voltage along the whole line from V_0 to $V_0/2$. At the open end, $\Gamma = 1$, and a reflected V_1^- wave is sent back in the $-z$-direction with $V_1^- = V_1^+ = -V_0/2$. This reflected wave returns to the sending end at $t = 2\ell/c$, reducing the voltage on the line to zero.

From Fig. 9–25(d),

$$I_R = -I_1,$$

where

$$I_1 = I_1^+ = \frac{V_1^+}{R_0} = -\frac{V_0}{2R_0} \qquad \text{for} \quad 0 \le t < 2\ell/c.$$

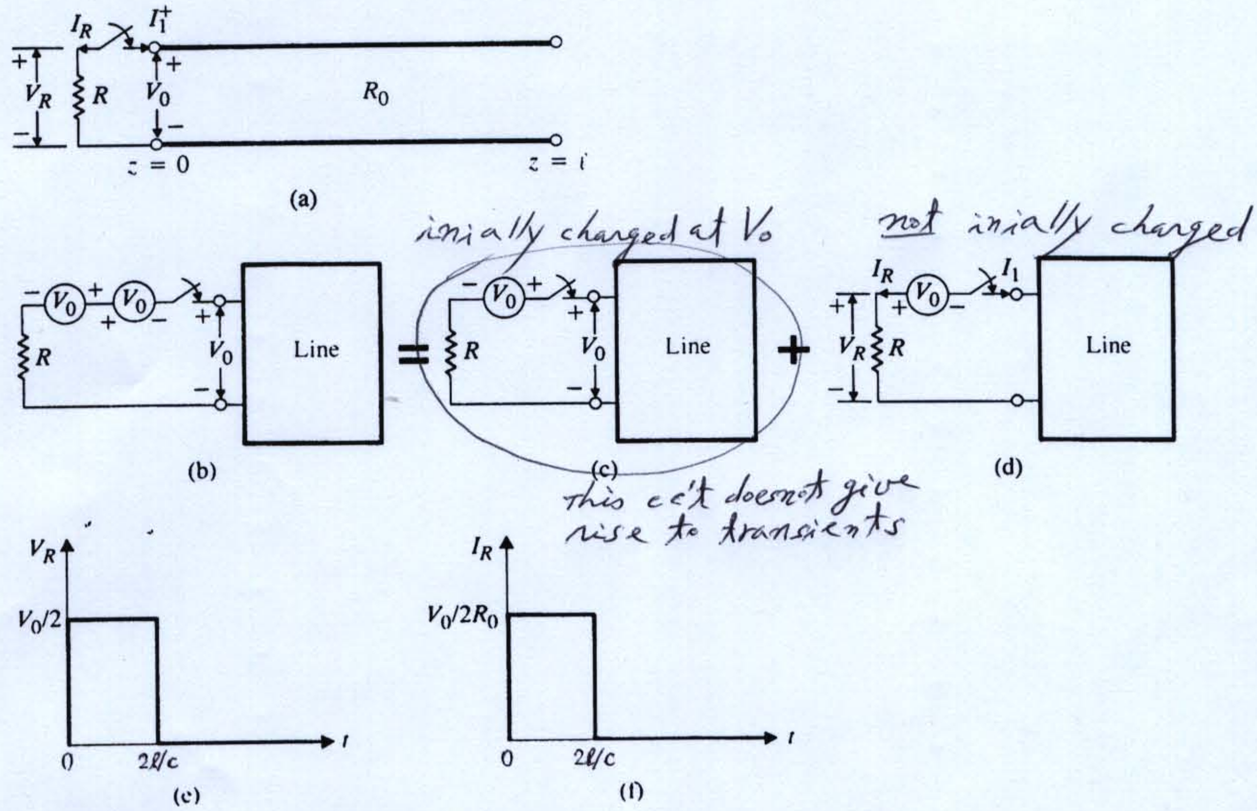


FIGURE 9–25
Transient problem of an open-circuited, initially charged line, $R = R_0$ (Example 9–12).

