4. Chapter Shear Flow Dispersion

◆ Taylor' Analysis of Dispersion (1953, 1954)

- laminar flow in pipe
- turbulent flow

→ apply Fickian model to dispersion

→ reasonably accurate estimate of the rate of longitudinal dispersion in rivers and estuaries

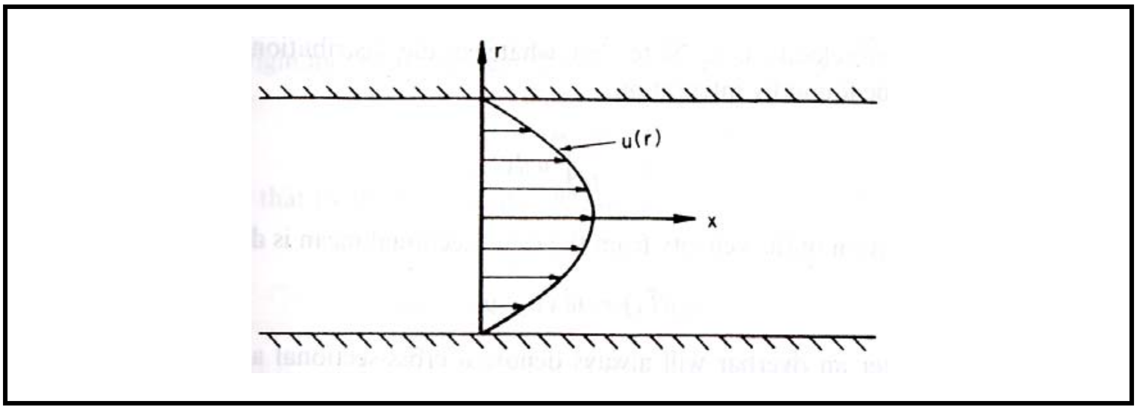
◆ Dispersion = spreading of a cloud of contaminants by the combined effects of shear flow and transverse diffusion

Shear flow = flows with velocity gradients

4.1 Dispersion in Laminar Shear Flow

4.1.1 Introductory Remarks

◆ Taylor's analysis (1953) in laminar flow in pipe



<Fig. 4.1> The parabolic velocity distribution in laminar pipe flow

1) rate of separation caused by the difference in advective velocity

$\gg$ separation by molecular motion

2) given enough time, any single molecular wander randomly throughout the cross section of the pipe because of molecular diffusion

3) velocity of any single molecule is equal to velocity of the stream line on which it is located, a function of cross-sectional position,

(4) because of molecular diffusion each molecule moves at random walk back and forth across the cross section.

→ motion of single molecule is the sum of a series of independent steps of random length.

(5) Fickian diffusion equation can describe the spread of particles along the axis of the pipes, except that since the step length and time increment are much different from those of molecular diffusion we expect to find a different value of different value of diffusion coefficient.

- turbulent diffusion coefficient $\varepsilon = < U^2 > T_L$

where U = velocity deviation

T_L = Lagrangian time scale

For laminar flow in pipe; $< U^2 > \propto {u_0}^2$

$$T_L \propto \frac{a^2}{D}$$

where u_0 = maximum velocity at the centerline of pipe

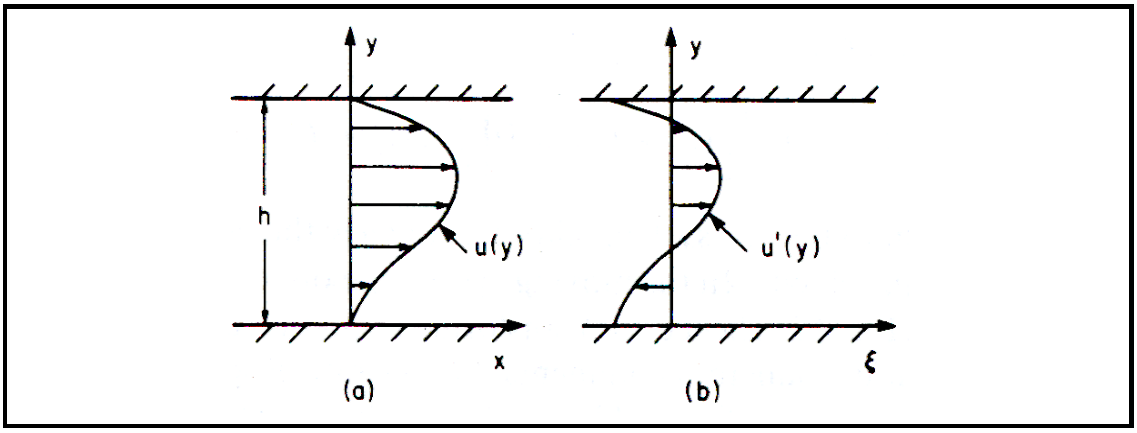
a = radius of pipe

D = molecular diffusion coefficient

- longitudinal dispersion coefficient

$$K \propto u_0^{\,2} \frac{a^2}{D} \tag{4.1}$$

4.1.2 A Generalized Introduction



(a) example velocity distribution (b) transformed coordinate system moving at the mean velocity

◆ 2-D laminar flow

• cross-sectional mean velocity

$$\overline{u} = \frac{1}{h}\int_0^h u dy$$

Velocity deviation : $u' = u(y) - \bar{u}$

Let flow carry a solute with concentration *C(x, y)* and molecular diffusion coefficient *D*

mean concentration: $\bar{C} = \frac{1}{h}\int_0^h C dy, \qquad \bar{C} = f(x) \neq f(y)$

concentration deviation: $C' = C(y) - \bar{C}, \;\; C' = C'(x, y)$

◆ 2-D diffusion equation with <u>only flow in *x*-direction (*v* =0)</u>

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial x^2} + D\frac{\partial^2 C}{\partial y^2}$$

$$\frac{\partial}{\partial t}(\bar{C} + C') + (\bar{u} + u')\frac{\partial}{\partial x}(\bar{C} + C') = D\left[\frac{\partial^2}{\partial x^2}(\bar{C} + C') + \frac{\partial^2}{\partial y^2}(\bar{C} + C')\right] \quad \text{(a)}$$

◆ transformation of coordinate system whose origin moves at the mean flow velocity

$$\xi = x - \bar{u}t \qquad \rightarrow \frac{\partial \xi}{\partial x} = 1 \qquad \frac{\partial \xi}{\partial t} = -\bar{u}$$

$$\tau = t \qquad \rightarrow \frac{\partial \tau}{\partial x} = 0 \qquad \frac{\partial \tau}{\partial t} = 1$$

Chain rule

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial \xi} \qquad \text{(b)}$$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = -\bar{u}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \tau} \qquad \text{(c)}$$

Substitute Eq. (b)-(c) into Eq. (a)

$$-\bar{u}\frac{\partial}{\partial \xi}(\bar{C} + C') + \frac{\partial}{\partial \tau}(\bar{C} + C') + \left(\bar{u} + u'\right)\frac{\partial}{\partial \xi}(\bar{C} + C') = D\left[\frac{\partial^2}{\partial \xi^2}(\bar{C} + C') + \frac{\partial^2 C'}{\partial y^2}\right]$$

$$u'\frac{\partial}{\partial \xi}(\bar{C} + C') + \frac{\partial}{\partial \tau}(\bar{C} + C') = D\left[\frac{\partial^2}{\partial \xi^2}(\bar{C} + C') + \frac{\partial^2 C'}{\partial y^2}\right]$$

◆ Now, neglect longitudinal diffusion because rate of spreading along the flow direction <u>due to velocity difference</u> greatly exceed that due to molecular diffusion.

$$D\frac{\partial^2}{\partial \xi^2}(\bar{C} + C') << u'\frac{\partial}{\partial \xi}(\bar{C} + C')$$

$$\therefore \frac{\partial \bar{C}}{\partial \tau} + \frac{\partial C'}{\partial \tau} + u'\frac{\partial \bar{C}}{\partial \xi} + u'\frac{\partial C'}{\partial \xi} = D\frac{\partial^2 C'}{\partial y^2} \qquad (4.3)$$

◆ Now introduce Taylor's assumption

→ discard three terms

$$u'\frac{\partial \bar{C}}{\partial \xi} = D\frac{\partial^2 C'}{\partial y^2} \qquad (4.4)$$

◆ Derivation of Eq. (4.4) using orders of magnitude analysis

Take average over the cross section of Eq. (4.3)

(apply the operator $\frac{1}{h}\int_0^h (\ )dy$)

$$\overline{\frac{\partial \bar{C}}{\partial \tau}} + \overline{\frac{\partial C'}{\partial \tau}} + \overline{u' \frac{\partial C'}{\partial \xi}} + \overline{u' \frac{\partial C'}{\partial \xi}} = \overline{D \frac{\partial^2 C'}{\partial y^2}}$$

Apply Reynolds rule of average

$$\frac{\partial \bar{C}}{\partial \tau} + \overline{u' \frac{\partial C'}{\partial \xi}} = 0 \qquad (4.5)$$

Subtract Eq.(4.5) from Eq.(4.3)

$$\frac{\partial C'}{\partial \tau} + u' \frac{\partial \bar{C}}{\partial \xi} + u' \frac{\partial C'}{\partial \xi} + \overline{u' \frac{\partial C'}{\partial \xi}} = D \frac{\partial^2 C'}{\partial y^2}$$

Assume $\bar{C}, C'$ are slowly varying functions and $\bar{C} >> C'$

Then $u' \frac{\partial \bar{C}}{\partial \xi} >> u' \frac{\partial C'}{\partial \xi}, \overline{u' \frac{\partial C'}{\partial \xi}}$

Thus we can drop $u' \frac{\partial C'}{\partial \xi}, \overline{u' \frac{\partial C'}{\partial \xi}}$

$$\frac{\partial C'}{\partial \tau} = D \frac{\partial^2 C'}{\partial y^2} - u' \frac{\partial \bar{C}}{\partial \xi} \qquad \text{(a)}$$

$-u^{'}\dfrac{\partial \overline{C}}{\partial \xi}$ = source term of variable strength

→ net addition by source term is zero because the average of $u^{'}$ is zero

Assume that $\dfrac{\partial \overline{C}}{\partial \xi}$ remains constant for a long time, so that the source is constant.

