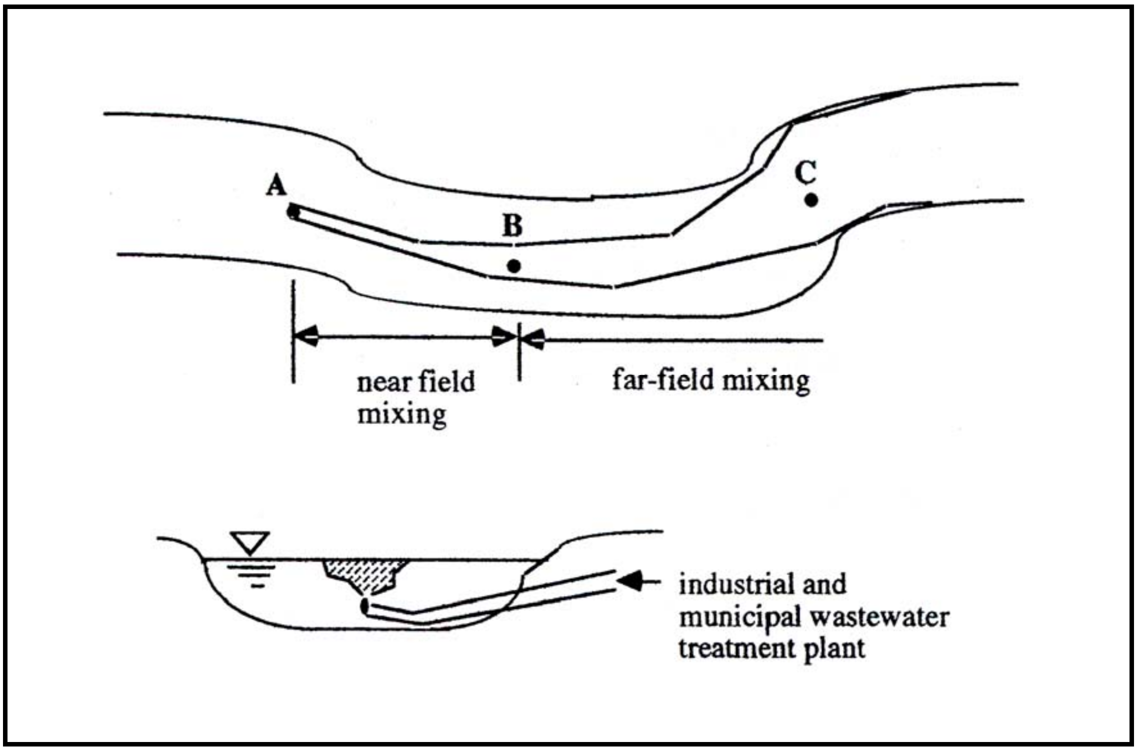
5. Chapter Mixing in Rivers



◆ Two phases of hydrodynamics mixing processes

1) Near field: mixing is controlled by the initial jet characteristics of momentum flux, buoyancy flux, and outfall geometry

2) Far field: source characteristics are less important, mixing is controlled by buoyant spreading motions and passive diffusion due to ambient turbulence

◆ Three stages in the mixing of a effluent into a river

Stage I: Near field mixing

→ initial momentum and buoyancy determine mixing near the outlet

→ vertical mixing is usually completed

→ Ch.9 Turbulent jets and plumes

Ch.10 Design of ocean wastewater discharge system

• Multiport diffuser

- linear structure consisting of many closely spaced ports, or nozzles, through which wastewater effluent is discharged at high velocity into the receiving water body
- attractive engineering solution to the problem of managing wastewater discharge in an environmentally sound way
- offer high degree of initial dilution
- optimally adapted to the assimilative characteristic of the water body

· thermal diffuser: heated water discharge from the once-through cooling systems of steam-electric power plants

· wastewater diffuser: wastewater discharge from the sewage treatment plants

◆ Three groups of parameters for jet analysis

1) receiving water flow patterns – ambient water depth, velocity, density stratification

2) pollutant discharge flow characteristics – discharge velocity (momentum), flow rate, density of pollutant (buoyancy)

3) diffuser characteristics - single/multi ports, submerged/surface discharge, alignment of port

- jet analysis model: CORMIX (Cornell Mixing Zone Export System)
  VISJET

◆ water-quality policy in USA

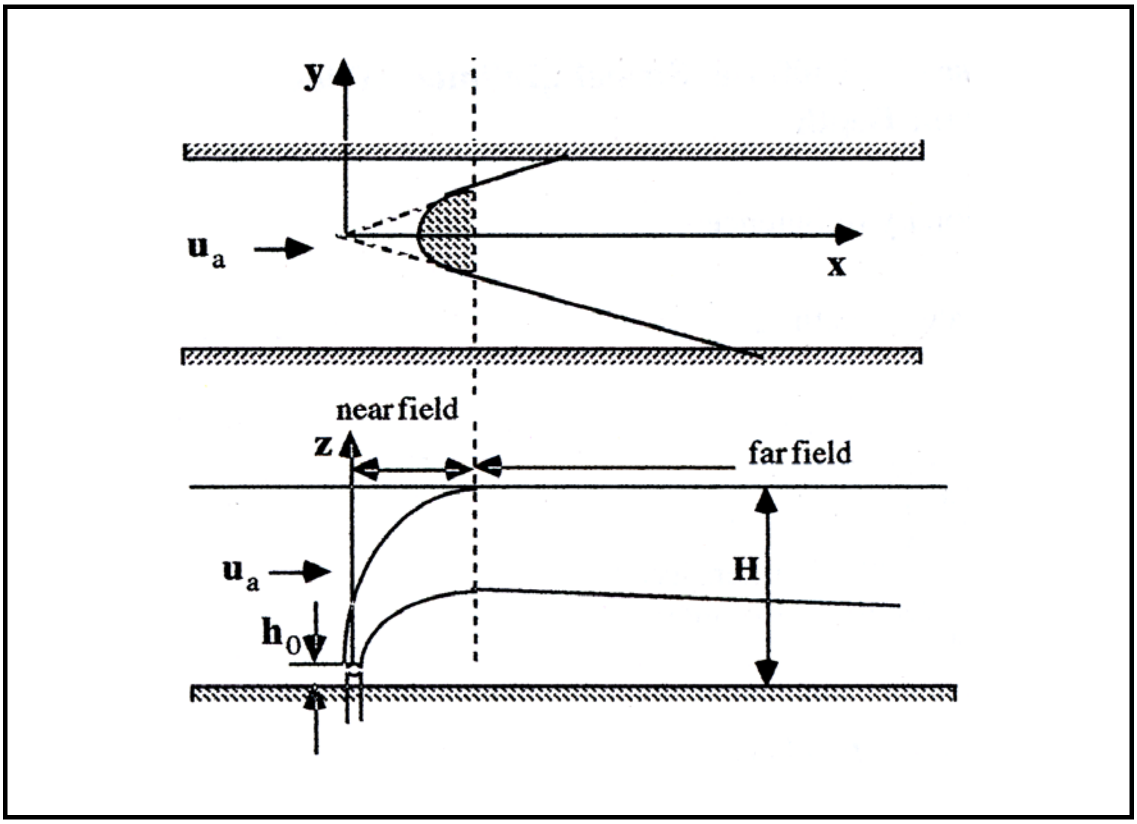
- "Technical support document for water quality-based toxics control", Office of Water, Washington, DC. (1991)

→ regulations on toxic control with higher initial mixing requirements
→ concept of regulatory mixing zone (RMZ)
= limited area or volume of water where initial dilution of an aqueous pollutant discharge occurs
- regulator = U.S. Environmental Protection Agency
- should predict the initial dilution of a discharge and extent of its mixing zone
- toxic dilution zone (TDZ) for toxic substances
- regularly mixing zone (RMZ) for conventional pollutants

◆ RMZ

streams, rivers	lakes, estuaries
Florida: RMZ ≤ 800m and ≤ 10% total length	≤ 125,600m² and ≤ 10% surface area
Michigan: RMZ ≤ 1/4 cross-sectional area	≤ 1000ft radius
West Virginia: RMZ ≤ 20～33% cross-sectional area and ≤ 5～10 times width	≤ 300ft any direction

◆ near field mixing ≤ regulatory mixing zone



Stage II: Lateral mixing

→ waste is mixed across the receiving channel primarily by turbulence in the receiving stream

Stage III: Longitudinal dispersion

→ process of longitudinal shear flow dispersion erases any longitudinal concentration variations

→ Taylor's analysis of longitudinal dispersion

Far field = Stage II+ Stage III

→ deal with a source of tracer without its own momentum or buoyancy

5.1 Turbulent Mixing in Rivers

5.1.1 The Idealized Case of a Uniform, Straight, Infinitely Wide Channel of Constant Depth

- mixing of source of tracer without its own momentum or buoyancy

→ homogeneous, stationary turbulence of ambient water

- important Lagrangian length scale ≈ depth

◆ Eq. (3.40) : $\varepsilon = \ell_L \left[\overline{u'^2} \right]^{\frac{1}{2}}$ (1)

where ε = turbulent mixing coefficient

ℓ_L = Lagrangian length scale ≈ d (a)

$\left[\overline{u'^2} \right]^{\frac{1}{2}}$ = intensity of turbulence

· Lauffer (1950) - Experimental Data

turbulent intensity $\propto$ shear stress on the wall

· Henderson (1966)