At $t = \ell/c$, I_1^+ reaches the open end, and the reflected I_1^- must make the total current there zero. Hence,

$$I_1^- = -I_1^+ = \frac{V_0}{2R_0},$$

which reaches the sending end at $t = 2\ell/c$ and reduces both I_1 and I_R to zero. Since $R = R_0$, there is no further reflection, and the transient state ends. As shown in Figs. 9–25(e) and 9–25(f), both V_R and I_R are a pulse of duration $2\ell/c$. We then have a way of generating a pulse by discharging a charged open-circuited transmission line, the width of the pulse being adjustable by changing ℓ. ▬

9–5.4 LINE WITH REACTIVE LOAD

When the termination on a transmission line is a resistance different from the characteristic resistance, an incident voltage or current wave will produce a reflected wave of the same time dependence. The ratio of the amplitudes of the reflected and incident waves is a constant, which is defined as the reflection coefficient. If, however, the termination is a reactive element such as an inductance or a capacitance, the reflected wave will no longer have the same time dependence (no longer be of the same shape) as the incident wave. The use of a constant reflection coefficient is not feasible in such cases, and it is necessary to solve a differential equation at the termination in order to study the transient behavior. We shall consider the effect on the reflected wave of an inductive termination and a capacitive termination separately in this subsection.

Figure 9–26(a) shows a lossless line with a characteristic resistance R_0, terminated at $z = \ell$ with an inductance L_L. A d-c voltage V_0 is applied to the line at $z = 0$ through a series resistance R_0. When the switch is closed at $t = 0$, a voltage wave of an amplitude

$$V_1^+ = \frac{V_0}{2} \tag{9–167}$$

travels toward the load. Upon reaching the load at $t = \ell/u = T$, a reflected wave $V_1^-(t)$ is produced because of mismatch. It is the relation between $V_1^-(t)$ and V_1^+ that we wish to find. At $z = \ell$, the following relations hold for all $t \geq T$:

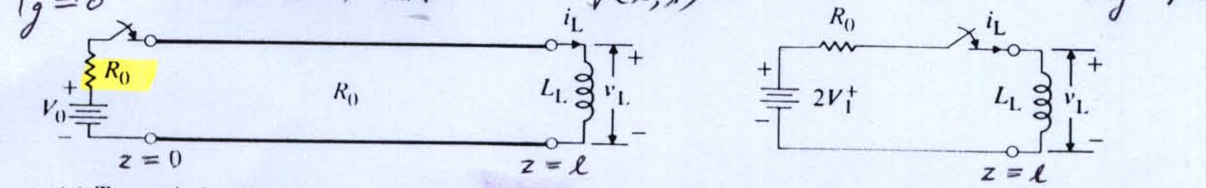
Boundary Condition: at Load End

$$v_L(t) = V_1^+ + V_1^-(t), \tag{9–168}$$

" $V(\ell, t)$

Since $R_g = R_0$

$\Gamma_g = 0$



(a) Transmission-line circuit with inductive termination (b) Equivalent circuit for the load end, $t \geq T$

FIGURE 9–26
Transient calculations for a lossless line with an inductive termination.

<Note> Here, we cannot use the concepts of Γ_L no longer since $Z_L \neq R_L$

For $t \geq T$,

$V_1(\ell, t) = f(t-\ell\sqrt{LC}) + g(t-\ell\sqrt{LC})$

$I(\ell, t) = \dfrac{\{f(t-\ell\sqrt{LC}) - g(t-\sqrt{LC})\}}{R_0}$

$\lambda_L = L_L i_L$

$$v_L(t) = V_1^+ + V_1^-(t) \tag{9-168}$$

$$i_L(t) = \frac{1}{R_0}[V_1^+ - V_1^-(t)], \tag{9-169}$$

$$v_L(t) = L_L \frac{di_L(t)}{dt}. \tag{9-170}$$

Eliminating $V_1^-(t)$ from Eqs. (9–168) and (9–169), we obtain

$$v_L(t) = 2V_1^+ - R_0 i_L(t). \tag{9-171}$$

It is seen that Eq. (9–171) describes the application of Kirchhoff's voltage law to the circuit in Fig. 9–26(b), which is then the equivalent circuit at the load end for $t \geq T$. In view of Eq. (9–170), Eq, (9–171) leads to a first-order differential equation with constant coefficients:

$$L_L \frac{di_L(t)}{dt} + R_0 i_L(t) = 2V_1^+, \qquad t \geq T. \tag{9-172}$$

The solution of Eq. (9–172) is

$$i_L(t) = \frac{2V_1^+}{R_0}[1 - e^{-(t-T)R_0/L_L}], \qquad t \geq T, \tag{9-173}$$

which correctly gives $i_L(T) = 0$ and $i_L(\infty) = 2V_1^+/R_0$. The voltage across the inductive load is

$$v_L(t) = L_L \frac{di_L(t)}{dt} = 2V_1^+ e^{-(t-T)R_0/L_L}, \qquad t \geq T. \tag{9-174}$$

The amplitude of the reflected wave, $V_1^-(t)$, can be found from Eq. (9–168):

$$\begin{aligned} V_1^-(t) &= v_L(t) - V_1^+ \\ &= 2V_1^+[e^{-(t-T)R_0/L_L} - \tfrac{1}{2}], \qquad t > T. \end{aligned} \tag{9-175}$$

This reflected wave travels in the $-z$-direction. The voltage at any point $z = z_1$ along the line is V_1^+ before the reflected wave from the load end reaches that point, $(t - T) < (\ell - z_1)/u$, and equals $V_1^+ + V_1^-(t - T)$ after that.

In Figs. 9–27(a), 9–27(b), and 9–27(c) are plotted $i_L(t)$, $v_L(t)$, and $V_1^-(t)$ at $z = \ell$ using Eqs. (9–173), (9–174), and (9–175). The voltage distribution along the line for $T < t_1 < 2T$ is shown in Fig. 9–27(d). Obviously, the transient behavior on a transmission line with a reactive termination is more complicated than that with a resistive termination.

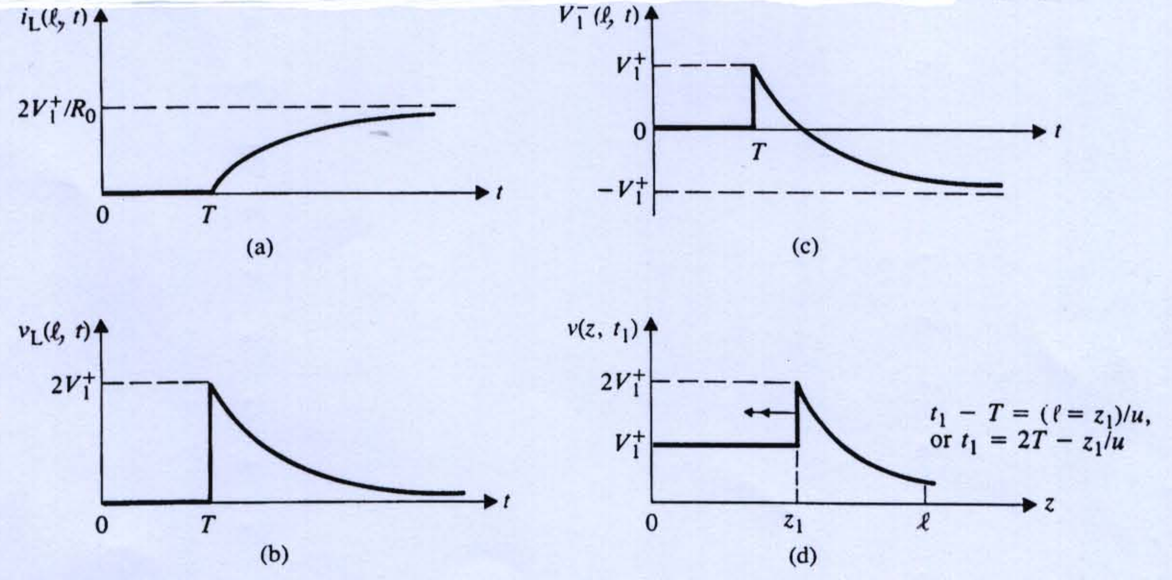


FIGURE 9–27
Transient responses of a lossless line with an inductive termination.