Then Eq. (a) can be assumed as steady state

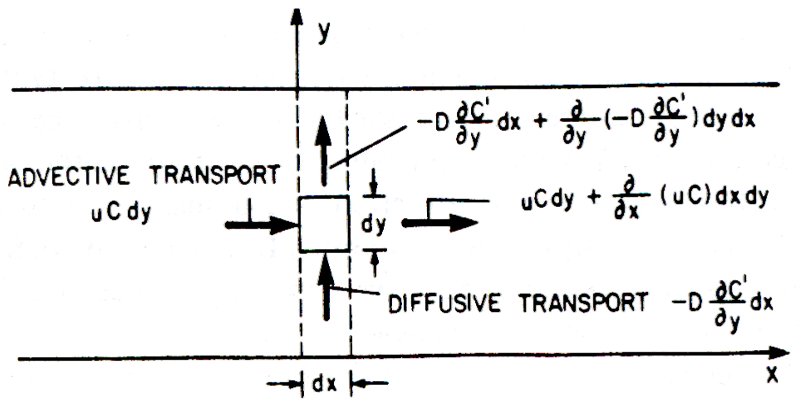
→ $\dfrac{\partial C^{'}}{\partial \tau} = 0$

Then

$$u^{'}\frac{\partial \overline{C}}{\partial \xi} = D\frac{\partial^2 C^{'}}{\partial y^2}$$

(A) (B)

→ cross sectional concentration profile $C^{'}(y)$ is established by a balance between longitudinal advective transport (A) and cross sectional diffusive transport (B)



<Fig. 4.3> The balance of advective flux versus diffusive flux

In balance, net transport = 0

$$u'\overline{C}dy - \left\{ u'\overline{C}dy + \frac{\partial}{\partial x}\left(u'\overline{C}\right)dxdy \right\} + \left\{ -D\frac{\partial C'}{\partial y}dx - \left[-D\frac{\partial C'}{\partial y}dx + \frac{\partial}{\partial y}\left(-D\frac{\partial C'}{\partial y} \right)dydx \right] \right\}$$

$$= -\frac{\partial}{\partial x}\left(u'\overline{C}\right)dxdy + \frac{\partial}{\partial y}\left(D\frac{\partial C'}{\partial y} \right)dydx = 0$$

$$\therefore \frac{\partial}{\partial x}\left(u'\overline{C}\right) = \frac{\partial}{\partial y}\left(D\frac{\partial C'}{\partial y} \right)$$

◆ Solution of Eq. (4.4)

$$\frac{\partial^2 C'}{\partial y^2} = \frac{1}{D}\frac{\partial C'}{\partial \xi}u' = \frac{1}{D}\frac{\partial C'}{\partial x}u'$$

Integrate twice w.r.t. *y*

$$C' = \frac{1}{D}\frac{\partial \overline{C}}{\partial x}\int_0^y \int_0^y u' dydy + C'(0) \qquad (4.6)$$

Consider mass transport in the streamwise direction

$$\dot{M} = \int_0^h \left[u'C' + \left(-D\frac{\partial C'}{\partial x} \right) \right] dy$$

$$\dot{M} = \int_0^h u'C' dy = \frac{1}{D}\frac{\partial \overline{C}}{\partial x}\int_0^h u' \int_0^y \int_0^y u' dydydy \qquad (4.7)$$

$$\int_0^h u' \left\{ C'(0) \right\} dy = 0$$

→ Total mass transport in the streamwise direction is proportional to the concentration gradient in that direction.

$$\dot{M} \propto \frac{\partial \bar{C}}{\partial x}$$

- similar to molecular diffusion $(q = -D\frac{\partial C}{\partial x})$

- but this is diffusion due to whole field of flow

q = rate of mass transport per unit area per unit time

$$q = \frac{\dot{M}}{h \times 1} = -K\frac{\partial \bar{C}}{\partial x} \qquad \text{(b)}$$

where h = depth = area per unit width of flow

K = longitudinal dispersion coefficient (= bulk transport coefficient) → express as the diffusive property of the velocity distribution (shear flow)

Then, (b) becomes

$$\dot{M} = -hK\frac{\partial \bar{C}}{\partial x} \qquad (4.8)$$

Compare Eq. (4.7) and Eq. (4.8)

$$K = -\frac{1}{hD}\int_0^h u' \int_0^y \int_0^y u' dy dy dy \qquad (4.9)$$

$$K \propto \frac{1}{D}$$

Now, we can express this transport process due to velocity distribution as a one-dimensional Fickian-type diffusion equation in moving coordinate system.

$$\frac{\partial \overline{C}}{\partial \tau} = K \frac{\partial^2 \overline{C}}{\partial \xi^2} \tag{4.10}$$

return to fixed coordinate system

$$\frac{\partial \overline{C}}{\partial t} + \overline{u} \frac{\partial \overline{C}}{\partial x} = K \frac{\partial^2 \overline{C}}{\partial x^2} \tag{4.11}$$

→ 1-D advection-dispersion equation

$\overline{C}$, $\overline{u}$ = cross-sectional average values

◆ Balance of advection and diffusion in Eq. (4.4) [Chatwin, 1970]

i) Initial period: $t < 0.4 \frac{h^2}{D}$

- advection > diffusion
- skewed longitudinal concentration distribution (Fig. 4.4(c))

ii) Taylor period: $t > 0.4 \frac{h^2}{D}$

- advection ≈ diffusion
- variance of dispersing cloud $\propto t$

$\sigma^2 = 2Kt$

$\frac{\partial \sigma^2}{\partial t} = 2K$

- initial skew degenerates into normal distribution

- longitudinal spreading follows Eq. (4.11)

4.1.3 A Simple Example

◆ laminar flow between two plates → Couette flow

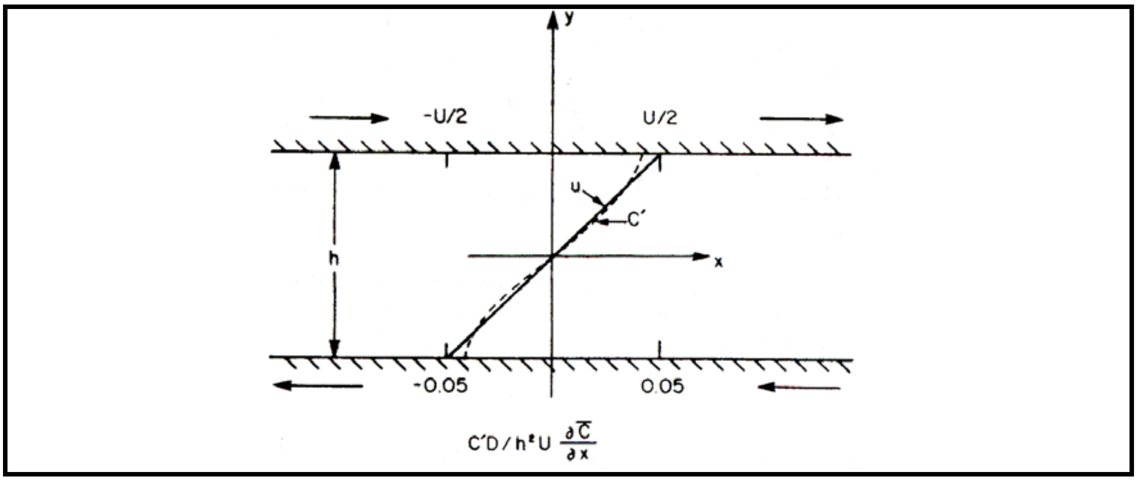


Fig. 4.5 Velocity profile and the resulting concentration profile

$$u(y) = U\frac{y}{h}$$

$$\bar{u} = \frac{1}{h}\int_{-\frac{h}{2}}^{\frac{h}{2}} U\frac{y}{h}dy = 0$$

$$\therefore u' = u$$

Suppose $t > \dfrac{h^2}{D}$ ~ tracer is well distributed

→ Taylor's analysis can be applied

From Eq.(4.6)

$$C'(y) = \frac{1}{D}\frac{\partial \bar{C}}{\partial x}\int_{-\frac{h}{2}}^{y}\int_{-\frac{h}{2}}^{y} u' dy dy + C'(0) \qquad \text{(a)}$$

$$= \frac{1}{D}\frac{\partial \bar{C}}{\partial x}\int_{-\frac{h}{2}}^{y}\int_{-\frac{h}{2}}^{y} \frac{Uy}{h} dy dy + C'(-\frac{h}{2})$$

$$=\frac{1}{D}\frac{\partial \bar{C}}{\partial x}\int_{-\frac{h}{2}}^{y}\left[\frac{U}{2h}y^2\right]_{-\frac{h}{2}}^{y}dy+C^{'}(-\frac{h}{2})$$

$$=\frac{1}{D}\frac{\partial \bar{C}}{\partial x}\int_{-\frac{h}{2}}^{y}\left[\frac{Uy^2}{2h}-\frac{Uh}{8}\right]dy+C^{'}(-\frac{h}{2})$$

$$=\frac{1}{D}\frac{\partial \bar{C}}{\partial x}\left[\frac{Uy^3}{6h}-\frac{Uh}{8}y\right]_{-\frac{h}{2}}^{y}+C^{'}(-\frac{h}{2})$$

$$=\frac{1}{D}\frac{\partial \bar{C}}{\partial x}\left[\frac{Uy^3}{6h}-\frac{Uh}{8}y+\frac{Uy^2}{48}-\frac{Uh^2}{16}\right]+C^{'}(-\frac{h}{2})$$

$$=\frac{1}{D}\frac{\partial \bar{C}}{\partial x}\frac{U}{2h}\left[\frac{y^3}{3}-\frac{h^2}{4}y-\frac{h^3}{12}\right]+C^{'}\left(-\frac{h}{2}\right)$$