- bottom shear stress is evaluated by a force balance

$$\tau_0 = \rho g d S$$

For dimensional reason, shear stress must be expressed as a velocity

→ shear velocity $\therefore u^* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{gdS}$ (b)

where S = Slope of the channel

Substitute (a) & (b) into (1)

$$\therefore \varepsilon \propto d\, u^*$$

1) ε_v = vertical mixing

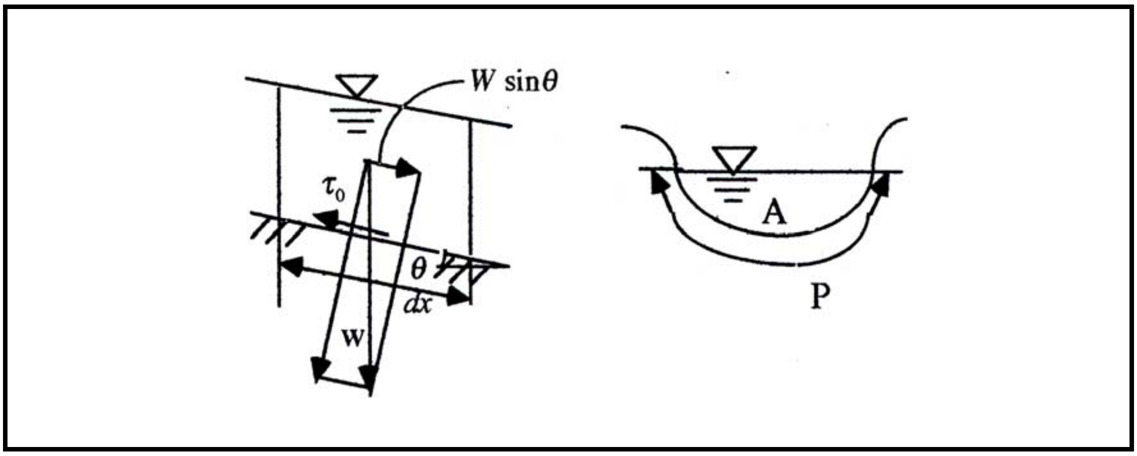
→ influence of surface and bottom boundaries

→ turbulence will not be isotropic

2) $\varepsilon_t = \varepsilon_l$ = transverse and longitudinal mixing

→ no boundaries to influence flow

[Re] Shear stress and shear velocity



For uniform flow

$$\tau_0 (Pdx) = \rho g A dx \sin\theta$$

$$\tau_0 = \rho g \frac{A}{P} \sin\theta$$

Set $S = \tan\theta \approx \sin\theta$

R = hydraulic radius $= \dfrac{A}{P}$

Then $\tau_0 = \gamma RS$

For very wide channel (b>>d)

$$R = \frac{bd}{b+2d} = \frac{d}{1+2\dfrac{d}{b}} \approx d$$

$$\therefore \tau_0 = \gamma dS$$

5.1.1.1 Vertical Mixing

(1) Vertical mixing coefficient in 3D model

→ no dispersion effect by shear flow

i) vertically varying coefficient:

- vertical mixing coefficient for momentum due to logarithmic law velocity profile → Eq. 4.43

$$\varepsilon_v = \kappa du^* (z/d)\left[1-\left(z/d\right)\right] \tag{5.2}$$

[Re] Velocity profiles:

- vertical profile of u-velocity → logarithmic

- vertical profile of v-velocity → linear → might be neglected because v-velocity is relatively small compared to u-velocity

· Reynolds analogy

→ The same coefficients can be used for transport of mass (pollutant) and momentum

→ verified by Jobson & Sayre (1970)

ii) depth-averaged coefficient

- average Eq. (5.2) over the depth, taking $\kappa = 0.4$

$$\overline{\varepsilon_v} = \frac{1}{d}\int_0^d \kappa d u^* \left(\frac{z}{d}\right)\left[1-\left(\frac{z}{d}\right)\right] dz$$

$$= \frac{\kappa}{6} d u^* = 0.067 d u^* \qquad (5.3)$$

[Cf] For atmospheric boundary layer: $\overline{\varepsilon_v} = 0.05 d u^*$

where d = depth of boundary layer

u^* = shear velocity at the surface of the earth

5.1.1.2 Transverse Mixing

(1) Transverse mixing coefficient in 3D model

→ no dispersion effect by shear flow, turbulence effect only

→ vertically varying coefficient

• For infinitely wide uniform channel, there is no transverse velocity profile

→ not possible to establish a transverse analogy of Eq. (5.2)

→ need to know velocity profiles:

- transverse profile of u-velocity → **parabolic**
- transverse profile of w-velocity → might be neglected because w-velocity is usually very small

(2) Transverse mixing coefficient in 2D model

→ dispersion effect by shear flow due to vertical variation of v-velocity

→ depth-averaged coefficient

→ rely on experiments

→ see Table 5.1 for results of 75 separate experiments

$$\varepsilon_t \approx 0.15du^* \tag{5.4}$$

5.1.1.3 Longitudinal Mixing

(1) Longitudinal mixing coefficient in 3D model

→ no dispersion effect by shear flow, turbulence effect only

→ longitudinal analogy of Eq. (5.2)

→ need to know velocity profiles:

- longitudinal profile of v-velocity → **linear**
- longitudinal profile of w-velocity → might be neglected because w-velocity is usually very small

(2) Longitudinal mixing coefficient in 2D model

→ depth-averaged coefficient

- longitudinal turbulent mixing is the same rate as transverse mixing because there is an equal lack of boundaries to inhibit motion

- However, longitudinal mixing by turbulent motion is unimportant because shear flow dispersion coefficient caused by the velocity gradient (vertical variation of *u*-velocity) is much bigger than mixing coefficient caused by turbulence alone

$$K_l = 5.93du^* \approx 40\varepsilon_t$$

[Re] $\therefore K = -\frac{1}{h}\int_0^h u' \int_0^y \frac{1}{\varepsilon_y}\int_0^y u' dydydy$

· Aris (1956)

coefficients due to turbulent mixing and shear flow are additive

$$K_l + \varepsilon_\ell \rightarrow K_L$$

→ can neglect the longitudinal turbulent mixing

(3) Longitudinal dispersion coefficient in 1D model

→ Section 5.2

5.1.2 Mixing in Irregular Channels and Natural Streams

5.1.2.1 Mixing in natural channels

1) depth may vary irregularly → pool & riffle sequences

2) channel is likely to curve

3) there may be large sidewall irregularities → groins, dikes

i) Vertical mixing coefficient

• These have not much influence on vertical mixing since scale of vertical motion is limited by the local depth, d

$$\therefore \varepsilon_v = 0.067 d u^*$$

ii) Transverse mixing

• Transverse mixing is strongly affected by the channel irregularities because they are capable of generating a wide variety of transverse motions (vertical variation of v-velocity)

• Transverse mixing in open channels with curves and irregular sides

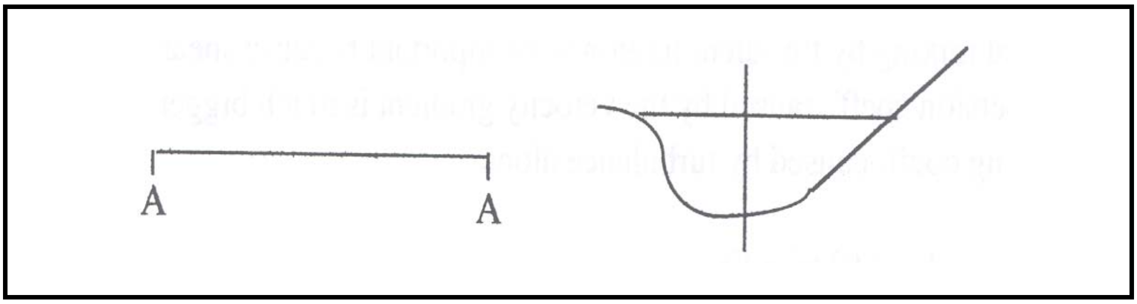
→ see Table 5.2

$$0.3 < \frac{\varepsilon_t}{d u^*} < 0.7$$

1) effect of channel irregularity:

the bigger the irregularity, the faster the transverse mixing

2) effect of channel curvature : secondary flow causes trasnsverse dispersion due to shear flow



- when a flow rounds a bend, the centrifugal forces induce a flow towards the outside bank at the surface, and a compensating reverse flow near the bottom.