We follow a similar procedure in examining the transient behavior of a lossless line with a capacitive termination, shown in Fig. 9-28(a). The same Eqs. (9-167), (9-168), (9-169), and (9-171) apply at $z = \ell$, but Eq. (9-170) relating the load current $i_L(t)$ and load voltage $v_L(t)$ must now be changed to

(9-171) $v_L(t) = 2V_1^+ - R_0 i_L(t)$

$$i_L(t) = C_L \frac{dv_L(t)}{dt}. \qquad (9\text{-}176)^{\dagger}$$

The differential equation to be solved at the load end is, by substituting Eq. (9-176) in Eq. (9-171),

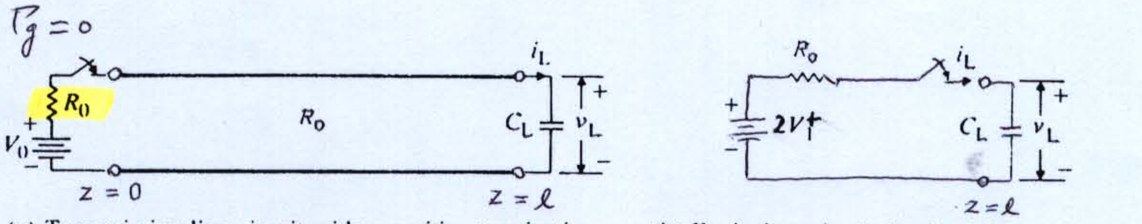
$$C_L \frac{dv_L(t)}{dt} + \frac{1}{R_0} v_L(t) = \frac{2}{R_0} V_1^+, \qquad t \geq T, \qquad (9\text{-}177)$$

where $V_1^+ = V_0/2$, as given in Eq. (9-167). The solution of Eq. (9-177) is

$$v_L(t) = 2V_1^+ \left[1 - e^{-(t-T)/R_0 C_L}\right], \qquad t \geq T. \qquad (9\text{-}178)$$

The current in the load capacitance is obtained from Eq. (9-176):

$$i_L(t) = \frac{2V_1}{R_0} e^{-(t-T)/R_0 C_L}, \qquad t \geq T. \qquad (9\text{-}179)$$



(a) Transmission-line circuit with capacitive termination. (b) Equivalent circuit for the load end, $t \geq T$.

FIGURE 9-28
Transient calculations

Using Eq. (9-178) in Eq. (9-168), we find the amplitude of the reflected wave as a function of t:

$$v_L(t) - V_1^+ = V_1^-(t) = 2V_1^+\left[\tfrac{1}{2} - e^{-(t-T)/R_0 C_L}\right], \qquad t \geq T. \qquad (9\text{-}180)$$