By symmetry $C^{'}=0$ @ $y=0$

$$0=\frac{1}{D}\frac{\partial \bar{C}}{\partial x}\frac{U}{2h}\left[-\frac{h^3}{12}\right]+C^{'}\left(-\frac{h}{2}\right)$$

$$C^{'}\left(-\frac{h}{2}\right)=\frac{1}{D}\frac{\partial \bar{C}}{\partial x}\frac{Uh^2}{24}$$

$$\therefore C^{'}(y)=\frac{1}{D}\frac{\partial \bar{C}}{\partial x}\frac{U}{2h}\left[\frac{y^3}{3}-\frac{h^2}{4}y\right] \qquad (4.21)$$

Dispersion coefficient. K

$$K=-\frac{1}{hD}\int_{-\frac{h}{2}}^{\frac{h}{2}}u'\underbrace{\int_{\frac{h}{2}}^{y}\int_{\frac{h}{2}}^{y}u'dydy}_{(A)}$$

From (a): $(A) = \dfrac{DC'(y)}{\dfrac{\partial \bar{C}}{\partial x}}\left[C'(y) - C'\left(-\dfrac{h}{2}\right)\right]$

$$\therefore K = -\frac{1}{hD}\int_{-\frac{h}{2}}^{\frac{h}{2}} u' \frac{D}{\dfrac{\partial \bar{C}}{\partial x}}\left[C'(y) - C'\left(-\frac{h}{2}\right)\right]dy$$

$$= -\frac{1}{h\dfrac{\partial \bar{C}}{\partial x}}\left[\int_{-\frac{h}{2}}^{\frac{h}{2}} u'C'dy + C'\left(-\frac{h}{2}\right)\int_{-\frac{h}{2}}^{\frac{h}{2}} u'dy\right]$$

$$= -\frac{1}{h\dfrac{\partial \bar{C}}{\partial x}}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{Uy}{h}\right)\left\{\frac{1}{D}\frac{\partial \bar{C}}{\partial x}\frac{U}{2h}\left(\frac{y^3}{3} - \frac{h^2}{4}y\right)\right\}dy$$

$$= -\frac{U^2}{2h^3D}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\frac{y^4}{3} - \frac{h^2y^2}{4}\right]dy$$

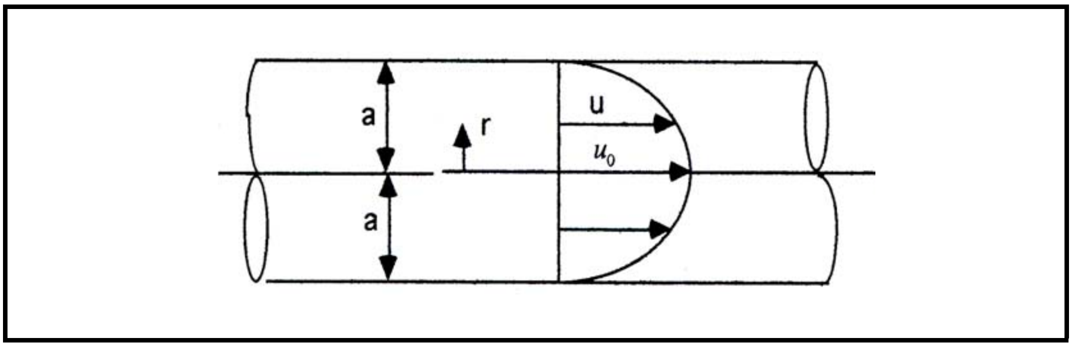
$$= -\frac{U^2}{2h^3D}\left[\frac{y^5}{15} - \frac{h^2y^3}{12}\right]_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$= \frac{U^2h^2}{120D}$$

Note that $K \propto \dfrac{1}{D}$

→ Larger lateral mixing coefficient makes C' to be decreased.

4.1.4 Taylor's Analysis of Laminar Flow in a Tube

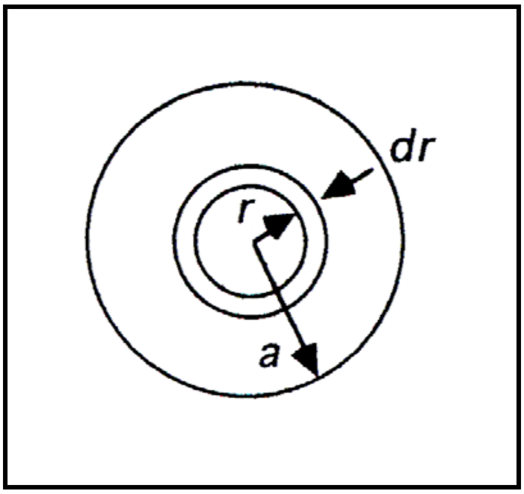


◆ axial symmetrical flow → Poiseuille flow

Tracer is well distributed over the cross section.

$$u(r) = u_0\left(1 - \frac{r^2}{a^2}\right) \quad \rightarrow \text{paraboloid} \qquad \text{(a)}$$

◆ mean velocity



$$dQ \cong u \cdot 2\pi r dr$$

$$\therefore Q = \int_0^a 2\pi r\left\{u_0\left(1 - \frac{r^2}{a^2}\right)\right\}dr$$

$$= 2\pi u_0 a^2 \int_0^1 \frac{r}{a}\left(1 - \frac{r^2}{a^2}\right)d\left(\frac{r}{a}\right)$$

$$= 2\pi u_0 a^2 \int_0^1 z(1 - z^2)dz$$

$$= 2\pi u_0 a^2 \left[\frac{z^2}{2} - \frac{z^2}{4}\right]_0^1$$

$$= \frac{\pi}{2} a^2 u_0$$

By the way, $Q = \bar{u} \cdot \pi a^2$

$$\therefore \bar{u} = \frac{u_0}{2}$$

◆ 2-D advection-dispersion equation in cylindrical coordinate

$$\frac{\partial C}{\partial t} + u_0\left(1 - \frac{r^2}{a^2}\right)\frac{\partial C}{\partial x} = D\left(\frac{\partial^2 C}{\partial r^2} + \frac{1}{r}\frac{\partial C}{\partial r} + \frac{\partial^2 C}{\partial x^2}\right) \qquad \text{(b)}$$

move coordinate system at velocity $\frac{u_0}{2}$

neglect $\frac{\partial C}{\partial t}$ and $\frac{\partial^2 C}{\partial x^2}$ as before

let $z = \frac{r}{a}, \xi = x - \bar{u}t, \tau = t$

Then (b) becomes

$$\frac{u_0 a^2}{D}\left(\frac{1}{2} - z^2\right)\frac{\partial \bar{C}}{\partial \xi} = \frac{\partial^2 C'}{\partial z^2} + \frac{1}{z}\frac{\partial C'}{\partial z}$$

$$\frac{\partial C'}{\partial z} = 0 \quad \text{at} \quad z = 1$$

Integrate twice w.r.t. z

$$C' = \frac{u_0 a^2}{8D}\left(z^2 - \frac{1}{2}z^4\right)\frac{\partial \bar{C}}{\partial x} + const \qquad \text{(c)}$$

$$K = -\frac{\dot{M}}{A\frac{\partial \bar{C}}{\partial x}} = -\frac{1}{A\frac{\partial \bar{C}}{\partial x}}\int_A u' C' dA$$

where $A = \pi a^2$, $dA = 2\pi r dr$

Substitute (a), (c) into (d), and then perform integration

$$K = \frac{a^2 u_0^{\ 2}}{192D}$$

[Example] Salt in water flowing in a tube

$D = 10^{-5} cm^2 / \sec$

$u_0 = 1 cm / \sec$

$a = 2mm$

$R_e = \frac{ud}{v} = \frac{(0.01)(0.004)}{1 \cdot 10^{-6}} = 40 << 2000$ → laminar flow

$K = \frac{a^2 u_0^{\ 2}}{192D} = \frac{(0.2)^2 (1)^2}{192(10^{-5})} = 21 cm^2 / \sec \approx 10^6 D$