→ secondary flow generates

→ Fischer (1969) predict a transverse dispersion coefficient based on the transverse shear flow

$$\frac{\varepsilon_t}{du^*} = 25\left(\frac{\overline{u}}{u^*}\right)^2\left(\frac{d}{R}\right)^2$$

where R = radius of curvature

◆ Yotsukura and Sayre(1976) → see Fig.5.3

$$\frac{\varepsilon_t}{du^*} \propto \left(\frac{\overline{u}}{u^*}\right)^2\left(\frac{W}{R}\right)^2$$

where W = channel width

straight, uniform channels $\varepsilon_t = 0.15du^*$

natural channels with side irregularities $\varepsilon_t = 0.4du^*$

meandering channels with moderate side irregularities $\varepsilon_t = \left(0.6 \pm 50\%\right)du^*$

5.1.2.2 2D depth-averaged model

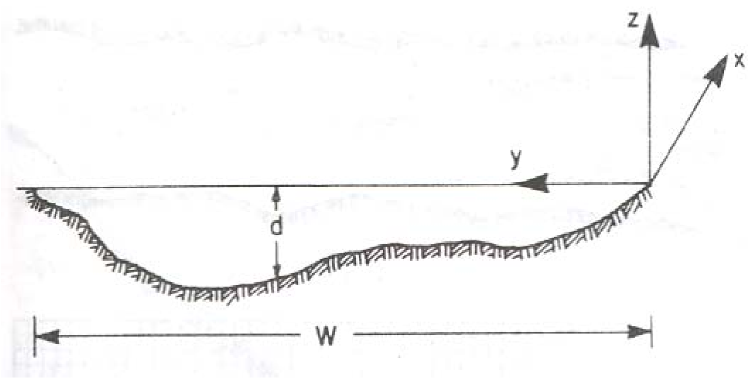
• Transverse dispersion coefficient in meandering channels

- Baek et al. (2006)
- Seo et al. (2008)
- Baek and Seo (2008)

• Transverse dispersion coefficient in natural streams

- Seo et al. (2006)
- Jeon et al. (2007)

5.1.3 Computation of concentration distributions



- compute the distribution of concentration downstream from an continuous effluent discharge in a flowing stream

$$\frac{\varepsilon_t}{\varepsilon_v} = \frac{0.6du^*}{0.067du^*} \approx 10$$

mixing time $T \propto \dfrac{(length)^2}{\varepsilon}$

$$\therefore \frac{T_t}{T_t} = \frac{(W)^2}{\varepsilon_t} / \frac{(d)^2}{\varepsilon_v} = \left(\frac{W}{d}\right)^2 \frac{\varepsilon_v}{\varepsilon_t} = \left(\frac{30}{1}\right)^2 \left(\frac{1}{10}\right) = 90 \approx 10^2$$

$$\therefore T_t \approx 10^2 T_v$$

→ vertical mixing is instantaneous compared to transverse mixing

→ assume that effluent is uniformly distributed over the vertical

→ analyze the two-dimensional spread from a uniform line source

◆ Maintained source in 2D: case of a rectangular channel of depth into which is discharged $\dot{M}$ units of mass (per time)

Recall Eq. (2.68)

$$C = \frac{\dot{M}/d}{\overline{u}\sqrt{4\pi\varepsilon_t \frac{x}{\overline{u}}}} \exp\left(-\frac{y^2 \overline{u}}{4\varepsilon_t x}\right)$$

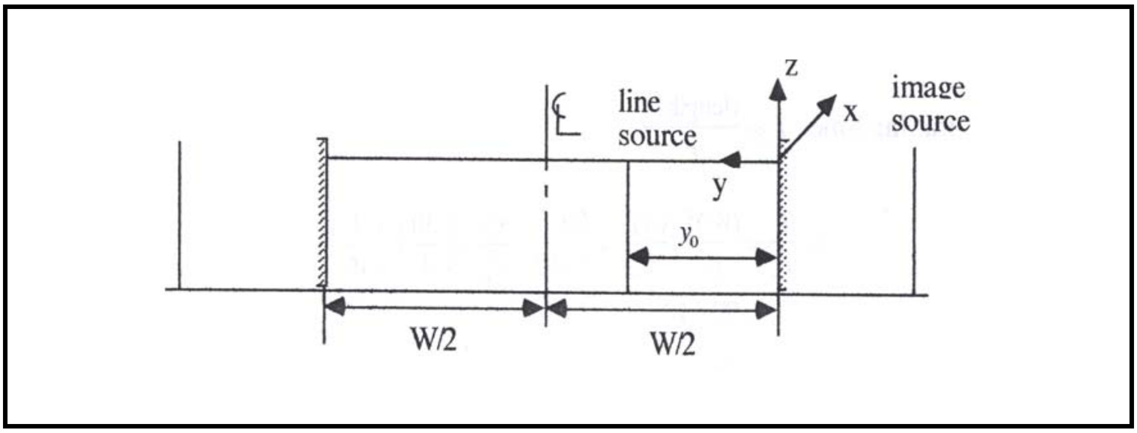
(5.7)

i) For very wide channel, when $t >> 2\varepsilon_t / \overline{u}^2$

→ use Eq. (5.7)

ii) For narrow channel, consider effect of boundaries

→ method of superposition



$$\frac{\partial C}{\partial y} = 0 \; at \; y = 0 \; and \; y = W$$

Define dimensionless quantities by setting

$$C_0 = \frac{\dot{M}}{\bar{u}dW} = \text{mass rate / volume of ambient water}$$

$\rightarrow$ concentration after cross-sectional mixing is completed

$$x' = \frac{x\varepsilon_t}{\bar{u}W^2}$$

$$y' = y/W$$

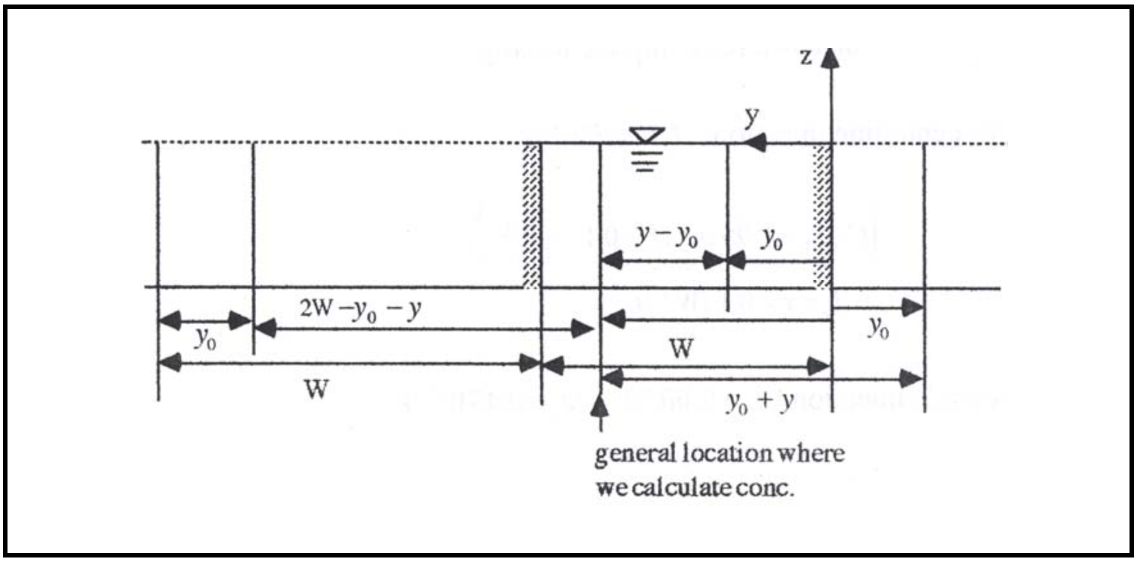
Then Eq. (5.7) becomes

$$C = \frac{\dfrac{\dot{M}}{\bar{u}dW}}{\sqrt{\dfrac{4\pi\varepsilon_t x'}{\bar{u}W^2}}} \exp\left(-\frac{(\dfrac{y}{W})^2}{\dfrac{4\varepsilon_t x}{\bar{u}W^2}} \right)$$

$$= \frac{C_0}{\sqrt{4\pi x'}} \exp\left(-\frac{y'^2}{4x'} \right)$$

$$\frac{C}{C_0} = \frac{1}{(4\pi x')^{1/2}} \exp\left(-\frac{y'^2}{4x'} \right)$$

If the source is located at $y = y_0 (y' = y'_0)$



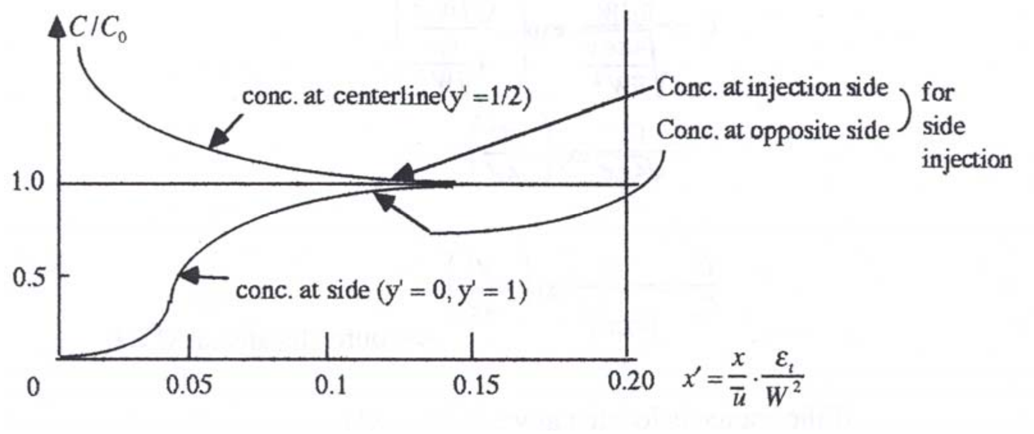
By superposition

$$\frac{C}{C_0} = \frac{1}{(4\pi x')^{\frac{1}{2}}}\left[e^{-\left\{\frac{(y'-y'_0)^2}{4x'}\right\}} + e^{-\left\{\frac{(y'+y'_0)^2}{4x'}\right\}} + e^{-\left\{\frac{(y'-2+y'_0)^2}{4x'}\right\}} + \bullet\bullet\bullet \right]$$

$$= \frac{1}{(4\pi x')^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \left\{ \exp\left[-(y' - 2 + y'_0)^2 / 4x' \right] + \exp\left[-\left(y' - 2n + y'_0 \right)^2 / 4x' \right] \right\} \quad (5.9)$$

