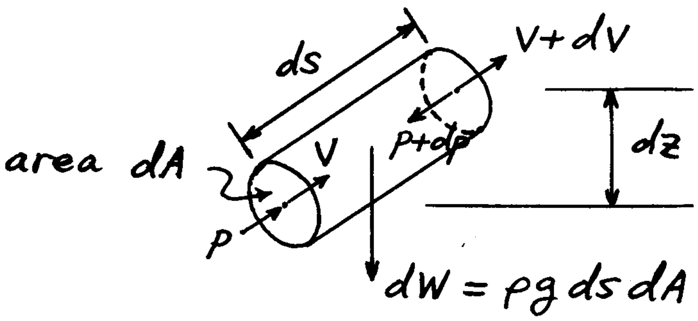
# Chap.5 Flow of an Incompressible Ideal Fluid

- Ideal fluid (or inviscid fluid) $\rightarrow$ viscosity=0
  $\rightarrow$ no friction b/w fluid particles
  or b/w fluid and boundary walls
- Incompressible fluid $\rightarrow$ $\rho$=const, $d\rho / dt = 0$

## 5.1 Euler's Equation (1-D steady flow)




- pressure force: $pdA - (p + dp)dA = -dpdA$
- body force: $-\rho g ds dA \left( \dfrac{dz}{ds} \right) = -\rho g dA dz$
- acceleration: $a = \dfrac{dV}{dt} = \dfrac{\partial V}{\partial t} + V \dfrac{\partial V}{\partial s} = V \dfrac{dV}{ds}$

Newton's 2nd law: $F = ma$

$$-dpdA - \rho gdAdz = \rho dAdsV\frac{dV}{ds}$$

$$\boxed{\frac{dp}{\rho} + gdz + VdV = 0} \leftarrow \text{1-D Euler's equation}$$

$$\frac{dp}{\gamma} + dz + \frac{V}{g}dV = 0$$

$$d\left(\frac{p}{\gamma} + z + \frac{V^2}{2g}\right) = 0 \;\rightarrow\; \frac{p}{\gamma} + z + \frac{V^2}{2g} = \text{const}$$

## 5.2 Bernoulli's Equation with Energy and Hydraulic Grade Lines

For 1-D steady flow of incompressible and uniform density fluid, integration of the Euler equation gives the Bernoulli equation as follows:

$$\underbrace{\frac{p}{\gamma}}_{\text{pressure head}} + \underbrace{\frac{V^2}{2g}}_{\text{velocity head}} + \underbrace{z}_{\text{elevation head}} = \underbrace{H}_{\text{total head}} = \text{const (along a streamline)}$$

정지유체 $(V = 0)$: $\frac{p}{\gamma} + z = \text{const}$ (2.6)

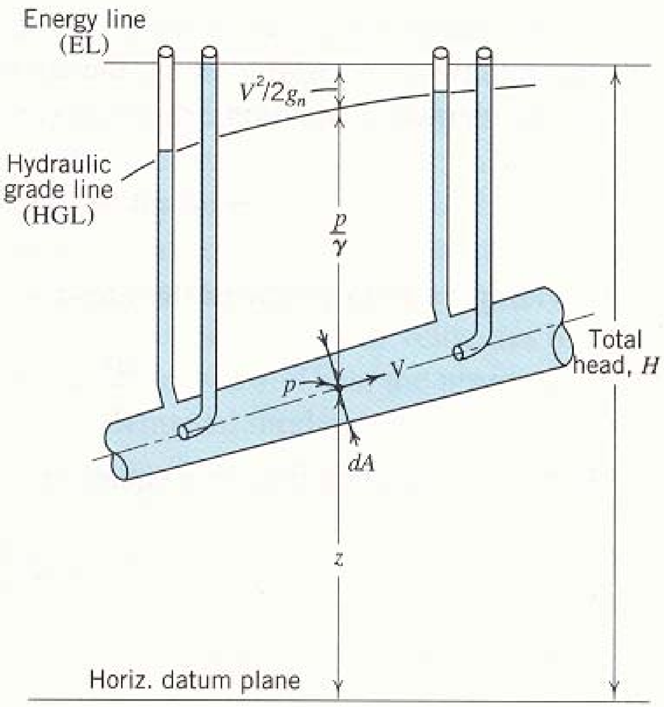


Fig. 5.2

Assumption: The stream tube is infinitesimally thin, so that it can be assumed to be a streamline.

$$\mathrm{EL} = z + \frac{p}{\gamma} + \frac{V^2}{2g} = H$$

$$\mathrm{HGL} = z + \frac{p}{\gamma}$$

$$\mathrm{EL} - \mathrm{HGL} = \frac{V^2}{2g}$$

## 5.3 1-D Assumption for Streamtubes of Finite Cross Section

Assumptions:

1. Streamlines are straight and parallel.
2. $V = \text{constant}$ across a cross section (ideal fluid $\rightarrow$ no friction)

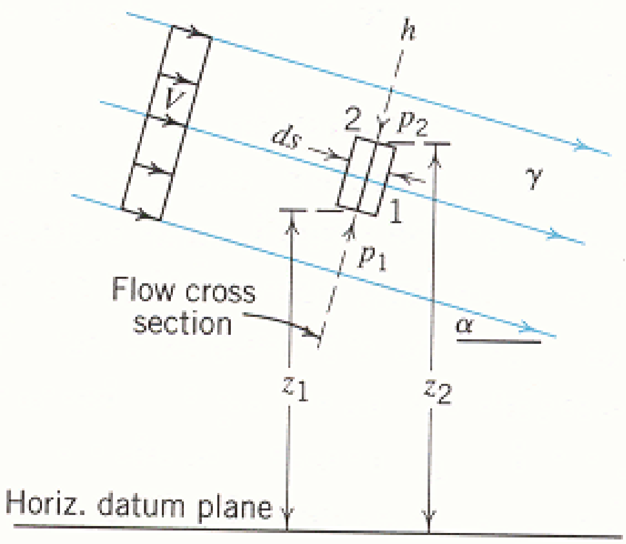



Fig. 5.3

Straight and parallel streamlines

$\rightarrow$ No cross-sectional velocity

$\rightarrow$ No cross-sectional acceleration

Newton's 2nd law in x-sectional direction:

$$F_c = \text{pressure force} + \text{body force} = 0$$

(∵ no x-sectional acceleration)