☞ Initial period

$t_0 = 0.4 \frac{a^2}{D} = \frac{0.4(0.2)^2}{(10^{-5})} = 1600 \sec = 27 \min$

$x_0 = \bar{u} t_0 = \frac{u_0}{2} t_0$

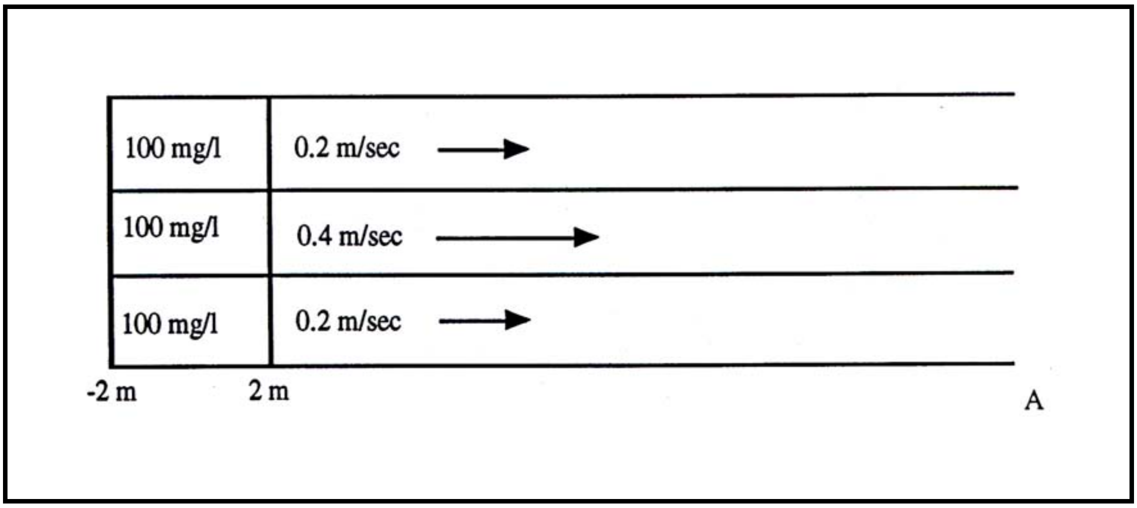
$= (0.5)(1600) = 800 cm$

$= \frac{800}{0.2} = 4000a$

$x > x_0 \rightarrow$ 1-D dispersion model can be applied

Homework Assignment No. 4-1

Due: Two weeks from today



A hypothetical river is 30m wide and consists of three "lanes", each 10m in width. The two outside lanes move at 0.2m/sec and the middle lane at 0.4m/sec. Every seconds complete mixing across the cross section of the river (but not longitudinally) occurs. An instantaneous injection of a conservative tracer results in a uniform of 100mg/ℓ in the water 2m upstream and downstream of the injection point. The concentration is initially zero elsewhere. As the tracer is carried downstream and is mixed across the cross-section of the stream, it also becomes mixed longitudinally, due to the velocity difference between lanes, even though there is no longitudinal diffusion within lanes. We call this type of mixing "dispersion".

1) Mathematically simulate the tracer concentration profile (concentration vs. longitudinal distance) as a function of time for several (at least four) values of including 10sec.

2) Compare the profiles and decide whether you think the effective longitudinal mixing increases or decrease as increases.

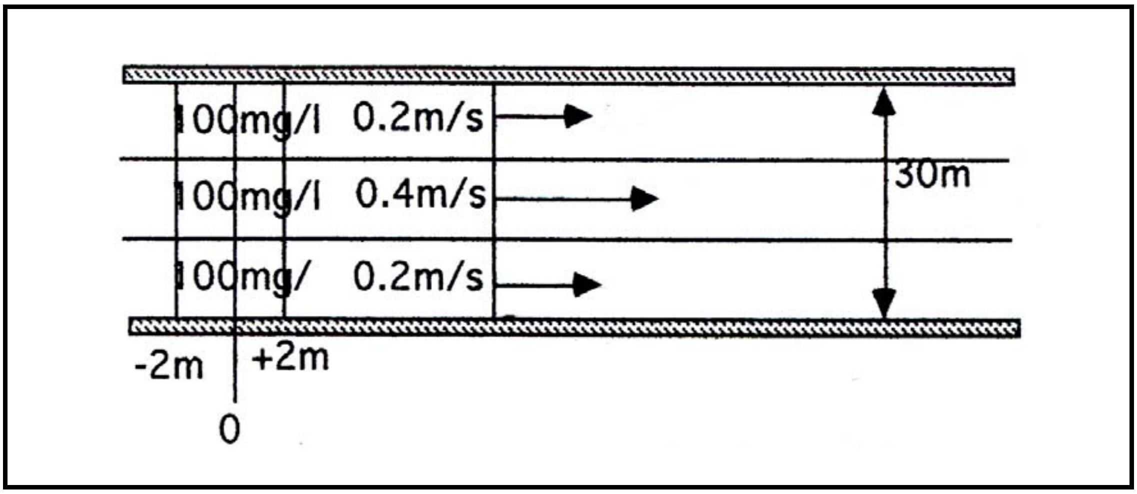
This "scenario" represents the one-dimensional unsteady-state advection and longitudinal dispersion of an instantaneous impulse of tracer for which the concentration profile follow the Gaussian plume equation

$$C = \frac{M}{\sqrt{4\pi Kt}} exp\left\{ -\frac{\left(x - Ut\right)^2}{4Kt} \right\}$$

in which x = distance downstream of the injection point, M = mass injected width of the stream, K = longitudinal dispersion coefficient, U = bulk velocity of the stream (flowrate/cross-sectional area), t = elapsed time since injection.

3) Using your best guess of a value for U, find a best-fit value for K for each and for which you calculated a concentration profile. Tabulate of plot the effective K as a function t_m of and make a guess of what you think the functional form is.

◆ Dispersion mechanism in a hypothetical river



1) 3 lanes of different velocities

2) Every seconds complete mixing occurs across the cross section of the river (but not longitudinally) occurs, after shear advection is completed.

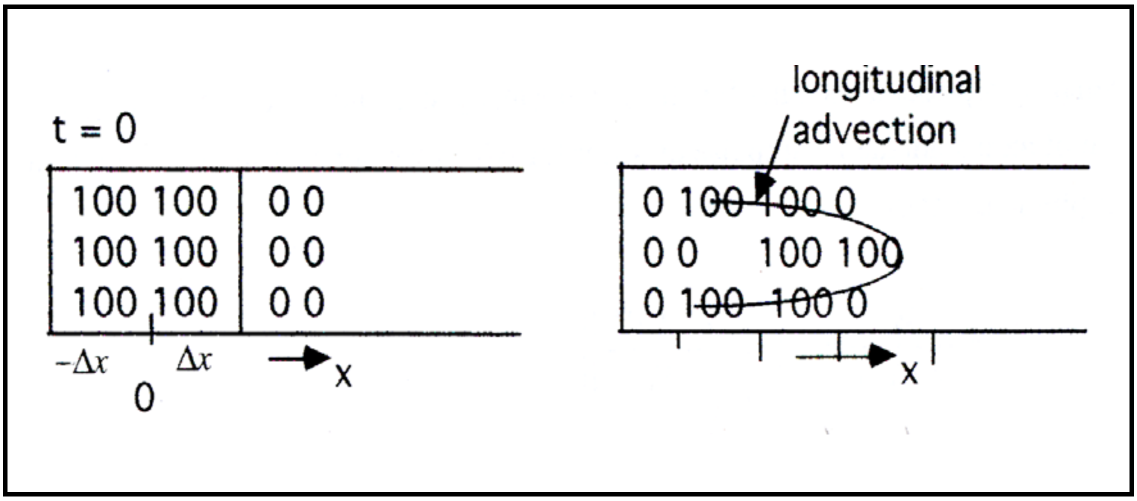
→ sequential mixing model

$$\frac{\partial}{\partial x}\left(\varepsilon_x \frac{\partial C}{\partial x}\right) \rightarrow 0$$

$$t_m \cong \frac{W^2}{\varepsilon_y}$$

3) Instantaneous injection

$t_m = 10$ s; $u_a = 0.2$ m/s; $\Delta x = 2$ m



4.1.5 Aris's Analysis

◆ Concentration moment method

◆ 2-D advective-diffusion equation in the moving coordinate system

$$\frac{\partial C}{\partial \tau} + u^{'} \frac{\partial C}{\partial \xi} = D\left(\frac{\partial^2 C}{\partial \xi^2} + \frac{\partial^2 C}{\partial y^2}\right) \tag{4.29}$$

Now, define the p_{th} moments of the concentration distribution

$$C_P(y) = \int_{-\infty}^{\infty} \xi^p C(\xi, y) d\xi$$

Define cross-sectional average of p_{th} moment

$$M_p = \frac{1}{A}\int_A C_P(y) dA = \overline{C_P}$$

Take the moment of Eq. (4.29) by applying the operator $\int_{-\infty}^{\infty} \xi^P (\ \ ) d\xi$

$$1) = \int_{-\infty}^{\infty} \xi^p \frac{\partial C}{\partial \tau} d\xi = \frac{\partial}{\partial \tau}\int_{-\infty}^{\infty} \xi^p C d\xi = \frac{\partial C_p}{\partial \tau}$$ ☜ Leibnitz rule

[Re] Leibnitz formula

$$\int_{u_0}^{u_1} \frac{\partial f}{\partial \alpha} dx = \frac{d}{d\alpha}\int_{u_0}^{u_1} f dx$$

$$2) = \int_{-\infty}^{\infty} \xi^p u^{'} \frac{\partial C}{\partial \xi} d\xi = u^{'}\int_{-\infty}^{\infty} \xi^p \frac{\partial C}{\partial \xi} d\xi$$ ☜ integral by parts

$$= u^{'}\left\{\left[\xi^p C\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} C_p \xi^{p-1} d\xi\right\}$$ ☜ $C|_{\xi=\pm\infty} = 0$

$$= -pu' \int_{-\infty}^{\infty} \xi^{p-1} C d\xi = -pu' C_{p-1}$$

$$3) = \int_{-\infty}^{\infty} \xi^{p} D \frac{\partial^2 C}{\partial \xi^2} d\xi = D \int_{-\infty}^{\infty} \xi^{p} \frac{\partial}{\partial \xi}\left(\frac{\partial C}{\partial \xi} \right) d\xi$$

☜ integral by parts

$$= D\left\{ \left[\xi^{p} \frac{\partial C}{\partial \xi} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial C}{\partial \xi} P \xi^{P-1} d\xi \right\}$$

$$= -Dp \int_{-\infty}^{\infty} \xi^{p-1} \frac{\partial C}{\partial \xi} d\xi$$

$$= -Dp \left\{ \left[\xi^{p-1} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} C(p-1) \xi^{p-1} d\xi \right\}$$

$$= Dp(p-1) \int_{-\infty}^{\infty} \xi^{p-2} C d\xi = Dp(p-1) C_{p-2}$$

$$4) = \int_{-\infty}^{\infty} \xi^{p} D \frac{\partial^2 C}{\partial y^2} d\xi = D \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} \xi^{p} C d\xi = D \frac{\partial^2 C_P}{\partial y^2}$$