① Continuous centerline discharge: $y'_0 = 1/2$

→ Fig.5.5



◆ Longitudinal distance for complete transverse mixing

For centerline injection, $L_c = 0.1\bar{u}W^2/\varepsilon_t$

$$\left(\frac{C}{C_0} = 0.95\, at\, x' = 0.1 = \frac{x\varepsilon_t}{\bar{u}W^2}\right)$$

$$\therefore L_c = x = 0.1\bar{u}W^2/\varepsilon_t$$

For side injection,

$$L = 0.1\bar{u}(2W)^2/\varepsilon_t = 0.4\bar{u}W^2/\varepsilon_t$$

[Ex 5.1] Spread of a plume from a point source

An industry discharges effluent;

C = 200ppm

Q = 3MGPD

$$= \frac{3\times 10^6\, GPD}{7.48 G/ft^3 \times 24 \times 3600} = 4.64\, CFS$$

Rate of mass input = $\dot{M}$ = QC

= 4.64(200ppm) = 928 CFSppm

Centerline injection in very wide, slowly meandering stream

$d = 30\,ft$; $\bar{u} = 2\,fps$; $u^* = 0.2\,fps$

Determine the width of the plume, and max. conc. at =1000ft downstream from discharge

[Sol]

For meandering stream,

$$\varepsilon_t = 0.6du^* = 0.6(30)(0.2) = 3.6\,ft^2/s$$

Use Eq.(5.7) for line source

$$C = \frac{\dot{M}}{\bar{u}d\left(\frac{4\pi\varepsilon_t x}{\bar{u}}\right)^{\frac{1}{2}}}\exp\left(-\frac{y^2\bar{u}}{4\varepsilon_t x}\right)$$

$$\exp\left(-\frac{y^2}{\frac{4\varepsilon_t x}{\bar{u}}}\right) \equiv \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

$$\sigma^2 = \frac{2\varepsilon_t x}{\bar{u}}, \qquad \sigma = \sqrt{\frac{2\varepsilon_t x}{\bar{u}}}$$

a) width of plume can be approximate by 4 σ. (95% of total mass)

$$b = 4\sigma = 4\sqrt{\frac{2\varepsilon_t x}{\overline{u}}} = 4\sqrt{\frac{2(3.6)(1000)}{2}} = 240\,ft$$

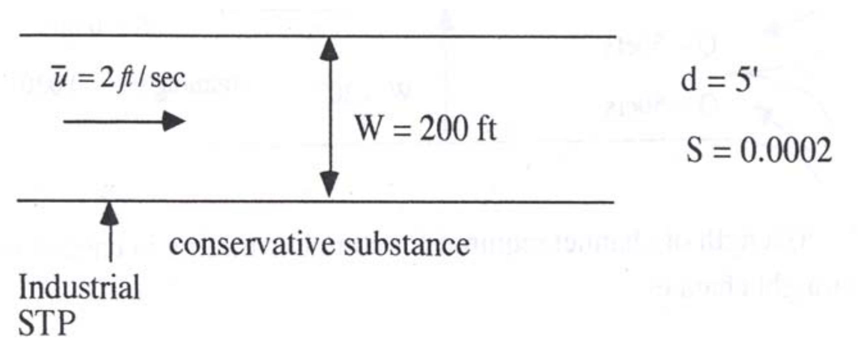
b) maximum concentration

$$C_{\max} = \frac{\dot{M}}{\overline{u}d\left(\frac{4\pi\varepsilon_t x}{\overline{u}}\right)^{\frac{1}{2}}} = \frac{928 CFSppm}{(2\,fps)(30\,ft)\left(\frac{4\pi \times 3.6\,ft^2/s \times 1000\,ft}{2\,ft/s}\right)^{\frac{1}{2}}}$$

[Ex 5.2] Mixing across a stream

→ consider boundary effect

Given :



Find: length of channel required for "complete mixing" as defined to mean that the concentration of the substance varies by no more than 5% over the cross section

[Sol]

Shear velocity

$$u^* = \sqrt{gdS} = \sqrt{32.2(5)(0.0002)} = 0.18\,ft/s$$

For uniform, straight channel

$$\varepsilon_t = 0.15du^*$$

$$= 0.15(5)(0.18) = 0.135\,ft^3/s$$

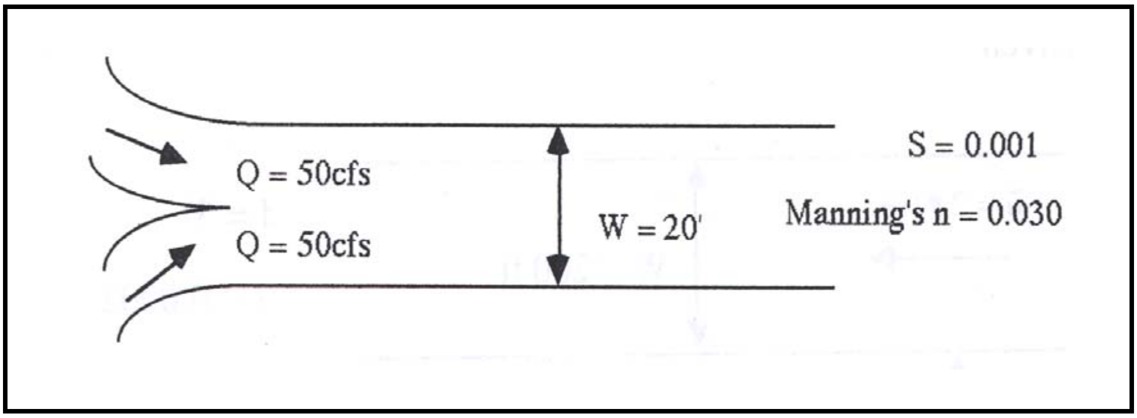
For complete mixing from a side discharge

$$L = 0.4\bar{u}W^2/\varepsilon_t$$

$$= 0.4(2)(200)^2/0.135 = 237,000\,ft \approx 45mile \approx 72km$$

[Ex 5.3] Blending of two streams

Given :



Find :

a) length of channel required for complete mixing for uniform straight channel

b) length of channel required for complete mixing for curved channel with a radius of 100ft

[Sol]

Manning's formula

$$\bar{u} = \frac{1.49}{n} R^{\frac{2}{3}} S^{\frac{1}{2}}$$

R = hydraulic radius = A/P

$$Q = A\bar{u} = \frac{1.49}{n} A R^{2/3} S^{1/2} = \frac{1.49}{n} \frac{A^{5/3}}{p^{2/3}} S^{1/2}$$

$$\therefore 100 = \frac{1.49}{0.030} \frac{(20d)^{5/3}}{(20+2d)^{2/3}} (0.001)^{1/2} = 145.41 \frac{d^{5/3}}{(10+d)^{2/3}}$$

$$d^{5/3} = 0.688(10+d)^{2/3}$$

$$d = 0.799(10+d)^{2/5}$$

By trial-error method, $d = 2.2\ ft$

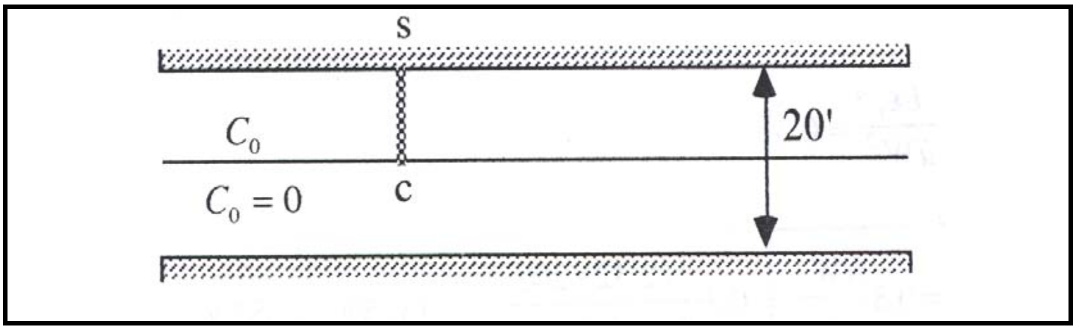
$$R = \frac{2.2(20)}{(20+4.4)} = 1.8$$

$$\bar{u} = \frac{1.49}{0.030}\left(\frac{2.2\times 20}{20+4.4}\right)^{2/3}(0.001)^{1/2} = 2.32\ ft/s$$

$$\therefore u^* = \sqrt{gRS} = \sqrt{32.2(1.8)(0.001)} = 0.24\ fps$$

$$\varepsilon_t = 0.15du^* = 0.15(2.2)(0.24) = 0.079\ \text{ft}^2/\text{s}$$

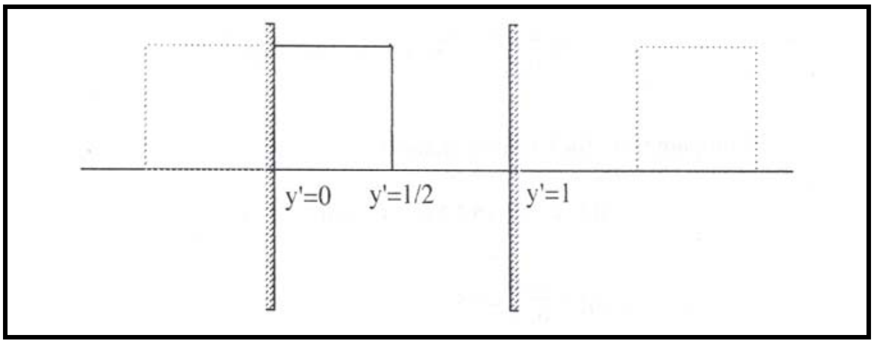
Think an upper bound first



i) For side injection only

$$L = 0.4\frac{\bar{u}W^2}{\varepsilon_t} = 0.4\frac{(2.32)(20)^2}{0.15(2.2)(0.24)} = 4687\ ft$$

ii) Consider sources ranging $y'_0 = 0 \sim 1/2 \rightarrow$ method of images



→ superposition of solutions for the step function

Eq.(2.33) for unbounded system

$$\frac{C}{C_0} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(erf\,\frac{y' + 1/2 + 2n}{\sqrt{4x'}} - erf\,\frac{y' - 1/2 + 2n}{\sqrt{4x'}} \right) \rightarrow \quad \text{Fig.5.9}$$

where $y' = y/W; x' = \dfrac{x\varepsilon_t}{\bar{u}W^2}$

From Fig.5.9, max. deviation in concentration is 5% of the mean when $x' \approx 0.3$.