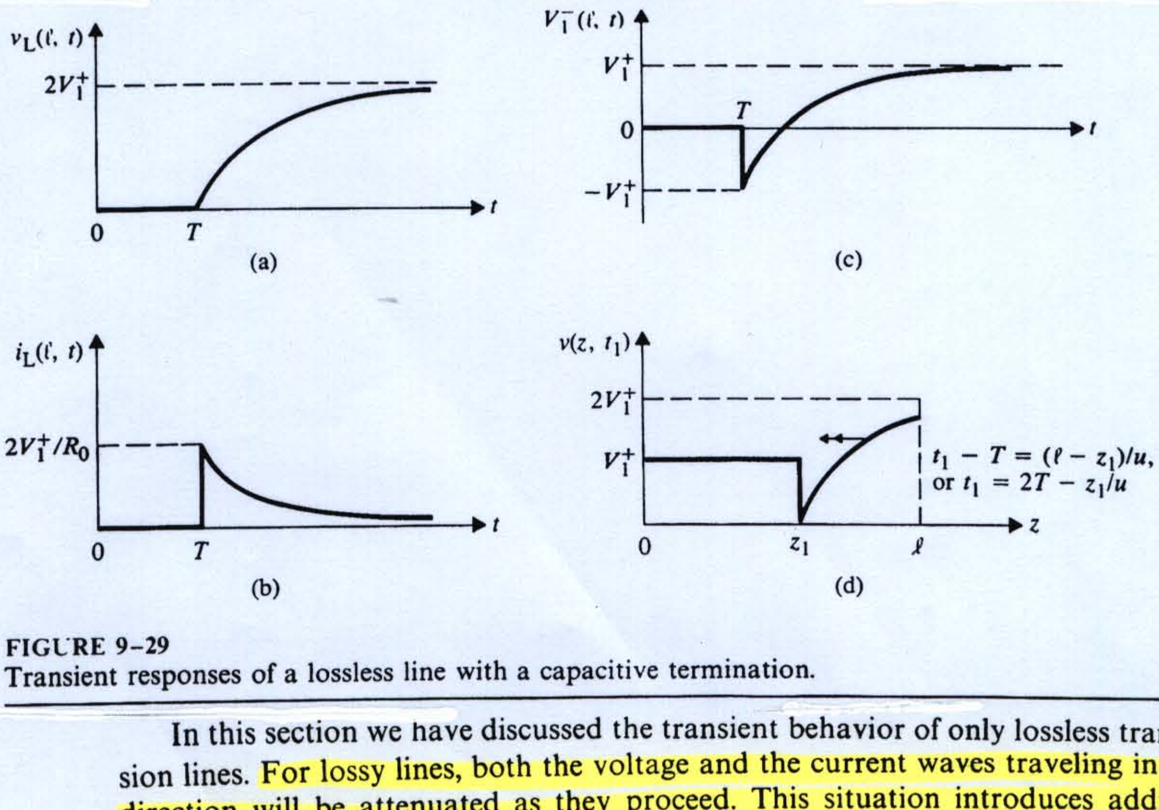


FIGURE 9-29
Transient responses of a lossless line with a capacitive termination.

In this section we have discussed the transient behavior of only lossless transmission lines. For lossy lines, both the voltage and the current waves traveling in either direction will be attenuated as they proceed. This situation introduces additional complication in numerical computation, but the basic concept remains the same.

2. Draw a straight line from O through P_2 to P_2' where the "wavelengths toward generator" reading is 0.364.
3. Draw a $|\Gamma|$-circle centered at O with radius $\overline{OP_2}$.
4. Move P_2' along the perimeter by 0.20 "wavelengths toward generator" to P_3' at $0.364 + 0.20 = 0.564$ or 0.064.
5. Join P_3' and O by a straight line, intersecting the $|\Gamma|$-circle at P_3.
6. Mark on line OP_3 a point P_i such that $\overline{OP_i}/\overline{OP_3} = e^{-2\alpha\ell} = 0.89$.
7. At P_i, read $z_i = 0.64 + j0.27$. Hence,

$$Z_i = 75(0.64 + j0.27) = 48.0 + j20.3 \quad (\Omega).$$

9–7 Transmission-Line Impedance Matching

$\Gamma = 0$

$S = 1$

Transmission lines are used for the transmission of power and information. For radio-frequency power transmission it is highly desirable that as much power as possible is transmitted from the generator to the load and as little power as possible is lost on the line itself. This will require that the load be matched to the characteristic impedance of the line so that the standing-wave ratio on the line is as close to unity as possible. For information transmission it is essential that the lines be matched because reflections from mismatched loads and junctions will result in echoes and will distort the information-carrying signal. In this section we discuss several methods for impedance-matching on lossless transmission lines. We note parenthetically that the methods we develop will be of little consequence to power transmission by 60 (Hz) lines inasmuch as these lines are generally very short in comparison to the 5 (Mm) wavelength and the line losses are appreciable. Sixty-hertz power-line circuits are usually analyzed in terms of equivalent lumped electrical networks.