Therefore Eq. (4.29) becomes

$$\therefore \frac{\partial C_p}{\partial \tau} - pu' C_{p-1} = D\left\{ p(p-1) C_{p-2} + \frac{\partial^2 C_P}{\partial y^2} \right\} \quad (4.33)$$

B.C.

$$D \frac{\partial C_P}{\partial y} = 0 \ at \ y = 0, h$$

Take cross-sectional average of Eq. (4.33)

$$\overline{\frac{\partial C_p}{\partial \tau}} - \overline{pu' C_{p-1}} = D\left\{ \overline{p(p-1) C_{p-2}} + \overline{\frac{\partial^2 C_P}{\partial y^2}} \right\}$$

$$\frac{\overline{\partial^2 C_P}}{\partial y^2} = \frac{\partial^2 \overline{C_P}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \overline{C_p}}{\partial y}\right) = 0$$ ☜ Reynolds average rule

$$\frac{dM_p}{d\tau} - \overline{pu^{'}C_{p-1}} = p(p-1)DM_{p-2} \qquad (4.34)$$

◆ Eq. (4.34) can be solved sequentially for = 0, 1, 2, …

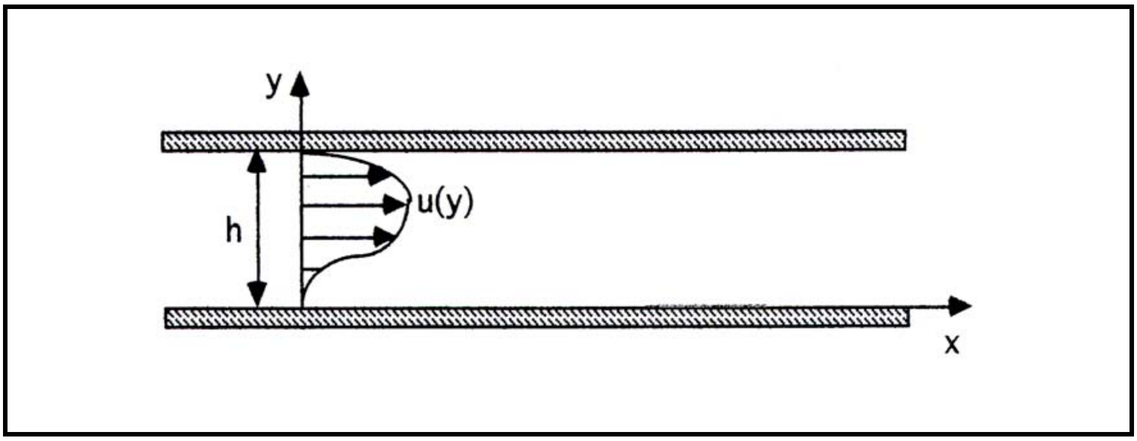
	Equation	Consequences as
$p=0$	$dM_0 / d\tau = 0$	Mass is conserved
	$M_0 \frac{1}{A}\int_A C_0(y)dA = \frac{1}{A}\int_A \int_{-\infty}^{\infty} C d\xi dA$	
	(4.33) → $\frac{\partial C_0}{\partial \tau} = D\frac{\partial^2 C_0}{\partial y^2}$	
$p=1$	$\frac{dM_1}{dt} = \overline{u^{'}C_0}$	$M_1 \rightarrow consant$
	(4.33) → $\frac{\partial C_1}{\partial \tau} - u^{'}C_0 = D\frac{\partial^2 C_1}{\partial y^2}$	
$p=2$	$\frac{dM_2}{dt} = \overline{2u^{'}C_1} + 2D\overline{C_0}$	$\frac{d\sigma^2}{dt} = 2K + 2D$
	→ molecular diffusion and shear flow dispersion are additive	

4.2 Dispersion in Turbulent Shear Flow

- Extend Taylor's analysis to turbulent flow

- Cross-sectional velocity profile in turbulent motion in the channel is different than in a laminar flow.

- Cross-sectional mixing coefficient is function of cross-sectional position.



2-D turbulent diffusion equation

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=\frac{\partial C}{\partial t}\left(\varepsilon_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(\varepsilon_y\frac{\partial C}{\partial y}\right) \qquad \text{(a)}$$

Where $C,u,v=$ time mean values; $C=\overline{C}=\frac{1}{T}\int_0^T cdt$

let $v=0$, turbulent fluctuation $v^{'}\neq 0$

assume $\frac{\partial C}{\partial t}\varepsilon_x\frac{\partial C}{\partial x}<<\frac{\partial}{\partial y}\varepsilon_y\frac{\partial C}{\partial y}$

Then (a) becomes

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}=\frac{\partial}{\partial y}\left(\varepsilon_y\frac{\partial C}{\partial y}\right) \qquad \text{(b)}$$

Now, decompose C and u into cross-sectional mean and deviation

$$\frac{\partial(\overline{C}+C^{'})}{\partial t}+\left(\overline{u}+u^{'}\right)\frac{\partial}{\partial x}(\overline{C}+C^{'})=\frac{\partial}{\partial y}\varepsilon_y\frac{\partial}{\partial y}\left(\overline{C}+C^{'}\right) \qquad \text{(c)}$$

Transform coordinate system into moving coordinate according to $\overline{u}$

$$\frac{\partial}{\partial \tau}\overline{C}+\frac{\partial C^{'}}{\partial \tau}+u^{'}\frac{\partial \overline{C}}{\partial \xi}+u^{'}\frac{\partial C^{'}}{\partial \xi}=\frac{\partial}{\partial y}\varepsilon_y\frac{\partial C^{'}}{\partial y}$$

Introduce Taylor's assumptions (discard three terms)

$$u^{'}\frac{\partial \overline{C}}{\partial \xi}=\frac{\partial}{\partial y}\varepsilon_y\frac{\partial C^{'}}{\partial y} \qquad (4.35)$$

Solution of Eq. (4.35) can be derived by integrating twice w.r.t. y

$$C^{'}=\frac{\partial \overline{C}}{\partial \xi}\int_0^y\frac{1}{\varepsilon_y}\int_0^y u^{'}dydy+C^{'}(0)$$

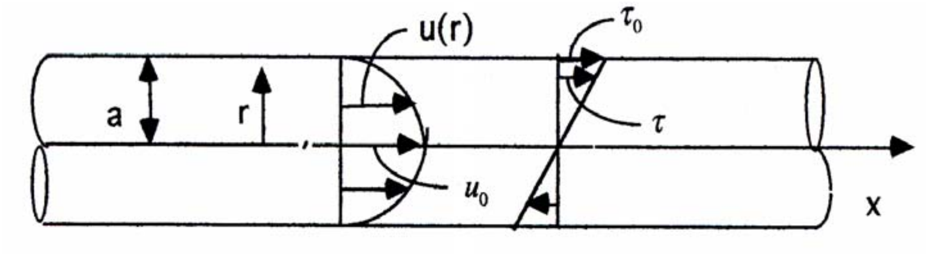
Mass transport in streamwise direction

$$\dot{M}=\int_0^h u^{'}C^{'}dy=\frac{\partial \overline{C}}{\partial \xi}\int_0^h u^{'}\int_0^y\frac{1}{\varepsilon_y}\int_0^y u^{'}dydydy$$

$$q=\frac{\dot{M}}{h}=-K\frac{\partial \overline{C}}{\partial \xi}$$

$$\therefore K=-\frac{1}{h}\int_0^h u^{'}\int_0^y\frac{1}{\varepsilon_y}\int_0^y u^{'}dydydy \qquad (4.36)$$

◆ Taylor's analysis of turbulent flow in pipe (1954)



Set $z = \frac{r}{a} \rightarrow \frac{dz}{dr} = \frac{1}{a}$

Then

$$u(z) = u_0 - u^* f(z) \qquad \text{(a)}$$

in which u^* = shear velocity $= \sqrt{\frac{\tau_0}{\rho}}$

f(z) = logarithmic function [Eq. (1.27)]

← velocity defect law

$$\therefore u = \bar{u} + \frac{3}{2}\frac{u^*}{\kappa} + \frac{2.30}{\kappa} u^* \log_{10} \frac{\zeta}{a}$$

in which κ = von Karman's constant ≈ 0.4

ς = distance from the wall

$$u = \bar{u} + 3.75u^* + 5.75u^* \log_{10} \frac{\zeta}{a}$$

$$\frac{u - \bar{u}}{u^*} = 3.75 + 2.5 \ln \frac{\zeta}{a}$$

◆ Reynolds analogy

→ mixing coefficients for momentum and mass transports are the same.

i) momentum flux through a surface

$$\frac{\tau}{\rho} = -\varepsilon \frac{\partial u}{\partial r}$$ ☜ Daily & Harleman (p. 56)

ii) mass flux - Fickian behavior

$$q = -\varepsilon \frac{\partial C}{\partial r}$$

$$\therefore \varepsilon = \frac{q}{-\frac{\partial C}{\partial r}} = \frac{\tau}{-\frac{\partial u}{\partial r}} \quad \text{(b)}$$

By the way,

$$\tau = \tau_0 \frac{r}{a} = z\tau_0 \quad \text{(c)}$$

Differentiate (a) w.r.t. *r*

$$\frac{\partial u}{\partial r} = -u^* \frac{df(z)}{dz} \frac{dz}{dr} = -u^* \frac{df}{dz} \frac{1}{a} \quad \text{(d)}$$