Read text

$$\frac{p_1}{\gamma} + z_1 = \frac{p_2}{\gamma} + z_2$$

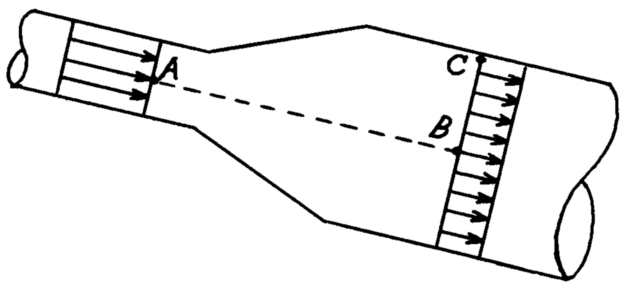
or $\dfrac{p}{\gamma} + z = \text{constant}$ across the x-section

Ex) Ideal fluid flowing in a pipe (IP 5.1)

$$\frac{p_A}{\gamma} + z_A + \frac{V_A^2}{2g} = \frac{p_B}{\gamma} + z_B + \frac{V_B^2}{2g} \quad (\because \text{same streamline})$$

$$\frac{p_B}{\gamma} + z_B + \frac{V_B^2}{2g} = \frac{p_C}{\gamma} + z_C + \frac{V_C^2}{2g} \quad (\because V_B = V_C)$$

Therefore, the Bernoulli's equation can be applied not only between A and B (on the same streamline) but also between A and C (on different streamlines) in the pipe shown below.



## 5.4 Applications of Bernoulli's Equation

(1) If $z \approx \text{const}$, $\frac{p}{\gamma} + \frac{V^2}{2g} \approx \text{const}$

$\therefore$ high velocity $\rightarrow$ low pressure

(2) Torricelli's equation

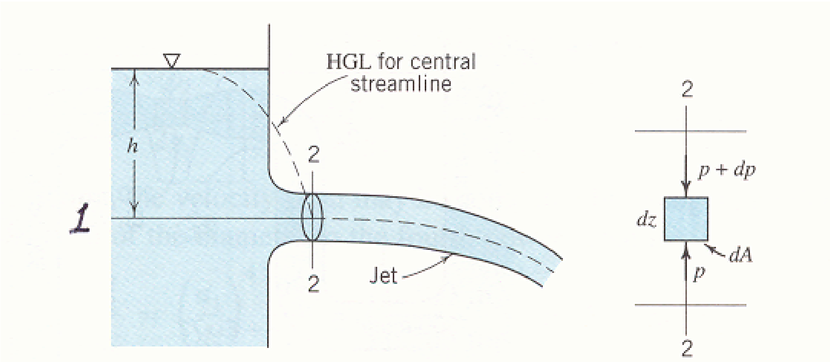



Fig. 5.4

Bernoulli equation b/w 1 and 2:

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

Using $z_1 = z_2$, $p_1 = \gamma h$, $p_2 = 0$, $V_1 \approx 0$,

$$h = \frac{V_2^2}{2g} \rightarrow \boxed{V_2 = \sqrt{2gh}} \leftarrow \text{Torricelli equation}$$

For application of Torricelli's equation, see IP 5.2 in text.

Note:

In the reservoir ($V = 0$),

$= 0$

Water surface = EL $\left(= \frac{p}{\gamma} + \cancel{\frac{V^2}{2g}} + z\right)$

= HGL $\left(= \frac{p}{\gamma} + z\right)$

= const. in static fluid (reservoir)

$= z\,|_{\text{surface}}$ ($\because$ $p = 0$ at surface)

(3) Cavitation

Cavitation occurs if $p_{\text{abs}} = p_{\text{gage}} + p_{\text{atm}} \le p_v$

or $p_{\text{gage}} \le -(p_{\text{atm}} - p_v)$

At initiation of cavitation,

$$p_{\text{gage}} = -(p_{\text{atm}} - p_v)$$

$p_c$ (critical gage pressure in text p. 134)

(4) Pitot tube (for measuring velocity)

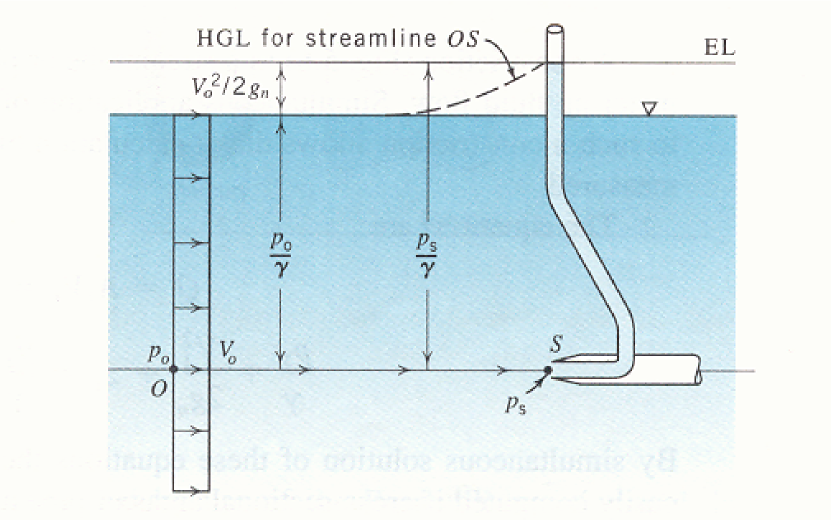


Fig. 5.5

$$\frac{p_O}{\gamma} + \frac{V_O^2}{2g} + z_O = \frac{p_S}{\gamma} + \frac{V_S^2}{2g} + z_S = H \ (= \mathrm{EL})$$

$$V_O = \sqrt{2g\left(\frac{p_S}{\gamma} - \frac{p_O}{\gamma}\right)}$$

can be measured

$\therefore$ $V_O$ can be known.