$$\therefore x' = \frac{L\varepsilon_t}{\bar{u}W^2} = 0.3$$

$$\therefore \therefore L = 0.3\frac{\bar{u}W^2}{\varepsilon_t} = 0.3\frac{(2.32)(20)^2}{0.15(2.2)(0.24)} = 3515\,ft < 4687\,ft$$

For curved channel

$$\frac{\varepsilon_t}{du^*} = 25\left(\frac{\bar{u}}{u^*}\right)^2\left(\frac{d}{R}\right)^2$$

$$\therefore \varepsilon_t = 25\left(\frac{2.23}{0.24}\right)^2\left(\frac{2.2}{100}\right)^2(2.2)(0.24) = 0.60\,ft^2/s$$

Compare to ε_t for uniform channel

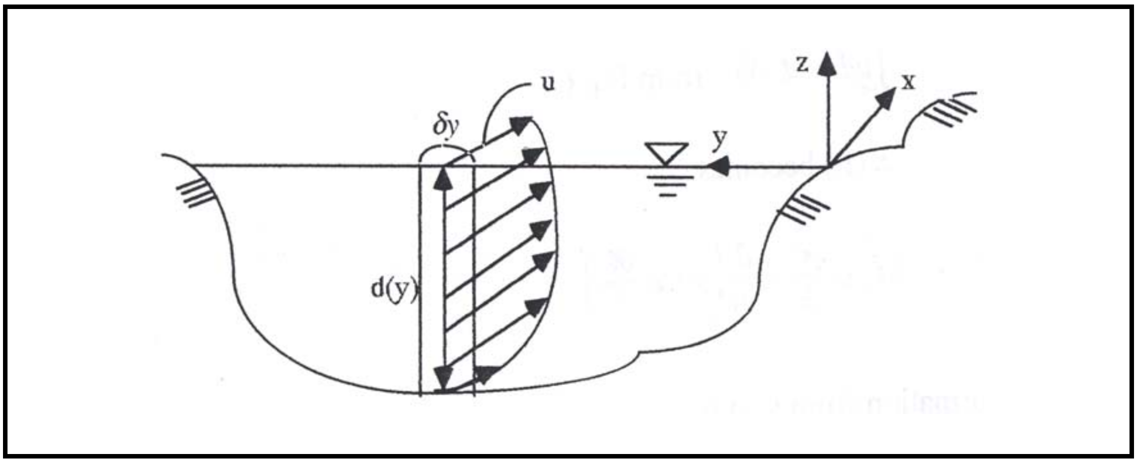
$$\varepsilon_t = 0.15du^* = 0.15(2.2)(0.24) = 0.079\ ft^2/s$$

$$\varepsilon_t\big|_C / \varepsilon_b\big|_u = \frac{0.60}{0.08} = 7.5$$

$$L = 0.3\frac{\bar{u}W^2}{\varepsilon_t} = \frac{0.3(2.32)(20)^2}{0.60} = 464\,ft$$

5.1.4 Complication in Real Streams

- Use of the Cumulative Discharge Method



In rivers the downstream velocity varies across the cross section

→ cumulative discharge method by Yotsukura and Sayre (1976)

$\bar{u}$ = cross-sectional average velocity

$\hat{u}$ = velocity averaged over depth at some value of y

$$\hat{u} = \frac{1}{d(y)} \int_{-d(y)}^{0} u dz \qquad \text{(a)}$$

$$dq = d(y) \delta y \hat{u} \qquad \text{(b)}$$

∴ cumulative discharge

$$q(y) = \int_0^y dq = \int_0^y d(y) \hat{u} dy \qquad \text{(c)}$$

$$q(y) = 0 \quad at \quad y = 0$$

$$q(y) Q \quad at \quad y = W$$

• Depth-averaged 2D equation for transverse diffusion

- assume steady-state, neglect longitudinal mixing and *v*-velocity

$$\frac{\partial C}{\partial t} + u\frac{\partial C}{\partial x} = \frac{\partial}{\partial y}\left(\varepsilon_t \frac{\partial C}{\partial y}\right) \tag{d}$$

Integrate (d) over depth

$$\int_{-d}^{0} u\frac{\partial C}{\partial x} dz = \int_{-d}^{0} \frac{\partial}{\partial y}(\varepsilon_t \frac{\partial C}{\partial y}) dz \tag{e}$$

From Eq.(a)

$$\int_{-d}^{0} u dz = d\left(y\right)\hat{u}$$

Eq. (e) becomes

$$d\left(y\right)\hat{u}\frac{\partial C}{\partial x} = \frac{\partial}{\partial y}\left(d\left(y\right)\varepsilon_t \frac{\partial C}{\partial y}\right)$$

$$\frac{\partial C}{\partial x} = \frac{1}{d\left(y\right)\hat{u}}\frac{\partial}{\partial q}\left(d\left(y\right)\varepsilon_t \frac{\partial C}{\partial y}\right) \tag{f}$$

Transformation from y to q

$$\frac{\partial}{\partial y} = \frac{\partial q}{\partial y}\frac{\partial}{\partial q} = d\left(y\right)\hat{u}\frac{\partial}{\partial q} \tag{g}$$

Substituting Eq. (g) into Eq.(f) yields

$$\frac{\partial C}{\partial x} = \frac{1}{d\left(y\right)\hat{u}}\frac{\partial}{\partial q}\left(d\left(y\right)\varepsilon_t \frac{\partial C}{\partial y}\right)$$

$$\therefore \frac{\partial C}{\partial x} = \frac{\partial}{\partial q}\left(d\left(y\right)\varepsilon_t d\left(y\right)\widehat{u}\frac{\partial C}{\partial q}\right) = \frac{\partial}{\partial q}\left(d^2\left(y\right)\varepsilon_t \widehat{u}\frac{\partial C}{\partial q}\right)$$

$$\therefore \frac{\partial C}{\partial x} = \varepsilon_q \frac{\partial^2 C}{\partial q^2}$$

where $\varepsilon_q = d^2 \varepsilon_t \widehat{u} \cong$ constant

→ Gaussian solution (curve) in the *x*-*q* coordinate system

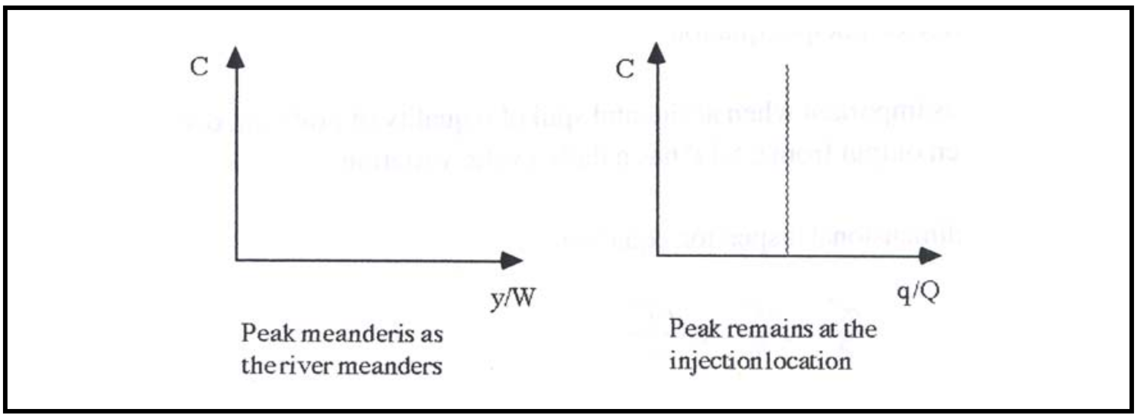
◆ Advantage of *x*-*q* coordinate system

- A fixed value of is attached to a fixed streamline, so that the coordinate system shifts back and forth within the cross section along with the flow.

→ simplifies interpretation of tracer measurements in meandering stream

→ see Fig.5.10

→ Transformation from transverse distance to cumulative discharge as the independent variable essentially transforms meandering river into an equivalent straight river.



5.2 Longitudinal Dispersion in Rivers

· After a tracer has mixed across the cross section, final stage in the mixing process is the reduction of longitudinal gradients by longitudinal dispersion.