Divide (c) by (d)

$$\frac{\tau}{\frac{\partial u}{\partial r}} = \frac{z\tau_0}{-u^* \frac{df}{dz} \frac{1}{a}} \quad \text{(e)}$$

Substitute (e) into (b)

$$\therefore \varepsilon = -\frac{\tau}{\rho \frac{\partial u}{\partial r}} = \frac{z\tau_0}{\rho u^* \frac{df}{dz}\frac{1}{a}} = \frac{(\tau_0/\rho)(za)}{u^* \frac{df}{dz}} = \frac{azu^*}{\frac{df}{dz}}$$

Now, tabulate u(r), $u^{'} = u - \overline{u}, \varepsilon(r)$

Taylor's equation in radial coordinates

$$u^{'} \frac{\partial \overline{C}}{\partial \xi} = \varepsilon \left[\frac{\partial^2 C^{'}}{\partial r^2} + \frac{1}{r}\frac{\partial C^{'}}{\partial r} \right] \tag{4.39}$$

Numerically integrate Eq. (4.39) to obtain C

Numerically integrate Eq. (4.36) to find *K*

$$K = 10.1 a u^* \tag{4.40}$$

in which a = pipe radius

u^* = shear velocity

◆ Elder's application of Taylor's method (1959)

· Assumptions

- Turbulent flow down an <u>infinitely wide inclined plane</u>
- assume von Karman logarithmic velocity profile

$$u' = \frac{u^*}{\kappa}(1 + \ln y') \qquad \text{(a)}$$

in which $u' = u - \overline{u} \quad \rightarrow \quad \frac{du}{dy} = \frac{u^*}{\kappa}\frac{1}{y'}\frac{1}{d}$

$y' = y/d$

d = depth of channel

$$\tau = \rho\varepsilon\frac{du}{dy} = \tau_0(1 - y') \qquad \text{(b)}$$

$$\varepsilon = \frac{\tau_0}{\rho}\frac{(1 - y')}{\frac{du}{dy}} = \frac{\tau_0}{\rho}\frac{(1 - y')}{\frac{u^*}{\kappa}\frac{1}{y'}\frac{1}{d}} = \kappa y'(1 - y')du^* \qquad \text{(c)}$$

Substitute Eq. (a) and Eq. (c) into Eq. (4.36) and integrate

$$C' = \frac{\partial\overline{C}}{\partial x}\frac{d}{\kappa^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{d - y}{d}\right)^n - 0.648\right) \qquad (4.44)$$

$$K = \frac{0.404}{\kappa^3}du^* \qquad (4.45)$$

Input $\kappa = 0.41$

$$K = 5.93du^* \qquad (4.46)$$

◆ General form for the longitudinal dispersion coefficient

Set

$$y' = \frac{y}{h} \rightarrow y = hy',\ dy = hdy' \qquad \text{(a)}$$

$$u'' = \frac{u'}{\sqrt{\overline{u'^2}}} \rightarrow u' = u''\sqrt{\overline{u'^2}} \qquad \text{(b)}$$

$$\varepsilon' = \frac{\varepsilon}{E} \rightarrow \varepsilon = \varepsilon' E \qquad \text{(c)}$$

in which u' = velocity deviation from cross-sectional mean velocity

$$\sqrt{\overline{u'^2}} = \left\{\frac{1}{h}\int_0^h (u')^2 dy\right\}^{\frac{1}{2}}$$

= intensity of the velocity deviation

= different from turbulent intensity

=measure of how much the turbulent averaged velocity deviates throughout the cross section from its cross-sectional mean

E = cross-sectional average of ε

Substitute (a) ~ (c) into Eq. (4.36)

$$K = -\frac{1}{h}\int_0^1 u''\sqrt{\overline{u'^2}}\int_0^{y'}\frac{1}{\varepsilon' E}\int_0^{y'} u''\sqrt{\overline{u'^2}}h^3 dy'dy'dy'$$

$$= -\frac{1}{h}\sqrt{\overline{u'^2}}\frac{1}{E}\sqrt{\overline{u'^2}}h^3\int_0^1 u''\int_0^{y'}\frac{1}{\varepsilon'}\int_0^{y'} u'' dy'dy'dy'$$

$$= \frac{\overline{u'^2}h^2}{E}\left(-\int_0^1 u''\int_0^{y'}\frac{1}{\varepsilon'}\int_0^{y'} u'' dy'dy'dy'\right) \qquad \text{(d)}$$

Set $I = -\int_0^1 u'' \int_0^{y'} \frac{1}{\varepsilon'} \int_0^{y'} u'' dy' dy' dy'$ (4.48)

Then (d) becomes

$$K = \frac{h^2 \overline{u'^2}}{E} I \quad (4.47)$$

◆ Range of values of for flows of practical interest

$$I = 0.054 \sim 0.10 \;\; \rightarrow \;\; I \cong 0.10$$

Flow	Velocity profile	Charac. length, h	I	K
(i)laminar flow in a tube	$u = u_0(1 - \frac{r^2}{a^2})$	a	0.0625	$\frac{a^2 u_0^{\ 2}}{192D}$
(ii)laminar flow at depth down on inclined plane	$u = u_0\left[2\left(\frac{y}{d}\right) - \frac{y^2}{d^2}\right]$	d	0.0952	$\frac{8}{945}\frac{d^2 u_0^{\ 2}}{D}$
(iii)laminar flow with a linear velocity profile across a spacing	$u = U\frac{y}{h}$	h	0.10	$\frac{U^2 h^2}{120D}$
(iv)turbulent flow in a pipe	empirical	a	0.054	$10.1\ au^*$
(v)turbulent flow at depth down an inclined plane	$u = \overline{u} + \frac{u^*}{\kappa}(1 + \ln\frac{y}{d})$	d	0.067	$\frac{0.404}{\kappa} du^*$

4.3 Dispersion in Unsteady Shear Flow

◆ Unsteady flow

- reversing flow in a tidal estuary; wind driven flow in a lake caused by a passing storm

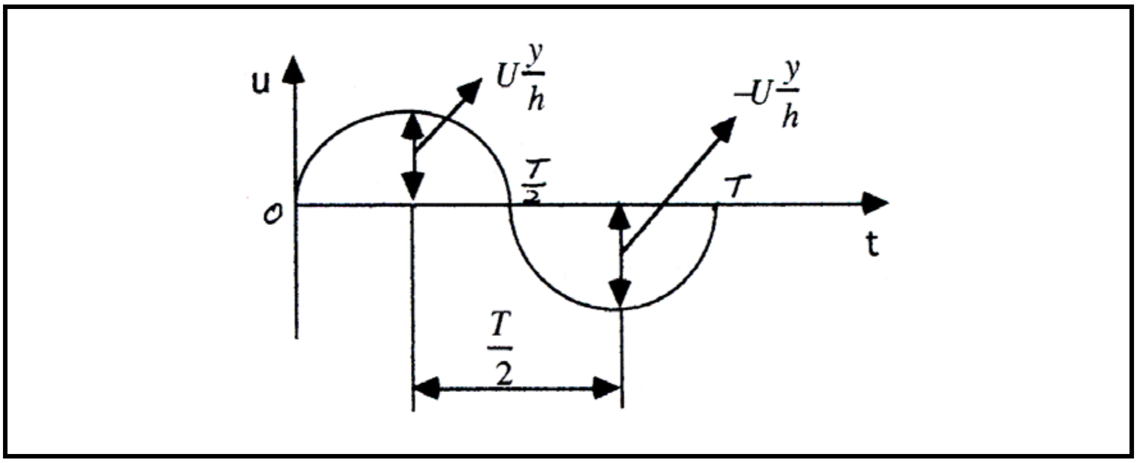
= steady component + oscillatory component

◆ Application of Taylor's analysis to an oscillatory shear flow

(i) linear velocity profile with a sinusoidal oscillation

$$u = U\frac{y}{h}\sin\left(\frac{2\pi t}{T}\right)$$

where T = period of oscillation



◆ 'flip-flop' sort of flow

- reversing instantaneously between $u = U\frac{y}{h}$ and $-u = U\frac{y}{h}$ after every time interval $\frac{T}{2}$

→ after each reversal the concentration profile has to be reversed

→ substitute – y for y in Eq. (4.21)

→ but enough time bigger than mixing time ($T_c \approx h^2 / D$) is required before the concentration profile is completely adopted to a new velocity profile.

(1) $T >> T_C$

- concentration profile will have sufficient time to adopt itself to the velocity profile in each direction
- time required for to reach the profile given by Eq.(4.21) is short compared to the time during which has that profile.

→ dispersion coefficient will be <u>the same as that in a steady flow</u>

(2) $T << T_C$

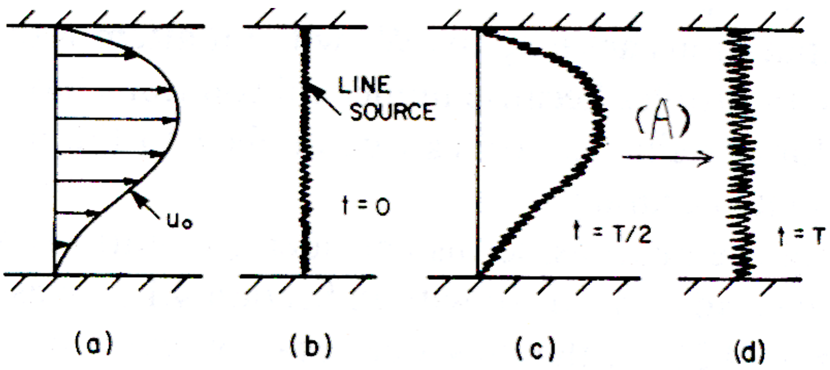
- period of reversal is very short compared to the cross-sectional mixing time
- concentration profile does not have time to respond to the velocity profile
- C' will oscillate around the mean of the symmetric limiting profiles, which is C'=0.