9–7.1 IMPEDANCE MATCHING BY QUARTER-WAVE TRANSFORMER

A simple method for matching a resistive load R_L to a lossless transmission line of a characteristic impedance R_0 is to insert a quarter-wave transformer with a characteristic impedance R_0' such that

$Z_i = \dfrac{R_0'^2}{Z_L}$ (9-114) $\xleftarrow[R_0' \to R_0]{}$

$$R_0' = \sqrt{R_0 R_L}. \tag{9–194}$$

Since the length of the quarter-wave line depends on wavelength, this matching method is frequency-sensitive, as are all the other methods to be discussed.

$\Downarrow$

$Z_i = R_0$

EXAMPLE 9–17 A signal generator is to feed equal power through a lossless air transmission line with a characteristic impedance 50 (Ω) to two separate resistive loads, 64 (Ω) and 25 (Ω). Quarter-wave transformers are used to match the loads to the 50 (Ω) line, as shown in Fig. 9–36. (a) Determine the required characteristic impedances of the quarter-wave lines. (b) Find the standing-wave ratios on the matching line sections.

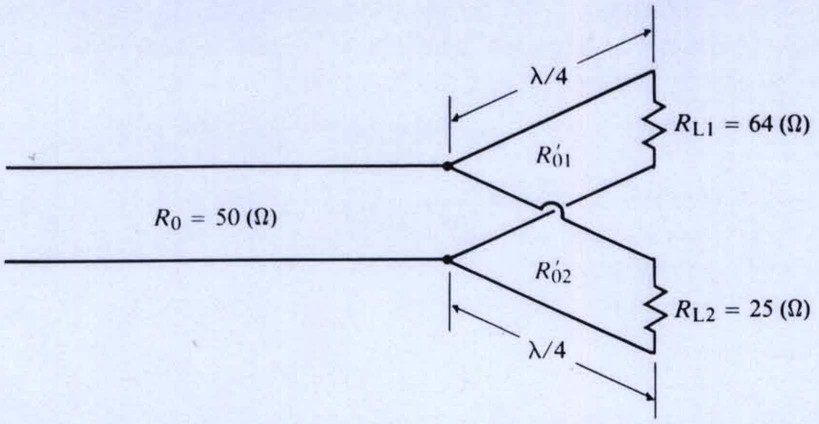


FIGURE 9–36
Impedance matching by quarter-wave lines (Example 9–17).

Solution

a) To feed equal power to the two loads, the input resistance at the junction with the main line looking toward each load must be equal to $2R_0$. $R_{i1} = R_{i2} = 2R_0 = 100$ (Ω):

$$R'_{01} = \sqrt{R_{i1}R_{L1}} = \sqrt{100 \times 64} = 80 \quad (\Omega),$$
$$R'_{02} = \sqrt{R_{i2}R_{L2}} = \sqrt{100 \times 25} = 50 \quad (\Omega).$$

b) Under matched conditions there are no standing waves on the main transmission line ($S = 1$). The standing-wave ratios on the two matching line sections are as follows.

Γ_L not small

Matching section No. 1:

$$\Gamma_1 = \frac{R_{L1} - R'_{01}}{R_{L1} + R'_{01}} = \frac{64 - 80}{64 + 80} = -0.11,$$

$$S_1 = \frac{1 + |\Gamma_1|}{1 - |\Gamma_1|} = \frac{1 + 0.11}{1 - 0.11} = 1.25.$$

Matching section No. 2:

$$\Gamma_2 = \frac{R_{L2} - R'_{02}}{R_{L2} + R'_{02}} = \frac{25 - 50}{25 + 50} = -0.33,$$

$$S_2 = \frac{1 + |\Gamma_2|}{1 - |\Gamma_2|} = \frac{1 + 0.33}{1 - 0.33} = 1.99.$$

Ordinarily, the main transmission line and the matching line sections are essentially lossless. In that case, both R_0 and R'_0 are purely real, and Eq. (9–194) will have no solution if R_L is replaced by a complex Z_L. Hence quarter-wave transformers are not useful for matching a complex load impedance to a low-loss line.

