# 5 Determinant (for a square matrix $A$)

$= detA = |A|$

## 5.1 Properties of Determinant (Strang, page 234)

**1.** $detI = 1$.

**2.** $det$ changes sign when two rows are exchanged.

$$\text{ex)} \quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix} = bc - ad$$

**3.** $det$ is a linear ftn of each row separately, when all other rows stay fixed.

$$\text{ex)} \quad \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Note Rules 1-3 completely determine the number $detA$.

Other Properties (These follow from above)

**4.** If two rows of $A$ are equal, then $detA = 0$.

$$\text{ex)} \quad \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0 \quad \text{(use rule 2)}$$

**5.** Subtracting a multiple of one row from another row does not change $detA$.

$$\text{ex)} \quad \begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Note The usual elimination steps do not change $detA$.

$$\therefore \quad \underline{detA \quad = \quad detU}$$

(If rows are exchanged, $detA = \pm detU$)

**6.** $A$ matrix with a row of zeros has $detA = 0$.

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0 \qquad \text{and} \qquad \begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$$

( rule 5 & 4 )

**7.** If $A$ is triangular then,

$$detA = product\ of\ diagonal\ entries.$$

( Triangular matrix $\underset{elimination}{\longrightarrow}$ diagonal matrix $\rightarrow$ rule 3 & 1)

**8.** If $A$ is singular then $detA = 0$.
If $A$ is invertible then $detA \neq 0$.

If $A$ is singular then $U$ has a zero row.
$\Rightarrow detA = detU = 0$
If $A$ is invertible then $U$ has nonzero pivots along its diagonal.
$\Rightarrow detA = \pm detU = \pm(product\ of\ the\ pivots)$

**9.** $|AB| = |A||B|$

ex) $$\begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix} \begin{vmatrix} a_2 & b_2 \\ c_2 & d_2 \end{vmatrix} = \begin{vmatrix} a_1a_2 + b_1b_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{vmatrix}$$

$$(detA)(detA^{-1}) = detI = 1$$

$$\therefore \quad |A^{-1}| = \frac{1}{|A|}$$

**10.** $|A^T| = |A|$

factorization of $A$ : $PA = LU$ where $P$ = row exchange matrix
$\Rightarrow \quad A^TP^T = U^TL^T$

$$\left\{ \begin{array}{c} |P||A| = |L||U| \\ |A^T||P^T| = |U^T||L^T| \end{array} \right\} \quad (1)$$

$$|U| = |U^T| \qquad , \qquad |L| = |L^T| = 1$$

$$P^T = P^{-1} \qquad \Rightarrow \qquad |P^T||P| = 1$$

$$(\ |P^T| = |P| = \ both\ 1\ or\ -1\ )$$

$$\therefore \quad (1) \quad \text{yields} \qquad |A^T| = |A|$$

<u>Note</u> Thanks to Rule 10, every rule for the rows applies also to the columns.
ex) $\cdot$ *det* changes sign when two columns are reversed.

· A zero column makes $det = 0$.
· A column is multiplied by $t$, so is $detA$.
· $det$ is a linear ftn of column separately.

## 5.2 Determinant by Cofactors

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$$= ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= ad - bc \ .$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & & a_{23} \\ a_{31} & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ 0 & & a_{33} \end{vmatrix}$$

$$+ \begin{vmatrix} & a_{12} & \\ a_{21} & & a_{23} \\ a_{31} & & 0 \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ 0 & a_{32} & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & a_{22} & \\ a_{31} & 0 & \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33}$$

$$+a_{31}a_{12}a_{23} + a_{21}a_{32}a_{13} - a_{31}a_{22}a_{13}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(-a_{21}a_{33} + a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

"Cofactors"

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= \sum_{j=1}^{3} a_{1j}C_{1j}$$

Cofactor $\quad C_{1j} = (-1)^{1+j}\ detM_{1j} \quad$ where $\quad M_{1j}$ is a submatrix of A without row 1 and column $j$.

We can do the same for row $i$, not just row 1.

> For any row $i$,
>
> $$detA = \sum_{j=1}^{n} a_{ij} C_{ij}$$
>
> where $C_{ij} = (-1)^{i+j}\ detM_{ij}$
>
> $M_{ij}$ = submatrix (order n-1) without row $i$ and column $j$

$$\begin{aligned} detA &= \text{dot product of any row } i \text{ with its cofactors} \\ &= \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{(expand down a column)} \end{aligned}$$

Ex

$$|A| = \begin{vmatrix} \underline{2} & \underline{-1} & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1) \begin{vmatrix} -1 & \\ -1 & 2 \end{vmatrix}$$

$$= 2 \cdot 3 + (-2) = 4$$

$$\begin{vmatrix} -1 & -1 & \\ & 2 & -1 \\ & -1 & 2 \end{vmatrix} = (-1) \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = (-1) \cdot 3 = -3$$

$$\therefore \quad |A| = 2 \cdot 4 + (-3) = 5$$