1 Linear Equations and Matrices

1.1 Vectors and Matrices

- Matrix.
$\Longrightarrow$ An m by n matrix is a rectangular array of mn real(or complex) numbers arranged in m horizontal rows and n vertical columns.

- Vectors.
$\Longrightarrow$ A 1 by n or an n by 1 matrix is also called an n-vector.

- dot product(inner product) of two vectors

$$\vec{v} = \begin{bmatrix} v_1 \\ \hline v_2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ \hline w_2 \end{bmatrix}$$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 = \|v\| \|w\| cos\gamma$$

- length (norm) of $\vec{v}$

$$\mathbf{v} = \|v\| = \sqrt{\vec{v} \cdot \vec{v}}$$

- unit vector $\vec{u}$

$$\vec{u} \quad : \quad \vec{u} \cdot \vec{u} = 1$$

$\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector

- Schwarz inequality $\|\vec{v} \cdot \vec{w}\| \leq \|\vec{v}\| \|\vec{w}\|$

- Triangle inequality $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

- i_{th} component of $Ax = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \ = \sum_{j=1}^{n} a_{ij}x_j$

- $(AB)_{ij}$ = (row i of A) $\cdot$ (column j of B)

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

often $AB \neq BA$
$[A(l \times m)]\ [B(m \times n)] = [M(l \times n)]$

- $A(BC) = (AB)C$
- $C(A+B) = CA + CB$
- $(A+B)C = AC + BC$
- Identity matrix $Ib = b$
- Elimination matrix E_{ij}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 - lb_1 \end{bmatrix}$$

$-l$ in the (i,j) position

$\Longrightarrow$ row $i -\ l \times$ row j.

- Row exchange matrix P_{ij}

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix}$$

identity marix with rows i, j reversed

$\Longrightarrow$ exchanges row i and j when multiplied.

- Augmented matrix
  $[A|b]$: elimination acts on whole rows of this matrix.

Q. How can we use the above ideas to solve linear equations ?

Ex.

$$\begin{aligned} x + 2y + 2z &= 1 \\ 4x + 8y + 9z &= 3 \\ 3y + 2z &= 1 \end{aligned}$$

$\Longrightarrow$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{array}\right]$$

$\Longrightarrow$ multiply E_{21}

$$\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$\Longrightarrow$ multiply P_{32}

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

upper triangular system

$\Longrightarrow$ *back substitution*

$$z = -1,\ y = 1,\ x = 1$$

1.2 Linear Equations: <u>Elimination</u>

(Strang, p35)

$$\begin{pmatrix} x - 2y = 1 & \text{(1a)} \\ 3x + 2y = 11 & \text{(1b)} \end{pmatrix}$$

$$\Downarrow \quad \text{(1b)-3*(1a)}$$

$$\begin{pmatrix} x - 2y = 1 & \text{(1a)} \\ 8y = 8 & \text{(1b)} \end{pmatrix} \qquad \begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 8 \end{bmatrix}$$

"upper triangular system"

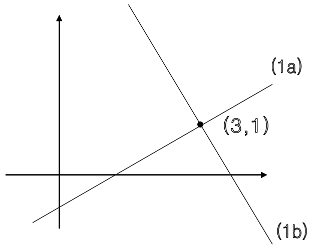
$$\begin{pmatrix} \text{Last equation : } y = 1 \\ \text{substitute into the upper equation : } x - 2 = 1 \end{pmatrix}$$

$\Longrightarrow$ "back substitution"

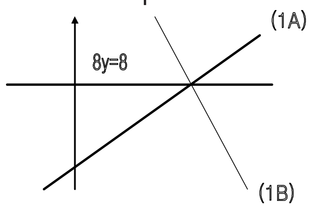
- **pivot** : first nonzero in the row that does the elimination
("1": the coefficient of x is the first pivot in the example above)
- **multiplier** $= \frac{entry \quad to \quad eliminate}{pivot}$
("$\frac{3}{1}$" in the example above)
- The pivots are on the diagonal of the triangle after the elimination.

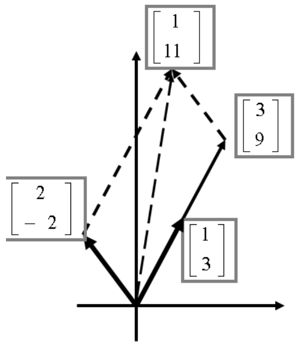
$$\begin{pmatrix} x - 2y = 1 & \quad (1a) \\ 3x + 2y = 11 & \quad (1b) \end{pmatrix}$$

$$x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$



row picture





column picture

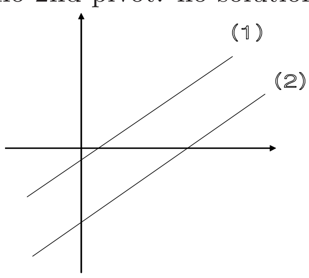
Ex 1.

$$\left(\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 11 \end{aligned} \right.$$

$$\Downarrow$$

$$\left(\begin{aligned} x - 2y \quad &= 1 \\ 0y \quad &= 8 \end{aligned} \right.$$

no 2nd pivot. no solution



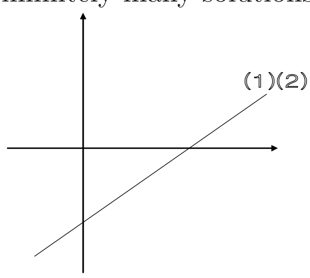
Ex 2.

$$\left(\begin{aligned} x - 2y &= 1 \\ 3x - 6y &= 3 \end{aligned} \right.$$

$$\Downarrow$$

$$\left(\begin{aligned} x - 2y \quad &= 1 \\ 0y \quad &= 0 \end{aligned} \right.$$

infinitely many solutions



Ex 3.

$$\left(\begin{aligned} 0x + 2y &= 4 \\ 3x - 2y &= 5 \end{aligned} \right.$$

$\Downarrow$ row exchange

$$\left(\begin{aligned} 3x - 2y \quad &= 5 \\ 2y \quad &= 4 \end{aligned} \right.$$