In the following subsection we will discuss a method for matching an arbitrary load impedance to a line by using a single open- or short-circuited line section (a

9–7.2 SINGLE-STUB MATCHING (lossless)

We now tackle the problem of matching a load impedance Z_L to a lossless line that has a characteristic impedance R_0 by placing a single short-circuited stub in parallel with the line, as shown in Fig. 9–39. This is the ***single-stub method*** for impedance matching. We need to determine the length of the stub, ℓ, and the distance from the load, d, such that the impedance of the parallel combination to the right of points B–B' equals R_0. Short-circuited stubs are usually used in preference to open-circuited stubs because an infinite terminating impedance is more difficult to realize than a zero terminating impedance for reasons of radiation from an open end and coupling effects with neighboring objects. Moreover, a short-circuited stub of an adjustable length and a constant characteristic resistance is much easier to construct than an open-circuited one. Of course, the difference in the required length for an open-circuited stub and that for a short-circuited stub is an odd multiple of a quarter-wavelength.

The parallel combination of a line terminated in Z_L and a stub at points B–B' in Fig. 9–39 suggest that it is advantageous to analyze the matching requirements in terms of admittances. The basic requirement is

$$Y_i = Y_B + Y_s$$
$$= Y_0 = \frac{1}{R_0}. \tag{9–197}$$

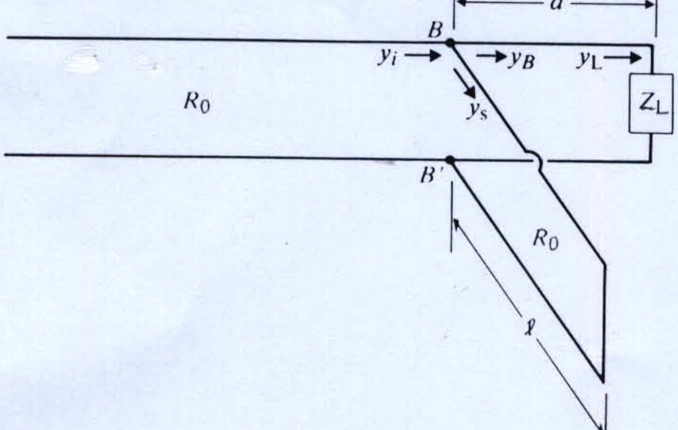
In terms of normalized admittances, Eq. (9–197) becomes

$$1 = y_B + y_s, \tag{9–198}$$

where $y_B = R_0 Y_B$ is for the load section and $y_s = R_0 Y_s$ is for the short-circuited stub. However, since the input admittance of a short-circuited stub is purely susceptive, y_s is purely imaginary. As a consequence, Eq. (9–198) can be satisfied only if

$$y_B = 1 + jb_B \tag{9–199}$$

$$y_s = -jb_B \tag{9–200}$$



$\boxed{\text{Solution of (9–199)}}$

$$Z_i(z') = Z_0 \frac{(1+\Gamma e^{-j2\beta z'})}{(1-\Gamma e^{-j2\beta z'})} \quad (9\text{–}190)$$

$$\Downarrow$$

$$1 + jb_B = \frac{(1+\Gamma e^{-j2\beta d})}{(1-\Gamma e^{-j2\beta d})} \quad (*)$$

FIGURE 9–39 where $\Gamma = (Z_L - R_0)/(Z_L + R_0)$
Impedance matching by single-stub method.