# (5) Overflow structure (e.g. spillway of a dam)

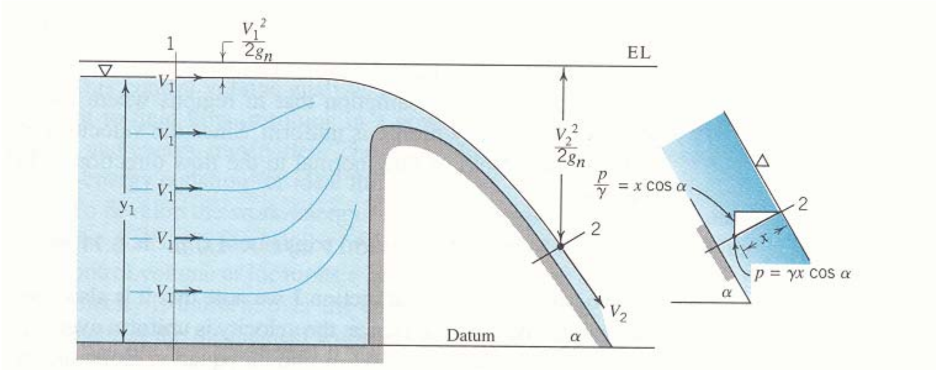
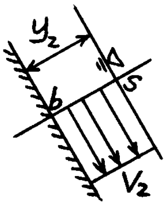


Fig. 5.7

Pressure and velocity on the spillway surface at 2?



$$V_s = V_b = V_2 \ (\because \text{ideal fluid})$$

Bernoulli equation at points $s$ and $b$:

$$\frac{p_s}{\gamma} + \frac{V_s^2}{2g} + z_s = \frac{p_b}{\gamma} + \frac{V_b^2}{2g} + z_b \leftarrow$$

See also § 5.3 $p/\gamma + z = \text{const}$ across a x-section

$$\therefore p_b = \gamma(z_s - z_b)$$

Continuity: $V_1 y_1 = V_2 y_2$

Bernoulli equation at water surfaces at 1 and 2

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + y_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_s$$

2 equations for 2 unknowns ($y_1$ and $V_1$)

$\rightarrow$ can be solved for $y_1$ and $V_1$.

## 5.5 Work-Energy Equation

- Addition or extraction of energy by flow machines
  - pump: add energy to flow system
    (or a pump does work on fluid)
  - turbine: extract energy from flow system
    (or fluid does work on a turbine)

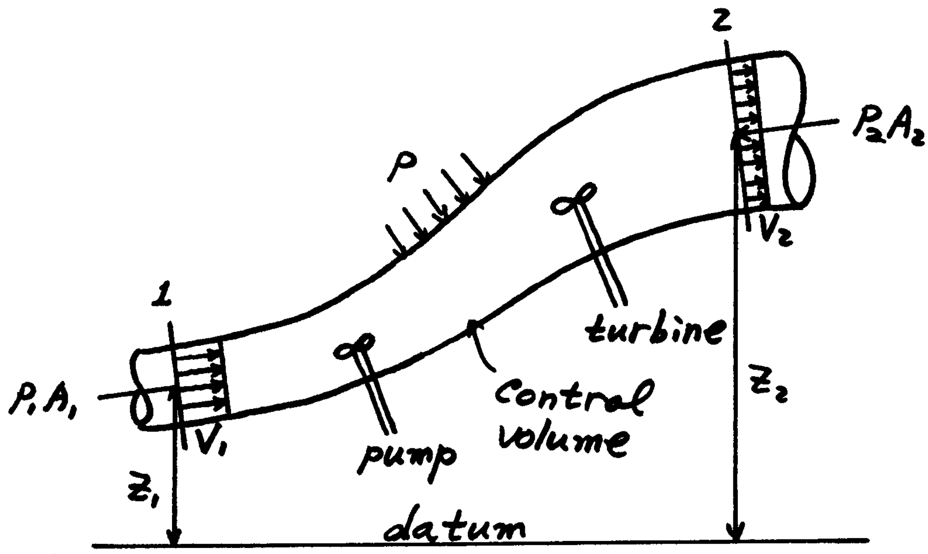
- Mechanical work-energy principle

  Work done on a fluid system is equal to the change in the mechanical energy (potential + kinetic energy) of the system

$$dW = dE \quad \rightarrow \quad \frac{dW}{dt} = \frac{dE}{dt}$$

Note: Work, energy, and heat have the same unit (J=N·m)

- Control volume analysis (Reynolds transport theorem)



Reynolds transport theorem for mechanical energy:

$$\frac{dE}{dt} = \frac{\partial}{\partial t}\left(\iiint_{CV} \rho\left(gz + \frac{V^2}{2}\right) d\forall\right)$$
$$+ \iint_{CS_{out}} \rho\left(gz + \frac{V^2}{2}\right)\vec{v}\cdot\vec{n}dA + \iint_{CS_{in}} \rho\left(gz + \frac{V^2}{2}\right)\vec{v}\cdot\vec{n}dA$$
$$= \rho\left(gz_2 + \frac{V_2^2}{2}\right)V_2A_2 - \rho\left(gz_1 + \frac{V_1^2}{2}\right)V_1A_1$$
$$= \left(z_2 + \frac{V_2^2}{2g}\right)\rho g V_2 A_2 - \left(z_1 + \frac{V_1^2}{2g}\right)\rho g V_1 A_1$$
$$= Q\gamma\left[\left(z_2 + \frac{V_2^2}{2g}\right) - \left(z_1 + \frac{V_1^2}{2g}\right)\right]$$