· longitudinal dispersion

1) may be neglected when effluent is discharged at a constant rate

→ Streeter-Phelps equation

2) is important when accidental spill of a quantity of pollutant occurs and when output from a STP has a daily cyclic variation

· 1D dispersion equation

$$\frac{\partial \overline{C}}{\partial t} + \overline{u}\frac{\partial \overline{C}}{\partial x} = K\frac{\partial^2 \overline{C}}{\partial x^2}$$

→ apply shear flow dispersion theory to evaluate the longitudinal dispersion coefficient K

5.2.1 Theoretical Derivation of Longitudinal Dispersion Coefficient

◆ Elder's analysis of dispersion

- due to vertical variation of u-velocity (logarithmic profile)

$$u(z) = \overline{u} + \frac{u^*}{\kappa}\left\{1 + \ln\left[z + d / d\right]\right\}$$

$$K = 5.93 d u^*$$

◆ experimental results shows $K >> 5.93du^*$ → Table 5.3

1) Godfrey and Frederick (1970) – natural streams

$$\frac{K}{du^*} = 140 \sim 500$$

2) Fischer (1967) - Laboratory channel

$$\frac{K}{du^*} = 150 \sim 392$$

- Fischer (1968) - Green-Duwamish River

$$\frac{K}{du^*} = 120 \sim 160$$

3) Yotsukura et al. (1970) - Missouri river

$$\frac{K}{du^*} = 7500$$

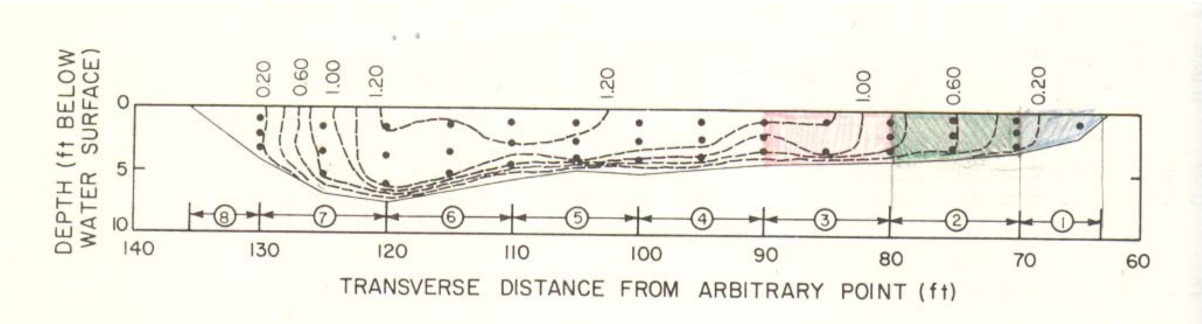
(4) Seo (1990) - Pool-Riffle lab. model

$$\frac{K}{du^*} = 5.7 \sim 11.5 \text{ for main flow zone only}$$

→ Elder's result does not apply to real stream dispersion (1D model)

◆ Fischer's model (1966, 1967)

He show that the reason that Elder's result does not apply to 1D model is because of <u>transverse variation</u> of across the stream → see Fig.5.11



vertical velocity profile, $u(z)$ → logarithmic

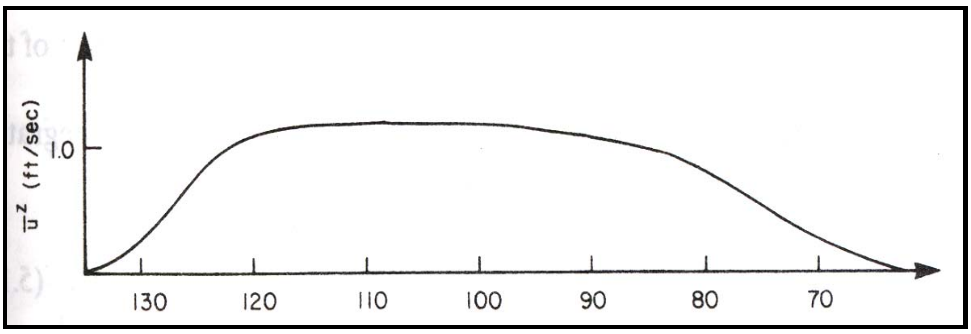
transverse velocity profile → parabolic, polynomial

- depth-averaged velocity at $y = y$

$$\hat{u}(y) = \frac{1}{d(y)} \int_{-d(y)}^{0} u(y,z)\,dz$$

- plot of $\hat{u}(y) vs\, y$ → Fig.5.12

~ shear flow velocity profile extending over the stream width W



Remember that longitudinal dispersion coefficient is proportional to the square of the distance over which the shear flow profile extends.

Eq. (4.26)

$$K = \frac{h^2 \overline{u'^2}}{E} I$$

$$K \propto h^2$$

where h = characteristic length

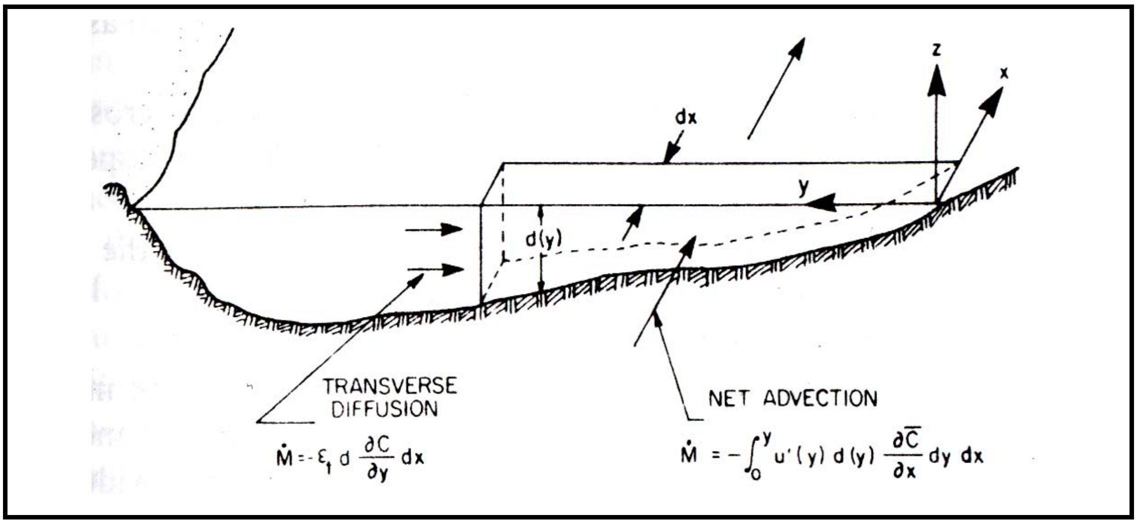
say that $W / d \approx 10$

$$\therefore K_W \approx 100 K_d$$

→ transverse profile *u(y)* is 100 or more times as important in producing longitudinal dispersion as the vertical profile.

→ The dispersion coefficient in a real stream (1D model) should be obtained by neglecting the vertical profile entirely and applying Taylor's analysis to the transverse velocity profile.

◆ Balance of diffusion and advection



Let $u^{'}(y) = \hat{u}(y) - \bar{u}$

$C^{'}(y) = \hat{C}(y) - \bar{C}$

$\bar{u}$ = cross-sectional average velocity

Equivalent of Eq. (4.35) is

$$u^{'}\left(y\right)\frac{\partial \bar{C}}{\partial x} = \frac{\partial}{\partial y}\varepsilon_t \frac{\partial C^{'}}{\partial y} \qquad \text{(a)}$$

Integrate Eq. (a) over the depth

$$\int_{-d}^{0} u^{'}\left(y\right)\frac{\partial \bar{C}}{\partial x}dz = \int_{-d}^{0}\frac{\partial}{\partial y}\varepsilon_t \frac{\partial C^{'}}{\partial y}dz \qquad \text{(b)}$$

$$\therefore u^{'}\left(y\right)d\left(y\right)\frac{\partial \bar{C}}{\partial x} = \frac{\partial}{\partial y}d\left(y\right)\varepsilon_t \frac{\partial C^{'}}{\partial y} \qquad \text{(c)}$$

Integrate Eq. (c) w.r.t. y

$$\int_{0}^{y} u^{'}\left(y\right)d\left(y\right)\frac{\partial \bar{C}}{\partial x}dy = d\varepsilon_t \frac{\partial C^{'}}{\partial y} \qquad (5.15)$$

$$\frac{\partial C^{'}}{\partial y} = \frac{1}{d\varepsilon_t}\int_{0}^{y} u^{'}\left(y\right)d\left(y\right)\frac{\partial \bar{C}}{\partial x}dy \qquad \text{(d)}$$

Integrate Eq. (d) w.r.t. y

$$C^{'} = \int_{0}^{y}\frac{1}{d\varepsilon_t}\int_{0}^{y} u^{'}\left(y\right)d\left(y\right)\frac{\partial \bar{C}}{\partial x}dydy \qquad \text{(e)}$$

Eq. (4.27)

$$K = -\frac{1}{A\frac{\partial \bar{C}}{\partial x}}\int_{A} u^{'}C^{'}dA \qquad \text{(f)}$$

Substitute Eq. (e) into Eq. (f)

$$K = -\frac{1}{A}\frac{1}{\frac{\partial \overline{C}}{\partial x}}\frac{\partial \overline{C}}{\partial x}\int_A u'\int \frac{1}{d\varepsilon_t}\int du' dydydA$$

Substitute $dA = dy\,d$

$$K = -\frac{1}{A}\int_0^W u'd\int_0^y \frac{1}{\varepsilon_t d}\int_0^y du' dydydy \tag{5.16}$$

◆ Simplified equation

Let $d' = d / \overline{d}$; $u'' = \frac{u'}{\sqrt{\overline{u'^2}}}$; $\varepsilon'_t = \frac{\varepsilon'_t}{\overline{\overline{\varepsilon_t}}}$; $y' = \frac{y}{W}$

overbars mean cross-sectional average;