$\Downarrow$

Find d such that

$$1 = \mathrm{Re}\left[\frac{(1+\Gamma e^{-j2\beta d})}{(1-\Gamma e^{-j2\beta d})}\right]$$

$\Downarrow$

Determine b_B from (*)

$\boxed{\text{Solution of (9–200)}}$

$$\frac{Z_i}{R_0} = j\tan\beta\ell \text{ from (9–112) or from (9–190) with } \Gamma = -1$$
$$= -jb_B$$

$$y_s = -jb_B, \tag{9-200}$$

where b_B can be either positive or negative. Our objectives, then, are to find the length d such that the admittance, y_B, of the load section looking to the right of terminals B–B' has a *unity real part* and to find the length ℓ_B of the stub required to *cancel the imaginary part.*

9–7.3 DOUBLE-STUB MATCHING (lossless)

The method of impedance matching by means of a single stub described in the preceding subsection can be used to match any arbitrary, nonzero, finite load impedance to the characteristic resistance of a line. However, the single-stub method requires that the stub be attached to the main line at a specific point, which varies as the load impedance or the operating frequency is changed. This requirement often presents practical difficulties because the specified junction point may occur at an undesirable location from a mechanical viewpoint. Furthermore, it is very difficult to build a variable-length coaxial line with a constant characteristic impedance. In such cases an alternative method for impedance-matching is to use two short-circuited stubs attached to the main line at fixed positions, as shown in Fig. 9–41. Here, the distance d_o is fixed and arbitrarily chosen (such as $\lambda/16$, $\lambda/8$, $3\lambda/16$, $3\lambda/8$, etc.), and the lengths of the two stub tuners are adjusted to match a given load impedance Z_L to the main line. This scheme is the ***double-stub method*** for impedance matching.

In the arrangement in Fig. 9–41 a stub of length ℓ_A is connected directly in parallel with the load impedance Z_L at terminals A–A', and a second stub of length ℓ_B is attached at terminals B–B' at a fixed distance d_o away. For impedance matching with a main line that has a characteristic resistance R_0, we demand the total input admittance at terminals B–B', looking toward the load, to equal the characteristic conductance of the line; that is,

$$Y_i = Y_B + Y_{sB} = Y_0 = \frac{1}{R_0}. \tag{9-210}$$

In terms of normalized admittances, Eq. (9–210) becomes

$$1 = y_B + y_{sB}. \tag{9-211}$$

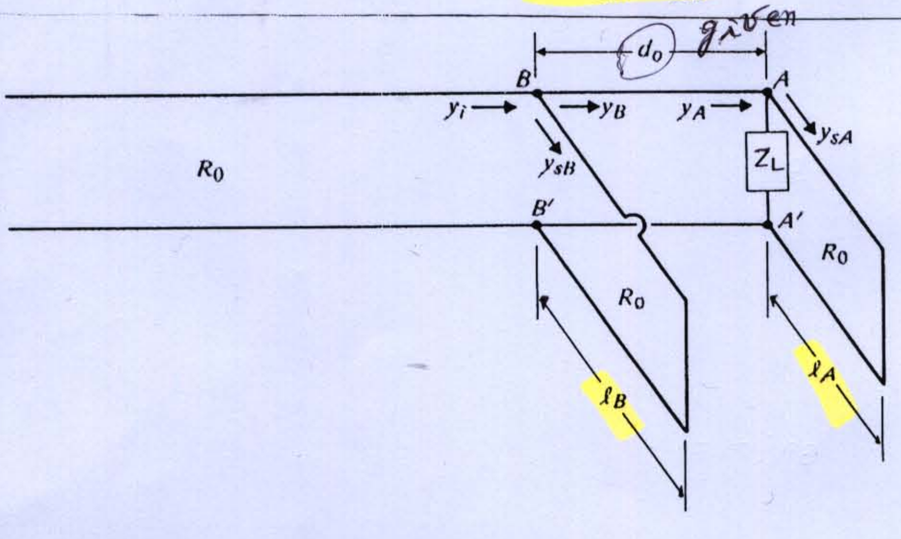


FIGURE 9–41
Impedance matching by double-stub method.

Now, since the input admittance y_{sB} of a short-circuited stub is purely imaginary, Eq. (9–211) can be satisfied only if

$$y_B = 1 + jb_B \tag{9-212}$$

and

$$y_{sB} = -jb_B. \tag{9-213}$$

Note that these requirements are exactly the same as those for single-stub matching.