Rate of work done on the flow system $\left(\frac{dW}{dt} = ?\right)$

1. flow work rate done by pressure force

$$= \iint_{CS} \vec{p}\cdot\vec{v}dA = p_1A_1V_1 - p_2A_2V_2 = Q\gamma\left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma}\right)$$

2. machine work rate done by pumps or turbines

$$= Q\gamma\left(E_p - E_t\right)$$

$E_p, E_t =$ work rate per unit weight of fluid

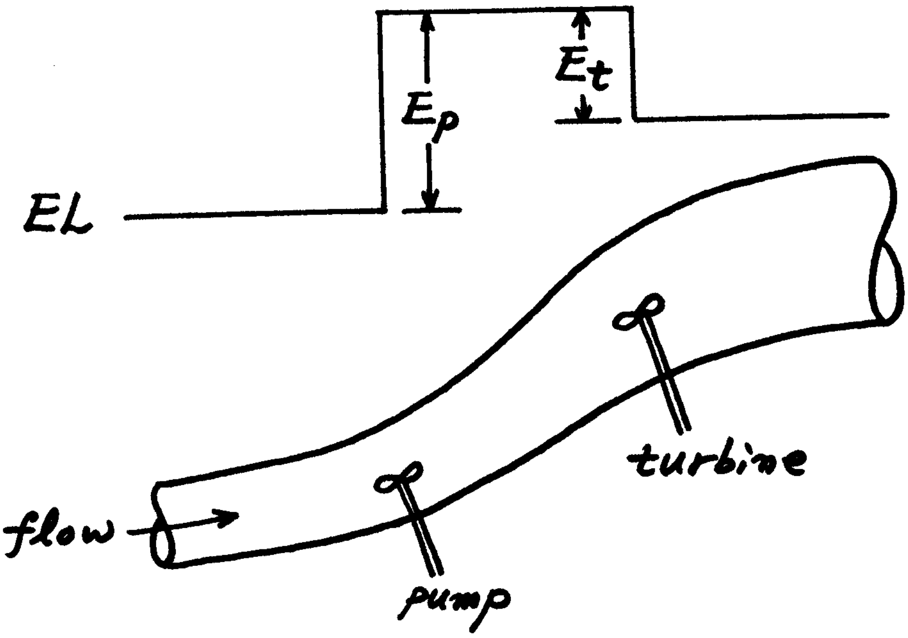
3. Shear work rate done by shearing forces on the control surface = 0 (∵ ideal fluid)

$$\therefore \frac{dW}{dt} = Q\gamma\left(\frac{p_1}{\gamma} - \frac{p_2}{\gamma} + E_p - E_t\right)$$

Since $\frac{dW}{dt}=\frac{dE}{dt}$, we get

$$\boxed{\frac{p_1}{\gamma}+z_1+\frac{V_1^2}{2g}+E_p-E_t=\frac{p_2}{\gamma}+z_2+\frac{V_2^2}{2g}}$$

Work-energy equation (per unit weight of fluid)




- Power of pump or turbine

Power = rate of work done

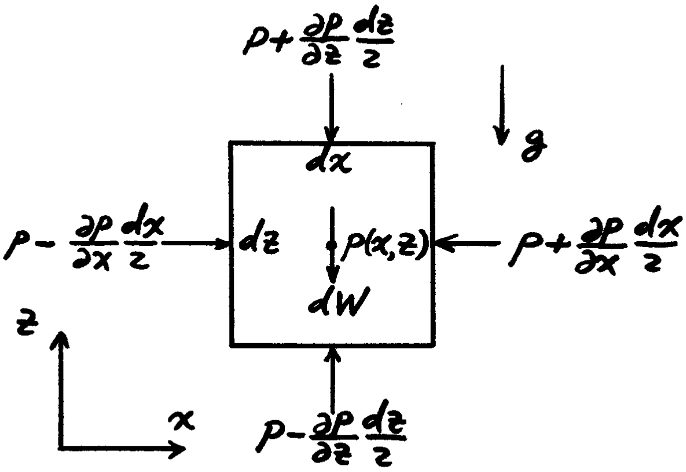
$E_p, E_t =$ rate of work done per unit weight of fluid

$\therefore$ Power of pump or turbine

$=\left(E_p \text{ or } E_t\right)\times Q\gamma$ (ft·lb/s or J/s=W)

↑ total weight of fluid passing the x-section per unit time

# 5.6 Euler's Equations (2-D Steady Flow)




Newton's 2nd law:

$x$-direction:

$$\left(p-\frac{\partial p}{\partial x}\frac{dx}{2}\right)dz-\left(p+\frac{\partial p}{\partial x}\frac{dx}{2}\right)dz=\rho dxdz\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)$$

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x}=u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}}$$

$z$-direction:

$$\left(p-\frac{\partial p}{\partial z}\frac{dz}{2}\right)dx-\left(p+\frac{\partial p}{\partial z}\frac{dz}{2}\right)dx-\rho g dxdz=\rho dxdz\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}\right)$$

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial z}=u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}+g}$$

Continuity: $\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0$

3 equations for 3 unknowns $(p,u,w)$

# 5.7 Bernoulli's Equation

Euler's equations:

$$\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}=u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)\times dx$$

$$\left(-\frac{1}{\rho}\frac{\partial p}{\partial z}=u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}+g\right)\times dz \quad +$$

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial z}dz\right)=u\frac{\partial u}{\partial x}dx+w\frac{\partial u}{\partial z}dx+u\frac{\partial w}{\partial x}dz+w\frac{\partial w}{\partial z}dz+gdz$$

$$=\left(u\frac{\partial u}{\partial x}dx+w\frac{\partial w}{\partial x}dx\right)+\left(u\frac{\partial u}{\partial z}dz+w\frac{\partial w}{\partial z}dz\right)$$

$$+\left(udz-wdx\right)\left(\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}\right)+gdz$$

