4 Rank & Solutions of Linear Systems

4.1 Rank

Recall G-J elimination. Elimination matrices are multiplied to put A into its reduced row echelon form.

$$i.e. \qquad \mathrm{E}\,[\mathrm{A} \quad \mathrm{I}\,] \;=\; [\mathrm{R} \quad \mathrm{E}]$$

If A is invertible,

$$\mathrm{E}\,[\mathrm{A} \quad \mathrm{I}\,] \;=\; [\,\underset{\mathrm{R}}{\mathrm{I}} \quad \underset{\mathrm{E}}{\mathrm{A}^{-1}}\,]$$

Example .

$$\mathrm{A} = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \longrightarrow \mathrm{R} = \begin{bmatrix} \underline{1} & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

only one pivot

Ax = 0 is <u>just one eqn.</u>, not three.

1 independent row
1 independent column

The "true" size of A is given by its rank.

Definition. rank(A) = the number of pivots.

Example . above example : rank(A) = 1

$$\text{Let} \quad \mathrm{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \text{then } \mathrm{A} = \begin{bmatrix} | & | & | \\ \mathrm{u} & 3\mathrm{u} & 10\mathrm{u} \\ | & | & | \end{bmatrix} = \mathrm{u}\begin{bmatrix} 1 & 3 & 10 \end{bmatrix}$$

<u>C(A) is 1-dim.</u>

Remark If A is $m \times n$ and rank(A)$= r$, then $r \leq m$, and $r \leq n$.

· A has full row rank if every row has a pivot.

($r = m$, No zero rows in R) $\begin{bmatrix} & & \end{bmatrix}$

· A has full column rank if every column has a pivot.

($r = n$, No free variables) $\begin{bmatrix} \\ \\ \end{bmatrix}$

$$
\begin{array}{lcl}
\underline{A}x = 0 & \longrightarrow & Rx = 0 \\
\llcorner\ m \times n & & \\
& & \left\{ \begin{array}{ll} r & \text{pivot columns.} \\ n-r & \text{free variables.} \end{array} \right. \\
& & i.e \\
& & \left\{ \begin{array}{ll} r & \text{indept eqns.} \\ n-r & \text{special solns (indept).} \end{array} \right.
\end{array}
$$

Example .

$$
\mathrm{A} = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 1 & 3 & 1 & 6 & -4 \end{bmatrix} \longrightarrow \mathrm{R} = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

rank(A) $= 2$
Ax $= 0$ (or Rx $= 0$) has two independent eqns.

Solution of

$$
\mathrm{Rx} = 0 = \begin{bmatrix} 1 & 3 & 0 & 2 & -1 \\ 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \quad \Leftarrow \left(\begin{array}{ll} x_1, x_3 & : \text{ pivot variables} \\ x_2, x_4, x_5 & : \text{ free variables} \end{array} \right.
$$

$$
\left\{ \begin{array}{ll}
(1) & \text{set } x_2 = 1,\ x_4 = x_5 = 0 \quad \text{then, } \ x_1 = -3 \\
& \quad \Rightarrow \ \mathrm{s}_1 = (-3,\ 1,\ 0,\ 0,\ 0) \\
(2) & \text{set } x_4 = 1,\ x_2 = x_5 = 0 \quad \text{then, } \ x_3 = -4,\ x_1 = -2 \\
& \quad \Rightarrow \ \mathrm{s}_2 = (-2,\ 0, -4,\ 1,\ 0) \\
(3) & \text{set } x_5 = 1,\ x_2 = x_4 = 0 \quad \text{then, } \ x_3 = 3,\ x_1 = 1 \\
& \quad \Rightarrow \ \mathrm{s}_3 = (\ 1,\ 0,\ 3,\ 0,\ 1)
\end{array} \right.
$$

$\rightarrow$ Three independent solns. $\quad (n = 5,\ r = 2)$
$N(\mathrm{A})$ is spanned by $\mathrm{s}_1,\mathrm{s}_2,\mathrm{s}_3$

4.2 Solving $Ax = b \neq 0$ (Strang, page 144)

- Suppose $m \times n$ matrix A has rank r.
  Then the $n - r$ special solns solve $\underline{Ax_h = 0}$.
  And suppose we found a soln for $\underline{Ax_p = b}$.
  Then

$$
A(x_h + x_p) = b
$$

$$
i.e. \qquad x_h + x_p \text{ is a soln for } Ax = b.
$$

- If A is a square, invertible matrix, the only vector in $N(A)$ is $x_h = 0$. And $Ax_p = b$ has only one soln $x_p = A^{-1}b$.

