3 Vector Space, Linear Independence

3.1 Vector Space

Definition. **Vector Space** : a nonempty set V of vectors such that with any two vectors $a,\ b\ \in V$,
all their linear combinations $c_1 a + c_2 b \in V$ (c_1, c_2 : any real numbers),
and all these vectors obey the following rules.

- **(a)** $a + b = b + a$.
- **(b)** $(a + b) + c = a + (b + c) = a + b + c$.
- **(c)** There is a unique zero vector such that $a + 0 = a$ for all a.
- **(d)** For each a, there is a unique vector-a such that $a + (-a) = 0$.
- **(e)** $1a = a$.
- **(f)** $c_1(c_2 a) = (c_1 c_2)a = c_1 c_2 a$.
- **(g)** $c_1(a + b) = c_1 a + c_1 b$.
- **(h)** $(c_1 + c_2)a = c_1 a + c_2 a$.

Example 1. $\mathbb{R}^n$, The vector space that consists only of a zero vector, the vector space of all real n by n matrices, ...

Definition. **Subspace** of a vector space is a set of vectors (including 0) that satisfies the following:
If v and w are vectors in the subspace and c is any scalar, then $v + w$, cv is in the subspace.

Example 2. three-dimensional space $\mathbb{R}^3$.

- A plane through (0,0,0) is a subspace of the full vector space $\mathbb{R}^3$.
- A line through (0,0,0)
- The single vector (0,0,0)
- The whole space $\mathbb{R}^3$

Definition. **Column space** of a matrix A : $C(A)$

$$= \text{all linear combinations of the columns}$$

$$= \text{span of the columns}$$

Note

- The combinations are all possible vectors Ax.
- To solve $Ax = b$, b needs to be a combination of the columns.

$$\therefore Ax = b \text{ is solvable iff } b \in C(A)$$

- If A is m-by-n, columns belong to $\mathbb{R}^m$.

$$\therefore C(A) \text{ is a subspace of } \mathbb{R}^m.$$

Example 3.

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \qquad Ax = x_1 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$C(A) =$ plane in $\mathbb{R}^3$.

- $Ax = b$ is solvable when b is on that plane.
- $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in $C(A)$ $\quad \therefore Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ solvable.

Example 4. In $\mathbb{R}^2$,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

- $C(I) = \mathbb{R}^2 \qquad (Ix = b$ always solvable $)$
- $C(A) =$ all linear combinations of $c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

  $= c_0 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad (c_0 \in \mathbb{R})$

  $\therefore C(A)$ is only a line.

  ($Ax = b$ solvable only when b is on the line.)
- $C(B) = \mathbb{R}^2 \qquad$ (every b is attainable)

  But $Bx = b = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ has multiple solutions.

  (2 eqns, 3 unknowns)

Definition. **Row space** of a matrix A : $R(A)$

= all linear combinations of the rows

= span of the rows

- If A is m-by-n, rows belong to $\mathbb{R}^n$.

  $\therefore$ The row space is a subspace of $\mathbb{R}^n$.

Note

- The row space of $A = C(A^T)$

Definition. Null space of a matrix $A : N(A)$

$= $ all solutions to $Ax = 0$

- If A is m-by-n, the solution vectors x are in $\mathbb{R}^n$.

  $\therefore N(A)$ is a subspace of $\mathbb{R}^n$.

- Consider $Ax = b$.

  If $b \neq 0$, then the solutions do not form a subspace.

  ($\because x = 0$ is only a solution if $b = 0$.)

Example 5.

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

- $N(A)$ is the plane through the origin

$$x + 2y + 3z = 0$$

- $s_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $s_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ lie on the plane

$$x + 2y + 3z = 0$$

All vectors on the plane are combinations of s_1 and s_2.

Example 6. $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$

$\Longrightarrow U = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$: all columns have pivots. A is invertible.

$N(A) = \{(0, 0)\}$

Example 7. $B = \begin{bmatrix} A \\ 2A \end{bmatrix}$

Extra rows impose more conditions on the vectors x in the null space.

$N(B) = \{(0, 0)\}$

Example 8. $C = [A \quad 2A] = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$

Now the solution vector x has 4 components.

$$\Longrightarrow U = \left[\begin{matrix} \underbrace{\begin{matrix} 1 & 2 \\ 0 & 2 \end{matrix}}_{\text{pivot cols}} & \underbrace{\begin{matrix} 2 & 4 \\ 0 & 4 \end{matrix}}_{\text{free cols}} \end{matrix}\right]$$

$$\Longrightarrow R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \text{ (}\textbf{reduced row echelon form}\text{)}$$

Special solutions to $Rx = 0$:

$$s_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad s_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

(The free variables $x_3,\ x_4$ can be given any values whatsoever. Then the pivot variables $x_1,\ x_2$ can be found by back substitution.)

All solutions are linear combination of s_1 and s_2.

Note

- If A has more columns than rows, then $Ax = 0$ has more unknowns than equations, and it has nonzero solutions. (There must be free columns without pivots.)

$$m\underbrace{\begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \end{bmatrix}}_{n}\begin{bmatrix} \quad \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

- $N(A) = N(U) = N(R)$

3.2 Linear Independence

Definition. Vectors $\vec{v}_1, \cdots, \vec{v}_n$ are **linearly independent**
if $x_1\vec{v}_1 + \cdots + x_n\vec{v}_n = 0$ only happens when all x's are zero.
(If a combination is 0 when the x's are not all zero, the vectors are dependent.)

- The columns of A are linear independent when the only solution to $A\vec{x} = 0$ is $\vec{x} = 0$.
- The columns of $A \in \mathbb{R}^{m\times n}$ are linear independent when the *rank* is $r = n$.
  - n pivots and no free variables.
  - only $\vec{x} = 0$ is in the nullspace.
- Any set of n vectors in $\mathbb{R}^n$ must be linearly dependent if n > m.

Definition. A basis for a vector space is a set of linearly independent vectors that span the space.

- $v_1, \cdots, v_n$ are a basis for $\mathbb{R}^n$ exactly when they are the columns of an $n \times n$ invertible matrix.

Q. Given m vectors in $\mathbb{R}^n$, how do you find a basis for the space they span ?

$$m \begin{bmatrix} - & v_1 & - \\ - & v_2 & - \\ - & \cdots & - \\ - & v_n & - \end{bmatrix}$$

$$\Downarrow$$

eliminate to find the nonzero rows.
or, put them in columns $\Longrightarrow$ find pivot columns.