Using

$$dp=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial z}dz$$

$$d(u^2+w^2)=\frac{\partial(u^2+w^2)}{\partial x}dx+\frac{\partial(u^2+w^2)}{\partial z}dz$$

$$=2u\frac{\partial u}{\partial x}dx+2w\frac{\partial w}{\partial x}dx+2u\frac{\partial u}{\partial z}dz+2w\frac{\partial w}{\partial z}dz$$

we get

$$\frac{dp}{\rho g}+\frac{d(u^2+w^2)}{2g}+dz=\frac{1}{g}(wdx-udz)\xi$$

Let $X_1 = (x_1, z_1)$ and $X_2 = (x_2, z_2)$.
Integrating from $X_1$ to $X_2$,

$$\left( \frac{p}{\gamma} + \frac{u^2 + w^2}{2g} + z \right) \Bigg|_{X_1}^{X_2} = \frac{1}{g} \int_{X_1}^{X_2} \xi (w dx - u dz)$$

If the path between $X_1$ and $X_2$ is a streamline ($u = dx/dt$, $w = dz/dt$), the RHS vanishes so that

$$\left( \frac{p}{\gamma} + \frac{u^2 + w^2}{2g} + z \right) \Bigg|_{X_1}^{X_2} = 0$$

Therefore,

$$\frac{p}{\gamma} + \frac{u^2 + w^2}{2g} + z = \text{const}(C) \text{ along a streamline.}$$

But, in general, the constant($C$) may be different for different streamlines.

On the other hand, for an irrotational flow, $\xi = 0$, thus

$$\left( \frac{p}{\gamma} + \frac{u^2 + w^2}{2g} + z \right) \Bigg|_{X_1}^{X_2} = 0$$

for any two points in the flow. Thus

$$\frac{p}{\gamma} + \frac{V^2}{2g} + z = H(\text{const})$$

over the whole flow field for irrotational flow.

## 5.8 Applications of Bernoulli's Equation

For irrotational flow of ideal incompressible fluid, the Bernoulli's equation applies over the whole flow field with a single energy line.

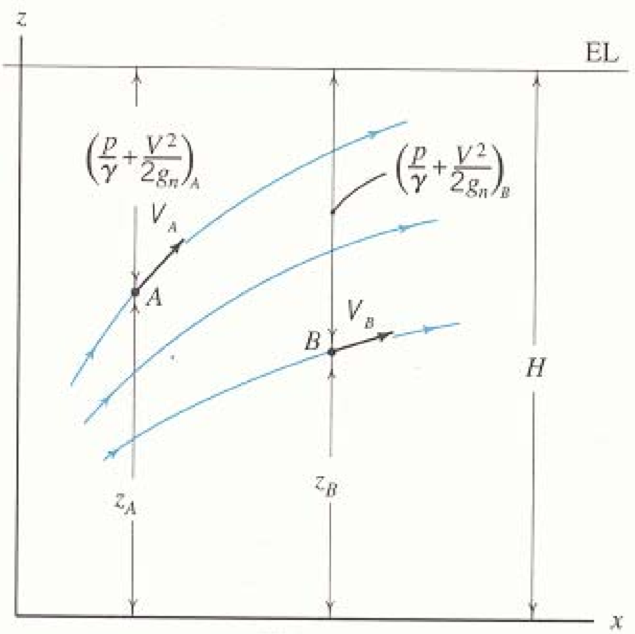



Fig. 5.10

$$H = \frac{p}{\gamma} + \frac{V^2}{2g} + z$$

We are interested in $p$ and $V$ at a point.

Exact velocity field → Exact pressure

→ Difficult to solve

↓

Semi-quantitative (approximate) approach

(1) Tornado or bathtub vortex: Higher pressure and lower velocity outwards from the center (Read text)

(2) Flow in a curved section in a vertical plane (Fig. 5.12)

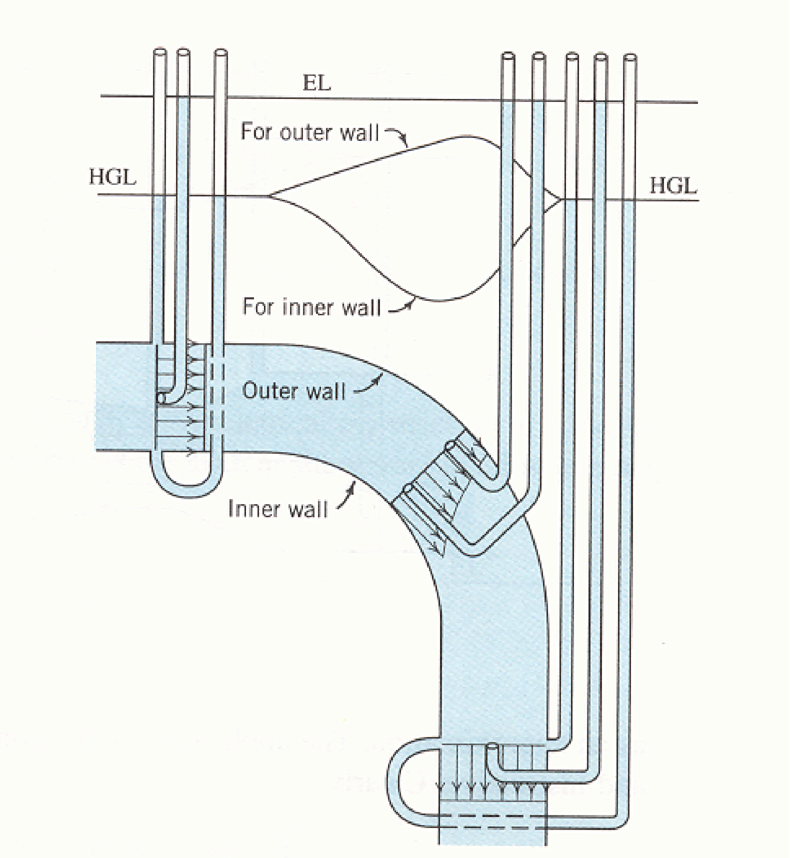


Fig. 5.12

- gravity effect >> centrifugal force (low velocity)
- outer wall: sparse streamlines → lower velocity → higher pressure
- inner wall: dense streamlines → higher velocity → lower pressure