→ dispersion coefficient tends toward zero

$T >> T_C$ dispersion as if flow were steady in either direction

$T << T_C$ no dispersion due to the velocity profile

◆ Fate of an instantaneous line source when $T << T_C$



(A) : there is no cross-sectional mixing before the flow reverse.

◆ Solution of Eq. (4.13) by Carslaw and Jaeger (1959)

$$\frac{\partial C'}{\partial \tau} - D\frac{\partial^2 C'}{\partial y^2} = -u'\frac{\partial \overline{C}}{\partial \xi}$$

$$u = u' = U\frac{y}{h}\sin\frac{2\pi t}{T} (\because \overline{u} = 0)$$

B.C. $\frac{\partial C'}{\partial y} = 0 \; at \; y = \pm\frac{h}{2}$

I.C. $C'(y,0) = 0$

- replace unsteady source term $u'\frac{\partial \overline{C}}{\partial \xi}$ by a source of constant strength by setting $t = t_0$

$$\frac{\partial C^*}{\partial \tau} - D\frac{\partial^2 C^*}{\partial y^2} = -U\frac{y}{h}\frac{\partial \overline{C}}{\partial x}\sin(\frac{2\pi t_0}{T})$$

$$\frac{\partial C^*}{\partial y} = 0 \ at \ y = \pm\frac{h}{2}$$

$$C^*(y,0) = 0$$

where C^* = distribution resulting from a suddenly imposed source distribution of constant strength

As diagrammed in Fig. 2.8, the solution for a series of sources of variable strength, can be obtained by

$$C'(y,t) = \int_0^t \frac{\partial}{\partial t}C^*(y,t-t_0;t_0)dt_0$$

For large t

$$C'(y,t) = \int_{-\infty}^t \frac{\partial}{\partial t}C^*(y,t-t_0;t_0)dt_0$$

By separation of variables / Fourier expansion

$$C' = \frac{2Uh^2}{\pi^3 D}\frac{T}{T_c}\frac{\partial \overline{C}}{\partial x}\sum_{n=1}^{\infty}\frac{(-1)^n}{\left(2n-1\right)^2}\sin(2n-1)\pi\frac{y}{h}$$

$$\times\left[\left(\frac{\pi}{2}\left(2n-1\right)\right)^2\frac{T}{T_c}\right]^{-\frac{1}{2}}\sin\left(\frac{2\pi t}{T}+\theta_{2n-1}\right)$$

where $\theta_{2n-1} = \sin^{-1}\left(-\left\{\left[\frac{1}{2}\pi\left(2n-1\right)^2\frac{T}{T_c}\right]^2+1\right\}^{-\frac{1}{2}}\right)$

◆ Average over the period of oscillation of K

$$K = \frac{1}{T}\int_0^T \left(-\int_{-\frac{h}{2}}^{\frac{h}{2}} u' C' \frac{dy}{h} \frac{\partial \overline{C}}{\partial x} \right) dt$$

$$= \frac{u^2}{\pi^4} \frac{h^2}{D} \left(\frac{T}{T_c} \right)^2 \sum_{n=1}^{\infty} (2n-1)^{-2} \left\{ \left[\frac{\pi}{2} (2n-1)^2 \left(\frac{T}{T_c} \right)^2 \right]^2 + 1 \right\}^{-1}$$

$$\rightarrow \begin{bmatrix} T << T_c, \; K \rightarrow 0 \\ T >> T_c, \; K_0 = \dfrac{1}{240} \dfrac{U^2 h^2}{D} \end{bmatrix}$$

[Re] Case of $T >> T_c$

For a linear steady velocity profile, $u = U \dfrac{y}{h} \sin\alpha$

$$K_{st} = \frac{1}{120} \frac{U^2 h^2}{D} \sin^2 \frac{\alpha}{D}$$

→ $K_0 = \dfrac{1}{240} \dfrac{U^2 h^2}{D}$ is an ensemble average of K_{st} over all values of α

Intermediate behavior → Fig.4.7

$$\frac{T}{T_c} = 0.1 \rightarrow K \approx 0.03 K_0$$

$$\frac{T}{T_c} = 1 \rightarrow K \approx 0.8 K_0$$

$$\frac{T}{T_c} = 10 \rightarrow K = K_0$$

(ii) Flow including oscillating and a steady component

$$u(y) = u_1(y)\sin 2\pi t / T + u_2(y)$$

$u_1 = u_2 = Uy/h$ $\rightarrow$ pulsating flow found in blood vessel

Assume that the results by separate velocity profile are additive.

Let $C' = C_1' + C_2'$ is solution to $\dfrac{\partial C'}{\partial t} + u(t)\dfrac{\partial \bar{C}}{\partial x} = \varepsilon \dfrac{\partial^2 C'}{\partial y^2}$

Then C_1' is solution to the equation

$$\frac{\partial C_1'}{\partial t} + u_1 \sin(2\pi t/T)\frac{\partial \bar{C}}{\partial x} = \varepsilon \frac{\partial^2 C_1'}{\partial y^2}$$

C_2' is solution to the equation

$$\frac{\partial C_2'}{\partial t} + u_2 \frac{\partial \bar{C}}{\partial x} = \varepsilon \frac{\partial^2 C_2'}{\partial y^2}$$

◆ cycle-averaged dispersion coefficient

$$\bar{K} = \frac{1}{T}\int_0^T -\frac{1}{h\dfrac{\partial \bar{C}}{\partial x}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(u_1 \sin\frac{2\pi t}{T} + u_2 \right)(C_1' + C_2')\,dydt$$

$$= -\frac{1}{h\dfrac{\partial \bar{C}}{\partial x}} \left[\frac{1}{T}\int_0^T \int_{-\frac{h}{2}}^{\frac{h}{2}} u_1 C_1' \sin\frac{2\pi t}{T}\,dydt + \int_{-\frac{h}{2}}^{\frac{h}{2}} u_2 C_2'\,dy \right]$$

$$= K_1 + K_2$$

where K_1 = result of oscillatory profile; K_2 = result of steady profile

4.4 Dispersion in Two Dimensions

◆ 2-D flow: velocity vector rotates with depth

$$\vec{u} = \vec{i}u(z) + \vec{j}v(z)$$

where $u =$ component of velocity $\vec{u}$ in the x direction

$v =$ component of velocity $\vec{u}$ in the y direction

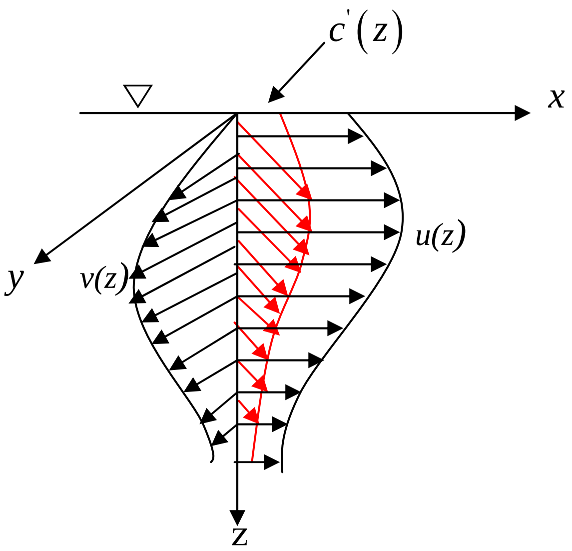


Fig. 4.8 skewed shear flow in the surface layer of Lake Huron

- Taylor's analysis applied to a skewed shear low with velocity profiles

· 2-D form of Eq. (4.10)

$$u'\frac{\partial \bar{C}}{\partial x} + v'\frac{\partial \bar{C}}{\partial y} = \frac{\partial}{\partial z}\varepsilon\frac{\partial C'}{\partial z} \tag{4.61}$$

$$\frac{\partial C'}{\partial z} = 0 \quad \text{at} \quad z = 0, h \quad \text{(water surface \& bottom)}$$

Integrate (4.61) w.r.t. z twice

$$C^{'}\left(z\right)=\int_{0}^{z}\frac{1}{\varepsilon}\int_{0}^{z}\left(u^{'}\frac{\partial\overline{C}}{\partial x}+v^{'}\frac{\partial\overline{C}}{\partial y}\right)dzdz \tag{4.62}$$

· Bulk dispersion tensor

$$\dot{M}_{x}=\int_{0}^{h}u^{'}C^{'}dz=-hK_{xx}\frac{\partial\overline{C}}{\partial x}-hK_{xy}\frac{\partial\overline{C}}{\partial y}$$

$$\dot{M}_{y}=\int_{0}^{h}v^{'}C^{'}dz=-hK_{yx}\frac{\partial\overline{C}}{\partial x}-hK_{yy}\frac{\partial\overline{C}}{\partial y} \tag{4.63}$$

Substitute (4.62) into (4.63)

(a): $\int_{0}^{h}u^{'}\int_{0}^{z}\frac{1}{\varepsilon}\int_{0}^{z}\left(u^{'}\frac{\partial\overline{C}}{\partial x}+v^{'}\frac{\partial\overline{C}}{\partial y}\right)dzdzdz=h\left(-K_{xx}\frac{\partial\overline{C}}{\partial x}-K_{xy}\frac{\partial\overline{C}}{\partial y}\right)$

$$K_{xx}=-\frac{1}{h}\int_{0}^{h}u^{'}\int_{0}^{z}\frac{1}{\varepsilon}\int_{0}^{z}u^{'}dzdzdz \tag{4.64a}$$

$$K_{xy}=-\frac{1}{h}\int_{0}^{h}u^{'}\int_{0}^{z}\frac{1}{\varepsilon}\int_{0}^{z}v^{'}dzdzdz \tag{4.64b}$$