Example . Ax = b

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 1 & 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix}$$

Augmented matrix

$$[\,\mathrm{A} \quad |\, \mathrm{b}\,] = \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 1 & 3 & 1 & 6 & 7 \end{array}\right]$$

Elimination $\downarrow$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = [\,\mathrm{R} \quad |\, \mathrm{d}\,]$$

$\mathrm{Rx}_h = 0$: free variables x_2, x_4

$$\begin{array}{ll} (1) & \text{set } x_2 = 1,\ x_4 = 0 \quad \text{then, } \ x_1 = -3,\ x_3 = 0 \\ & \quad \Rightarrow\ \mathrm{s}_1 = (-3,\ 1,\ 0,\ 0) \\ (2) & \text{set } x_4 = 1,\ x_2 = 0 \quad \text{then, } \ x_1 = -2,\ x_3 = -4 \\ & \quad \Rightarrow\ \mathrm{s}_2 = (-2,\ 0, -4,\ 1) \end{array}$$

$$\begin{aligned} \mathrm{x}_h &= \text{lin. combination of } \mathrm{s}_1 \text{ and } \mathrm{s}_2 \\ &= c_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \end{aligned}$$

$$\mathrm{Rx}_p = \begin{bmatrix} 1 \\ 6 \\ 7 \end{bmatrix} \quad \Rightarrow \quad \text{particular soln } \mathrm{x}_p = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix}$$

complete soln: $\quad \mathrm{x} = \mathrm{x}_h + \mathrm{x}_p.$

Example .

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Augmented matrix

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array}\right]$$

Elimination $\downarrow$

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2 - b_1 \\ 0 & -1 & b_3 + 2b_1 \end{array}\right]$$

Elimination$\downarrow$

$$\left[\begin{array}{cc|c} 1 & 0 & 2b_1 - b_2 \\ 0 & 1 & b_2 - b_1 \\ \hline 0 & 0 & b_1 + b_2 + b_3 \end{array}\right] = [\, R \mid d \,]$$

- For Ax=b to be solvable, we need $b_1 + b_2 + b_3 = 0$. (Otherwise, x_p does not exist)
- The only particular soln $x_p = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix}$.
- Complete soln: $x = x_h + x_p = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Note that every col. has a pivot. $\Rightarrow$ $r = n$ "*full col. rank*"

$$\text{tall \& thin} \quad m \overset{n}{\begin{bmatrix} \\ A \\ \\ \end{bmatrix}} \Longrightarrow \text{Elimination } R = \begin{array}{c|c} \multicolumn{1}{c}{n} & \\ \left[\begin{array}{c} I \\ \hline 0 \end{array}\right] & \begin{array}{c} n \\ \hline m-n \end{array} \end{array}$$

x_h=0 is the only nullspace soln. (no free variables, no special solns)

Every matrix A with *full col. rank* has all these properties: $\boxed{\text{cols are lin indept.}}$

1. All cols are pivot cols.
2. No free variables or special solns.
3. N(A) contains only the zero vector.
4. If Ax=b has a soln (it might not) then it has **only one** sol.

Example .

$$\begin{aligned} x+y+z &= 3 \\ x+2y-z &= 4 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right] = [\, R \mid d \,]$$

Two pivot cols. One free col.

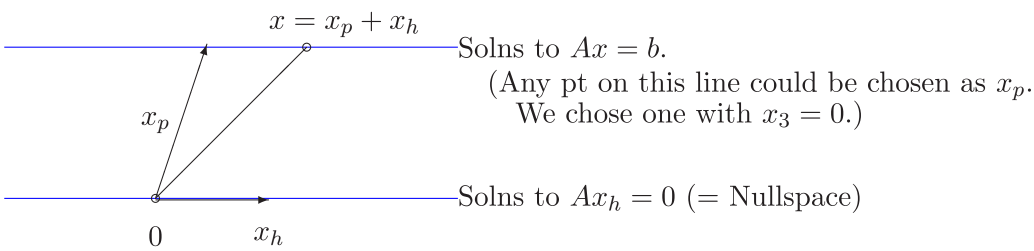
- $Rx_h = 0$: free variable x_3. Set $x_3 = 1$, then $x_1 = -3, x_2 = 2$.

$$\therefore s = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$$

- x_p : free variable $x_3 = 0$. x_p comes directly from d.

$$\therefore x_p = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

- complete soln: $x = x_p + x_h = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$



Every matrix A with *full row rank* ($r = m$) has all these properties: $\boxed{\text{rows are lin indept.}}$

1. All rows have pivots, and R has no zero rows.
2. Ax=b has a soln for **every** right side b.
3. C(A)=$\Re^m$.
4. $n - r = n - m$ special solns in N(A).