(3) Flow in a convergent-divergent section

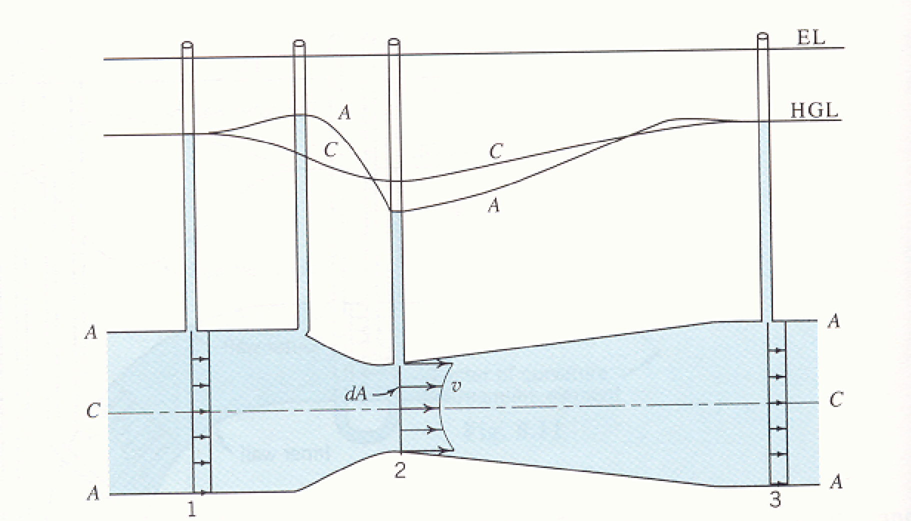


Fig. 5.13

At section 2:

Center: sparse streamlines → lower velocity
→ higher pressure

Wall: dense streamlines → higher velocity
→ lower pressure

$$\frac{p_U}{\gamma} + z_U + \frac{V_U^2}{2g} = \frac{p_L}{\gamma} + z_L + \frac{V_L^2}{2g}$$

$$V_U = V_L \;\rightarrow\; \frac{p_U}{\gamma} + z_U = \frac{p_L}{\gamma} + z_L$$

$$z_L < z_U \;\rightarrow\; p_U < p_L$$

If cavitation occurs, it will occur first at the upper wall.

(4) Stagnation points

Sharp corner of a surface → $V = 0$ (stagnation point)

At stagnation point, EL = HGL ($\because \; V = 0$)