$\overline{d}$ = cross-sectional average depth

Then

$$K = \frac{W^2\overline{u'^2}}{\overline{\varepsilon_t}} I \tag{5.17}$$

where

$$I = -\int_0^1 u''d'\int_0^{y'} \frac{1}{\varepsilon'_t d'}\int_0^{y'} u'' dy' dy' dy'$$

Compare with Eq. (4.26)

$$K = \frac{h^2\overline{u'^2}}{E} I$$

[Example 5.4]

Given: cross-sectional distribution of velocity (Fig.5.11) of Green-Duwamish at Renton Junction

$$\varepsilon_t = 0.133\, ft^2 / \sec$$

Find: longitudinal dispersion coefficient

Solution: divide whole cross section into 8 subarea

$$K = -\frac{1}{A}\int_0^W u' d \int_0^y \frac{1}{\varepsilon_t d}\int_0^y d u' dy dy dy$$

→ perform inner integral first

Column 2: transverse distance to the end of subarea

Column 4: $\Delta A = d\, \Delta y$

Column 46: $\Delta Q = \hat{u}\, \Delta A$

Column 8: $Relative\, \Delta Q = u'\, \Delta A$

Column 9: Cumulative of $Relative\, \Delta Q = u'\, \Delta A$

Column 11: $\int_0^y \frac{1}{\varepsilon_t d}\int_0^y d u' dy dy = \sum Col(10) \frac{\Delta y}{\varepsilon_t d}$

Column 13: $\int_0^W u' d \int_0^y \frac{1}{\varepsilon_t d}\int_0^y d u' dy dy dy = Col(8) \times Col(12)$

$$K = -\frac{1}{A} Cumulative\, of\ Col(13)$$

Homework Assignment #5-1

Due: Two weeks from today

1. Estimate the longitudinal dispersion coefficient by using the cross-sectional distribution of velocity measured in the field using Eq. (5.16). Take *S* (channel slope) = 0.00025 for natural streams.
2. Compare this result with Elder's analysis and Fischer's approximate formula, Eq. (5.19).

Table 1 Cross-sectional Velocity Distribution at Ottawa in the Fox River, Illinois

Station	*Y* from left bank (ft)	Depth, *d* (ft)	Mean Velocity (ft/sec)
1	0.00	0.0	0.00
2	4.17	1.4	0.45
3	7.83	3.0	0.68
4	11.50	3.7	1.05
5	15.70	4.7	0.98
6	22.50	5.3	1.50
7	29.83	6.2	1.65
8	40.83	6.7	2.10
9	55.50	7.0	1.80
10	70.17	6.5	2.40
11	84.83	6.3	2.55
12	99.50	6.8	2.45
13	114.17	7.4	2.20
14	132.50	7.3	2.65
15	150.83	7.1	2.70

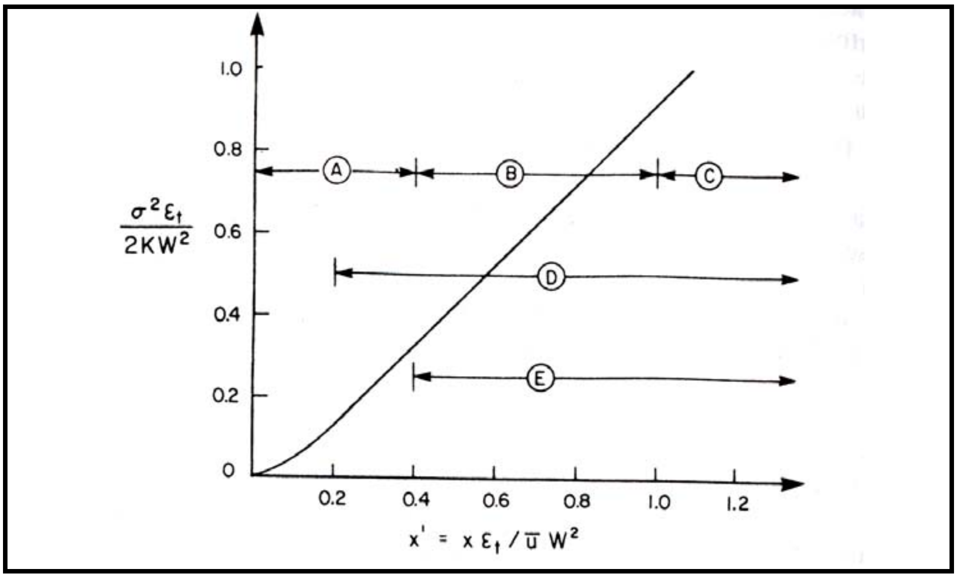
16	169.16	7.4	2.35
17	187.49	7.8	2.65
18	205.82	7.8	2.80
19	224.15	7.8	2.60
20	242.48	6.6	2.50
21	260.81	6.3	2.30
22	279.14	6.2	2.35
23	297.47	6.6	2.30
24	315.80	6.0	2.65
25	334.13	5.5	2.50
26	352.46	5.4	2.10
27	370.79	5.2	2.25
28	389.12	5.5	2.30
29	407.45	5.7	1.50
30	416.62	3.2	1.30
31	422.00	0.0	0.00

5.2.2 Dispersion in Real Streams

· Real streams have bends, sandbars, side pockets, pools and riffles, bridge piers, man-made revetments.

→ These irregularities contribute to dispersion.

5.2.2. 1 Limitation of Taylor's analysis



A) generation of skewed distribution: $x' (= \dfrac{x}{\bar{u}W^2/\varepsilon_t}) < 0.4$ (initial period)

B) decay of the skewed distribution: $0.4 < x' < 1.0$

C) approach to Gaussian distribution: $1.0 < x'$

D) zone of linear growth of the variance: $0.2 < x'$; $\dfrac{\partial \sigma^2}{\partial t} = 2D$

E) zone where use of the routing procedure is acceptable: $0.4 < x'$

5.2.2.2 Two-zone Models

· Irregularities in real streams

→ increase the length of the initial period

→ produce long tail on the observed concentration distribution due to detention of small amounts of effluent cloud and release slowly after the main cloud has passed

· Field studies

Godfray and Frederick (1974)
Nordin and Savol (1974)
Day (1975)
Legrand-Marcq and Laudelot (1985)

→ nonlinear behavior of variance for times beyond the initial period (increased faster than linearly with time)

$$\sigma^2 = f\left(t^{1.4}\right)$$

→ skewed concentration distribution
→ cannot apply Taylor's analysis

· Effect of storage zones (dead zones)

1) increase the length of the initial period

2) increase the magnitude of the longitudinal dispersion coefficient

• Two zone models

→ divide stream area into two zones

Flow zone: advection, dispersion, reaction, mass exchange

$$A_F \frac{\partial C_F}{\partial t} + U_F A_F \frac{\partial C_F}{\partial x} = \frac{\partial}{\partial x}\left(K A_F \frac{\partial C_F}{\partial y} \right) + F$$

Storage zone: vortex, dispersion, reaction, mass exchange

$$A_S \frac{\partial C_S}{\partial t} = -F$$

→ introduce auxiliary equation for mass exchange term F

Exchange model: $F = k(C_F - C_S)P$

Diffusion model: $F = -\varepsilon_y \left.\frac{\partial C_S}{\partial y}\right|_{y=0}$

i) Dead zone model

Hays et al (1967)

Valentine and Wood (1977, 1979), Valentine (1978)

Tsai and Holley (1979)

Bencala and Waters (1983), Jackman et al (1984)

ii) Storage zone model

Seo (1990), Seo and Maxwell (1991, 1992)

Seo and Yu (1993)

Seo & Cheong (2001), Cheong & Seo (2003)

◆ Effect of bends

1) increase the rate of transverse mixing

→ reduce the dispersion coefficient to some extent

2) transverse velocity profile induced by meandering flow increase longitudinal dispersion coefficient significantly

(3) effect of alternating series of bends

- depends on the ratio of the cross-sectional diffusion time to the time required for flow round the bend

$$\gamma = \frac{W^2 / \varepsilon_t}{L / \overline{u}}$$

where L= length of the curve

$\gamma \le 25 = \gamma_0$ → no effect due to alternating direction

$\gamma > 25 \rightarrow K = K_0 \dfrac{\gamma_0}{\gamma}$

5.2.3 Estimating and Using the Dispersion Coefficient

• Observation – calculation of observed values from field data

• Prediction – estimation of dispersion coefficient by theoretical or empirical equations

5.2.3.1 Observation of dispersion coefficient

1) Change of moment method

$$K = \frac{\sigma_{x2}{}^2 - \sigma_{x1}{}^2}{2(t_2 - t_1)} = \frac{U^2}{2} \frac{\sigma_{t2}{}^2 - \sigma_{t1}{}^2}{2(\overline{t_2} - \overline{t_1})}$$

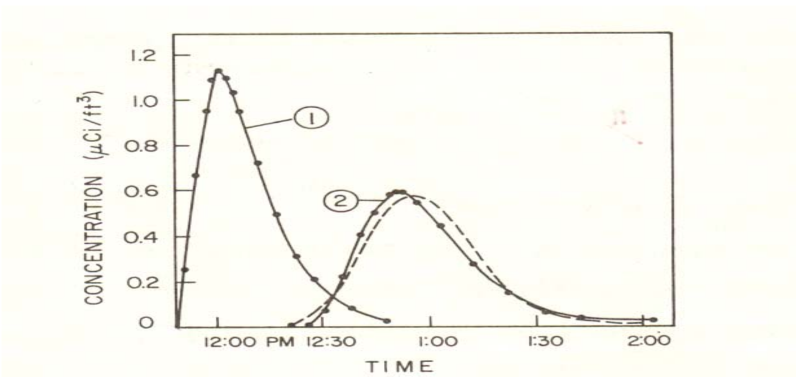
$\sigma_x{}^2$ = variance of C-x curve; $\sigma_t{}^2$ = variance of C-t curve;

$\overline{t_1}$ = centroid of C-t curve at $x = x_1$

- difficult to compute a meaningful value of variance when concentration distributions are skewed.