(b): $\int_{0}^{h}\upsilon^{'}\int_{0}^{z}\frac{1}{\varepsilon}\int_{0}^{z}\left(u^{'}\frac{\partial\overline{C}}{\partial x}+v^{'}\frac{\partial\overline{C}}{\partial y}\right)dzdzdz=h\left(-K_{yx}\frac{\partial\overline{C}}{\partial x}-K_{yy}\frac{\partial\overline{C}}{\partial y}\right)$

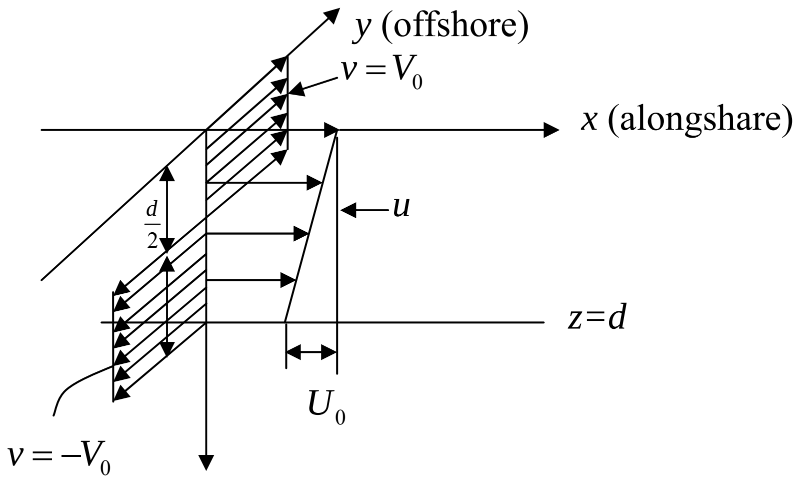
$$K_{yx}=-\frac{1}{h}\int_{0}^{h}v^{'}\int_{0}^{z}\frac{1}{\varepsilon}\int_{0}^{z}u^{'}dzdzdz \tag{4.64c}$$

$$K_{yy}=-\frac{1}{h}\int_{0}^{h}v^{'}\int_{0}^{z}\frac{1}{\varepsilon}\int_{0}^{z}v^{'}dzdzdz \tag{4.64d}$$

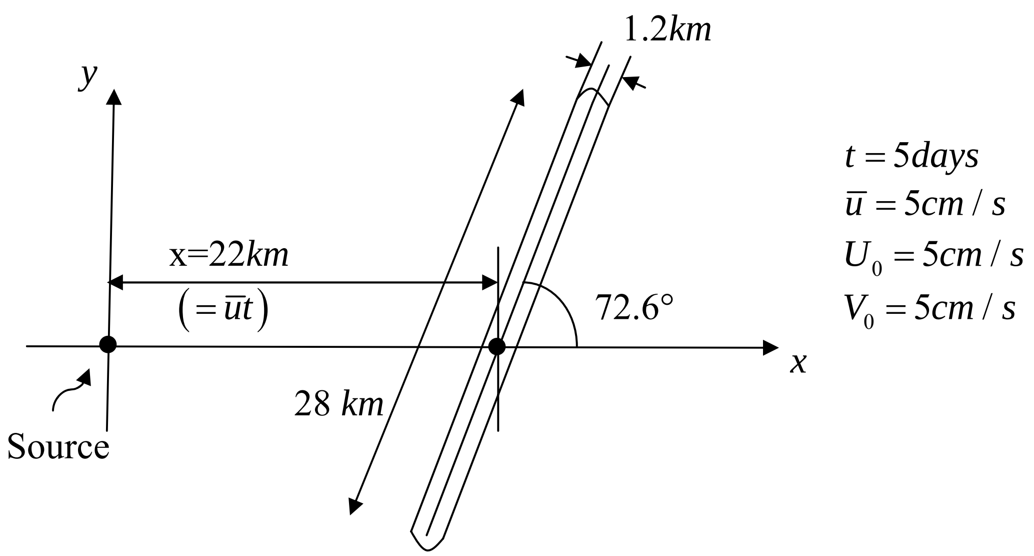
· $K_{xy},K_{yx}\sim$ depend on the interaction of the *x* and *y* velocity profiles

K_{yx}~ velocity gradient in the x direction can produce mass transport in the y direction and vice versa

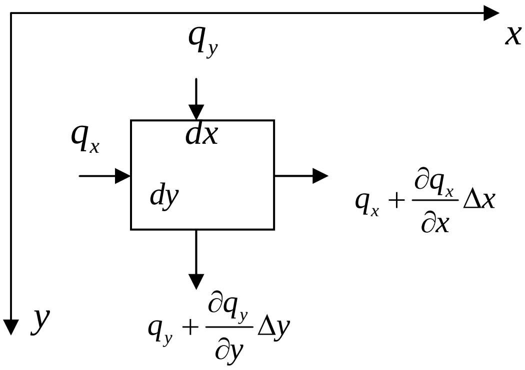
⊙Mean flow on a continental shelf discussed by Fischer (1978)



$$K = \frac{d^2}{\varepsilon}\begin{pmatrix} U_0^{\ 2}/120 & 5U_0V_0/192 \\ 5U_0V_0/192 & U_0^{\ 2}/120 \end{pmatrix} \tag{4.65}$$



◆ Governing Equation of 2-D Dispersion



(i) Conservation of mass

$$\frac{\partial C}{\partial t}\Delta x \Delta y = \left\{ q_x - \left(q_x + \frac{\partial q_x}{\partial x}\Delta x \right) \right\} \Delta y + \left\{ q_y - \left(q_y + \frac{\partial q_y}{\partial y}\Delta y \right) \right\} \Delta x$$

$$\therefore \quad \frac{\partial C}{\partial t} = -\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} \tag{1}$$

(ii) Apply Taylor's Analysis on 2-D shear flow

$$\dot{q}_x = \dot{M}_x = \left(\overline{u^{'}C^{'}} \right) xh = \int_0^h u^{'} C^{'} dz = \int u^{'} \int \frac{1}{\varepsilon} \int \left(u^{'} \frac{\partial \overline{C}}{\partial x} + v^{'} \frac{\partial \overline{C}}{\partial y} \right) dzdzdz$$

$$= -K_{xx} \frac{\partial \overline{C}}{\partial x} - K_{xy} \frac{\partial \overline{C}}{\partial y} \tag{2}$$

$$q_y = \dot{M}_y = \left(\overline{v^{'}c^{'}} \right) xh = \int_0^h v^{'} c^{'} dz = \int v^{'} \int \frac{1}{\varepsilon} \int \left(u^{'} \frac{\partial \overline{C}}{\partial x} + v^{'} \frac{\partial \overline{C}}{\partial y} \right) dzdzdz$$

$$= -K_{yx} \frac{\partial \overline{C}}{\partial x} - K_{yy} \frac{\partial \overline{C}}{\partial y} \tag{3}$$

(iii) Substitute (2) & (3) into (1)

$$\frac{\partial \overline{C}}{\partial t} = -\frac{\partial}{\partial x} \left(-K_{xx} \frac{\partial \overline{C}}{\partial x} - K_{xy} \frac{\partial \overline{C}}{\partial y} \right) - \frac{\partial}{\partial y} \left(-K_{yx} \frac{\partial \overline{C}}{\partial x} - K_{yy} \frac{\partial \overline{C}}{\partial y} \right)$$

(iv) Return to fixed coordinate system containing mean advective velocities

$$\frac{\partial \overline{C}}{\partial t}+\overline{u}\frac{\partial \overline{C}}{\partial x}+\overline{v}\frac{\partial \overline{C}}{\partial y}=\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial \overline{C}}{\partial x}+K_{xy}\frac{\partial \overline{C}}{\partial y}\right)+\frac{\partial}{\partial y}\left(K_{yx}\frac{\partial \overline{C}}{\partial x}+K_{yy}\frac{\partial \overline{C}}{\partial y}\right)$$

In general K_{xy} and K_{yx} are small compared with K_{xx} and K_{yy}. Thus, those two terms are often neglected. Then, 2-D depth-averaged transport equation becomes

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{yy}\frac{\partial C}{\partial y}\right)$$

[Cf] 2-D depth-averaged models (ASCE, 1988; vol.114, No.9)

· Scalar transport equation for Φ

$$\frac{\partial\left(H\overline{\Phi}\right)}{\partial t}+\frac{\partial\left(H\overline{U}\overline{\Phi}\right)}{\partial x}+\frac{\partial\left(H\overline{V}\overline{\Phi}\right)}{\partial y}=\frac{1}{\rho}\frac{\partial}{\partial x}\left(H\overline{J}_x\right)+\frac{1}{\rho}\frac{\partial}{\partial y}\left(H\overline{J}_y\right)$$

$$+\underbrace{\frac{1}{\rho}\frac{\partial}{\partial x}\int \rho U^{'}\Phi^{'}dz}_{dispersion}+\underbrace{\frac{1}{\rho}\frac{\partial}{\partial y}\int \rho V^{'}\Phi^{'}dz}_{dispersion}$$

where $\overline{J}_x=\int -\rho\overline{u^{'}\phi^{'}}dz$ turbulent diffusion in x-dir

$\overline{J}_y=\int -\rho\overline{u^{'}\phi^{'}}dz$ turbulent diffusion in y-dir

$u^{'}=u-U$ → Time fluctuation

$\phi^{'}=\phi-\Phi$

$$U' = U - \overline{U} \qquad \rightarrow \text{ depth deviation}$$

$$\Phi' = \Phi - \overline{\Phi}$$

If dispersion >> turbulent diffusion

Then neglect turbulent diffusion or incorporate turbulent diffusion into dispersion.