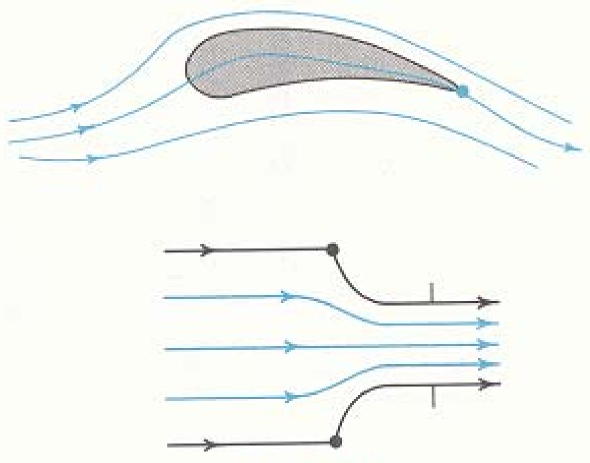


Fig. 5.14

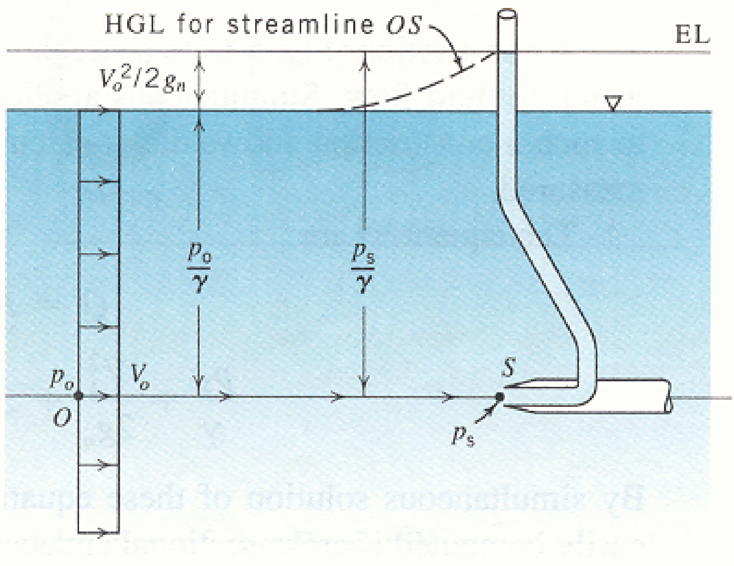



Fig. 5.5

(5) Sharp-crested weir

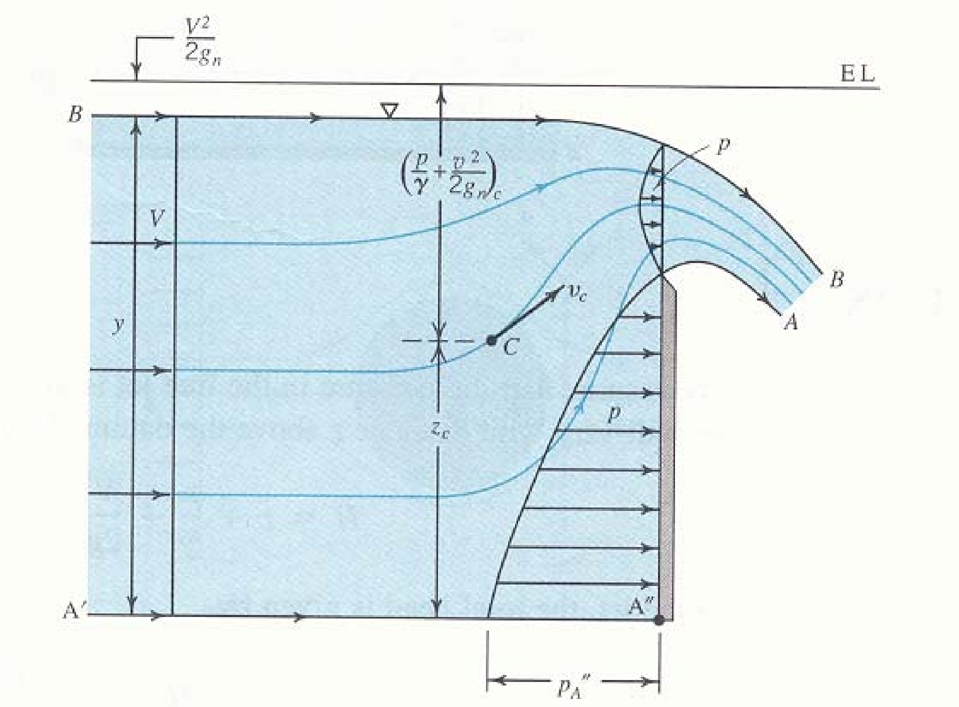



Fig. 5.15

At upstream side, pressure is hydrostatic.

$\therefore \quad p_{A'} = \gamma y$

At point $A''$ (stagnation point), $\frac{p_{A''}}{\gamma} = y + \frac{V^2}{2g}$

Pressure distribution above the weir:

Near free streamlines: dense streamlines
$\rightarrow$ higher velocity
$\rightarrow$ lower pressure

Near center: sparse streamlines $\rightarrow$ lower velocity
$\rightarrow$ higher pressure

Note: $p = 0$ along free streamlines

(6) Flow past a circular cylinder

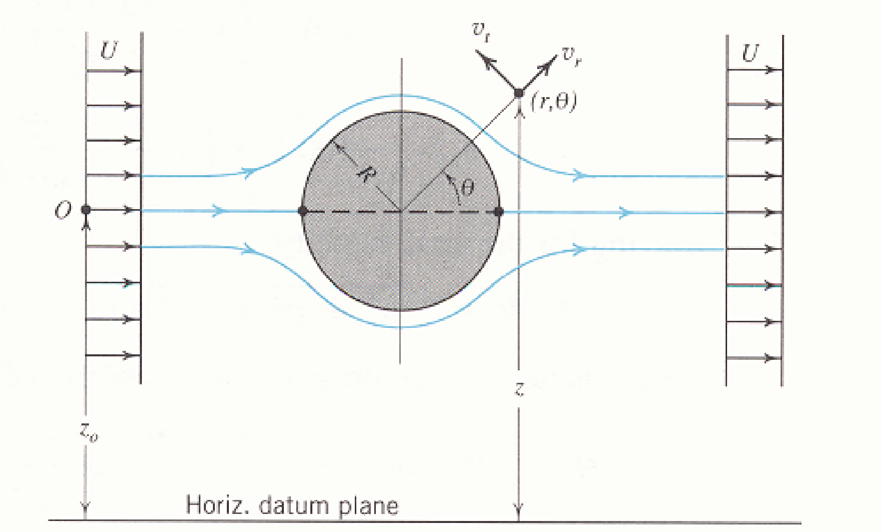



Fig. 5.16

$$v_r = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta, \quad v_t = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta$$

At windward stagnation point:

$$r = R \rightarrow v_r = 0, \quad \theta = \pi \rightarrow v_t = 0$$

At leeward stagnation point:

$$r = R \rightarrow v_r = 0, \quad \theta = 0 \rightarrow v_t = 0$$

At surface of the cylinder ($r = R$):

$$v_r = 0, \quad v_t = -2U\sin\theta$$

$$\therefore \quad \frac{p_o}{\gamma} + \frac{U^2}{2g} + z_o = \frac{p}{\gamma} + z + \frac{(-2U\sin\theta)^2}{2g}$$

$$\therefore \quad \frac{p}{\gamma} + z = fn(p_o, z_o, U, \theta)$$

(7) Flow through an orifice

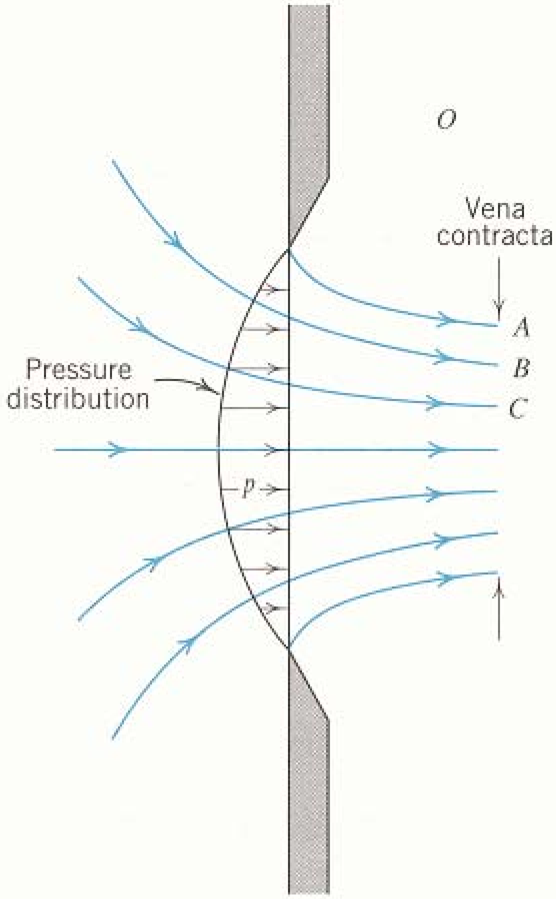



Fig. 5.17

Streamlines are straight and parallel in the vena contracta. $p = 0$ everywhere in the jet, but $v$ varies across the jet (Read page 131).