2) Routing procedure

- proposed by Fischer (1968)



→ match a downstream observation of passage of a tracer cloud to the prediction based on the upstream observation using an analytical solution
→ can use this procedure only when x' >0.4

• Predicted concentration distribution at downstream station is obtained according to the solution of one-dimensional Fickian dispersion model.

$$C^p(x_2,t) = \int_{-\infty}^{\infty} C(x_1,\tau) \frac{exp\left[-\dfrac{\bar{u}^2(\bar{t}_2 - \bar{t}_1 - t + \tau)^2}{4K(\bar{t}_2 - \bar{t}_1)}\right]}{\sqrt{4\pi K(\bar{t}_2 - \bar{t}_1)}} \bar{u} d\tau \qquad (5.20)$$

where
$\bar{t}_1$ = mean time of passage at the upstream station (x_1)
$\bar{t}_2$ = mean time of passage at downstream station (x_2)
τ = timelike variable of integration
$C(x_1,\tau)$ = upstream observed concentration-time curve

→ Compare $C^p(x_2,t)$ with $C(x_2,t)$ [= downstream observed concentration curve] until it fits together with varying dispersion coefficient K
→ best fit value is regarded as the observed dispersion coefficient

5.2.3.2 Prediction of dispersion coefficient

1) Theoretical equation

$$K = -\frac{1}{A}\int_0^W u' d \int_0^y \frac{1}{\varepsilon_t d}\int_0^y d u' dy dy dy \tag{5.16}$$

a) Seo and Baek (2004)

– use beta function for transverse profile of *u*-velocity

$$\frac{u}{U} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\left(\frac{y}{W}\right)^{\alpha-1}(1-\frac{y}{W})^{\beta-1}$$

$$K = \gamma \frac{U^2 W^2}{du^*}$$

2) Empirical equation

a) Fischer (1975)

$$K' = \frac{\overline{I u'^2 h^2}}{E} \tag{4.26}$$

Select $I = 0.07(0.054 \sim 0.10)$

$h = 0.7W\left(0.5 \sim 1.0W\right)$

$\overline{u'^2} = 0.2\bar{u}^2(0.17 \sim 0.25)$

$E = \varepsilon_t = 0.6du^*$

Then (4.26) becomes

$$K = 0.01 \frac{U^2 W^2}{du^*} \tag{5.19}$$

b) Seo and Cheong (1998)

- Dimensional analysis
- include dispersion by shear flow dispersion and storage effects

$$\frac{K}{du^*} = a \left(\frac{U}{u^*} \right)^b \left(\frac{W}{d} \right)^c$$

Fischer (1975):	a=0.011; b=2.0; c=2.0
Liu (1979):	a=0.18; b=0.5; c=2.0
Iwasa and Aya (1991):	a=2.0; b=0; c=1.5
Koussis and Rodrguez-Mirasol (1998):	a=0.6; b=0; c=2.0
Seo and Cheong (1998):	a=5.92; b=1.43; c=0.62

[Ex 5.5] Dispersion of slug

Given :

$M = 10 lb$ (Rhodamine WT dye); $\overline{u} = 0.90\, ft/s$; $W = 73\, ft$; $A = 338.6$

$\overline{d} = 4.46\, ft$, (weighted average), $u^* = 0.072\, ft/s$

$\varepsilon_t = 0.133\, ft^2/s$; $\dfrac{\varepsilon_t}{0.4d} = \dfrac{0.133}{0.4(4.64)} = 0.072$

Find :

(a) K by Eq. (5.19)

(b) length of initial zone in which Taylor's analysis does not apply

(c) length of dye cloud at the time that peak passes =20,000ft

(d) C_{peak} at x =20,000ft

[Solution]

(a) Fischer (1975) - Eq. (5.19)

$K = 0.011 \overline{u}^2 W^2 / du^*$

$= 0.011(0.90)^2 (73)^2 / (4.46)(0.072)$

$= 142.1\, ft^2/s$

$K(5.19)/K(5.16) = 142.1/77.5 = 1.83$

[Cf] K by Seo & Cheong (1998)

$$\frac{K}{du^*} = 5.92 \left(\frac{U}{u^*} \right)^{1.43} \left(\frac{W}{d} \right)^{0.62}$$

$K = 294\, ft^2/s$

→ include dispersion by shear flow dispersion and storage effects

(b) initial period

$$x = 0.4\bar{u}W^2 / \varepsilon_t = 0.4(0.90)(73)^2 / (0.133) = 14,424\,ft$$

(c)length of cloud

$$x' = x\varepsilon_t / \bar{u}W^2 = \frac{(20,000)(0.133)}{(0.90)(73)^2} = 0.55$$

- decay of skewed concentration distribution

→ assume Gaussian distribution

$$\frac{d\sigma^2}{dt} = 2K$$

From Fig.5.14

$$\frac{\sigma^2\varepsilon_t}{2KW^2} = (x' - 0.07)$$

$$\therefore \sigma^2 = 2K(W^2 / \varepsilon_t)(x' - 0.07)$$

$$= 2(142)(73)^2 / 0.133(0.55 - 0.07) = 5.46 \times 10^{-6}\ ft^2$$

$$\therefore \sigma = 2.337$$

length of cloud $= 4\sigma = 4(2,337) = 9,348\,ft$

(d) peak concentration

$$C_{\max} = \frac{M}{A\sqrt{4\pi Kx/\bar{u}}} = \frac{10}{(338.6)\sqrt{4\pi(142)(20,000)/(0.90)}} = 4.69 \times 10^{-6} lb / ft^3$$

$$= 4.69 \times 10^{-6} \times \frac{453.6g}{0.0283m^3} = 75.1 \times 10^{-3}\, g / m^3 (= mg / l = ppm)$$

$$= 75.1 ppb$$

Homework Assignment #5-2

Due: Two weeks from today

Concentration-time data listed in Table 2 are obtained from dispersion study by Godfrey and Fredrick (1970).

1) Plot concentration vs time

2) Calculate time to centroid, variance, skew coefficient.

3) Calculate dispersion coefficient using the change of moment method and routing procedure.

4) Compare and discuss the results.

Test reach of the stream is straight and necessary data for the calculation of dispersion coefficient are

$$\bar{u} = 1.70\, ft/s; \qquad W = 60\, ft;$$

$$d = 2.77\, ft; \qquad u^* = 0.33\, ft/s$$

Table-2 Time-concentration data for Copper Creek, Virginia

Section 1 x=630ft		Section 2 x =3310ft		Section 3 x =5670ft		Section 4 x =7870ft		Section 5 x =11000ft		Section 6 x =13550ft	
T (hr)	C/C_0	T (hr)	C/C_0	T (hr)	C/C_0	T (hr)	C/C_0	T (hr)	C/C_0	T (hr)	C/C_0
1111.5	0.00	1125.0	0.00	1138.0	0.00	1149.0	0.00	1210.0	0.00	1226.0	0.00
1112.5	2.00	1126.0	0.15	1139.0	0.12	1152.0	0.26	1215.0	0.05	1231.0	0.07
1112.5	16.50	1127.0	1.13	1140.0	0.30	1155.0	0.67	1220.0	0.25	1236.0	0.22
1113.0	13.45	1128.0	2.30	1143.0	1.21	1158.0	0.95	1225.0	0.52	1241.0	0.40

1113.5	7.26	1128.5	2.74	1145.0	1.61	1200.0	1.09	1228.0	0.64	1245.0	0.50
1114.0	5.29	1129.0	2.91	1147.0	1.64	1202.0	1.13	1231.0	0.70	1249.0	0.58
1115.0	3.37	1129.5	2.91	1149.0	1.56	1204.0	1.10	1234.0	0.72	1251.0	0.59
1116.0	2.29	1130.0	2.80	1153.0	1.26	1206.0	1.04	1237.0	0.71	1253.0	0.59
1117.0	1.54	1131.0	2.59	1158.0	0.86	1208.0	0.95	1240.0	0.65	1257.0	0.54
1118.0	1.03	1133.0	2.18	1203.0	0.53	1213.0	0.72	1244.0	0.55	1304.0	0.44
1120.0	0.40	1137.0	1.34	1208.0	0.30	1218.0	0.50	1248.0	0.45	1313.0	0.27
1124.0	0.10	1143.0	0.60	1213.0	0.17	1223.0	0.31	1258.0	0.24	1323.0	0.14
1128.0	0.04	1149.0	0.23	1218.0	0.10	1228.0	0.21	1308.0	0.12	1333.0	0.06
1133.0	0.02	1158.0	0.08	1228.0	0.04	1238.0	0.08	1318.0	0.06	1343.0	0.03
1138.0	0.00	1208.0	0.03	1238.0	0.01	1248.0	0.02	1333.0	0.03	1403.0	0.02
-	-	1218.0	0.00	1248.0	0.00	1300.0	0.00	1353.0	0.00	1423.0	0.00