## 5.9 Stream Function and Velocity Potential

- Bernoulli equation → Relationship b/w $p$, $V$ and $z$
- $z$ is known. Therefore, if $V$ ( $u$ and $v$ ) can be calculated, then $p$ can be calculated by the Bernoulli equation.
- Introduce stream function ($\psi$) and velocity potential ($\phi$), which are related to the velocity field ($u$ and $v$).
- PDE for $\psi(x,y)$ or $\phi(x,y)$ + boundary conditions → solve for $\psi$ or $\phi$ → $u$ and $v$ → $p$

(1) Stream function

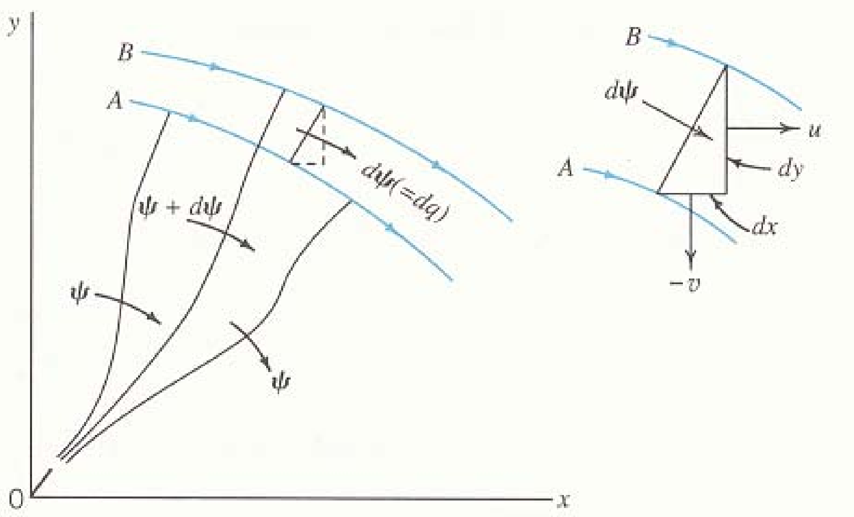
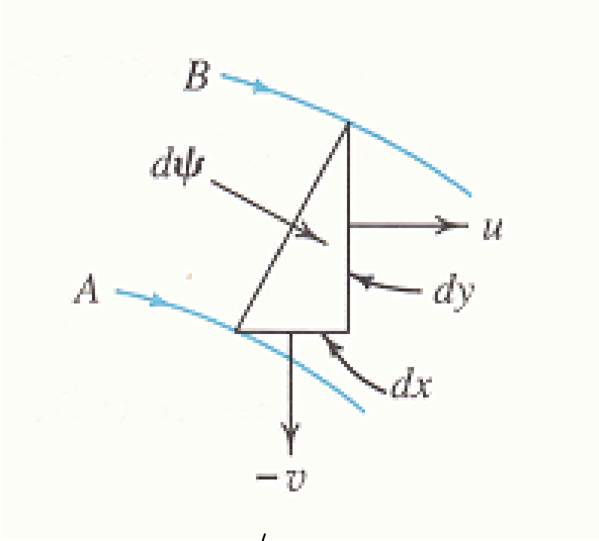



Fig. 5.18 Definition of the stream function.

- $\psi$ = flowrate b/w 0 and a streamline
- Flowrate $\psi$ b/w 0 and any point on streamline $A$ is the same (∵ no flow across a streamline) → $\psi$ = const on streamline $A$
- Flowrate b/w 0 and streamline $B$ = $\psi + d\psi$ → $d\psi(=dq)$ = flowrate b/w streamlines $A$ and $B$




physically

$$d\psi = udy - vdx = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

by definition

$$\therefore \quad u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

Now $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$

$\therefore$ $\psi$ (stream function) satisfies continuity equation for incompressible fluid.

For irrotational flow, $\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

or $\frac{\partial^2 \psi}{\partial^2 x} + \frac{\partial^2 \psi}{\partial^2 y} = \nabla^2 \psi = 0$ ← Laplace equation for $\psi(x, y)$

$\psi(x, y)$ can be solved with proper boundary conditions.

(2) Velocity potential

Define the velocity potential, $\phi(x, y)$, so as to satisfy

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}$$

Plug in continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 = -\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2}$$

$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0 \leftarrow$ Laplace equation for $\phi(x, y)$

Vorticity:

$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x} = 0 \rightarrow$ irrotational flow

$\therefore$ $\phi$ (velocity potential) is defined only in the irrotational flow.

Note:

1. irrotational flow = potential flow
   ($\because$ velocity potential exists in irrotational flow)
2. $\psi$ satisfies continuity for incompressible fluid
   $\rightarrow$ For irrotational flow, $\nabla^2 \psi = 0$
3. $\phi$ satisfies irrotationality
   $\rightarrow$ For incompressible fluid, $\nabla^2 \phi = 0$

(3) Relationship between $\phi$ and $\psi$

$$u = -\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \rightarrow \frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y}$$
$$v = -\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \rightarrow \frac{\partial\phi}{\partial y} = \frac{\partial\psi}{\partial x}$$

Cauchy-Riemann Conditions

Streamlines ( $\psi$ =const) and equipotential lines ($\phi$=const) are orthogonal (Read text p. 167-168).