8 Symmetric Matrices

8.1 Properties of Symmetric matrix

We know that an n×n symmetric matrix A has only real eigenvalues

$$\underbrace{(A-\lambda I)}_{\text{real}}x = 0\ ,$$

so the eigenvectors are also real. They are, in fact, *perpendicular*.

Example .

(symmetric)

$\cdot$ A= $\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ has an orthonormal basis of eigenvectors $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\rightarrow \lambda_1 = 8$ $\rightarrow \lambda_2 = 2$

$\cdot$ A= $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has an orthonormal basis of eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$\rightarrow \lambda_{1,2} = 1$

(Not symmetric)

$\cdot$ A= $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has only one eigenvector $\begin{bmatrix} c \\ 0 \end{bmatrix}$

Theorem **An n×n real symmetric matrix has an orthonormal basis of eigenvector for $\mathbb{R}^n$.**

$$\Uparrow$$
$$\mathrm{x}_i^{\mathrm{T}}\mathrm{x}_j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

Partial Proof (Eigenvectors of a real sym.matrix corresponding to different eigenvalues are perpendicular) :

Let $\mathrm{Ax} = \lambda_1\mathrm{x}$, $\mathrm{Ay} = \lambda_2\mathrm{y}$, $\lambda_1 \neq \lambda_2$, $\mathrm{A} = \mathrm{A}^{\mathrm{T}}$.

Then
$$\begin{array}{ccccccccc} (\lambda_1\mathrm{x})^{\mathrm{T}}\mathrm{y} & = & (\mathrm{Ax})^{\mathrm{T}}\mathrm{y} & = & \mathrm{x}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}\mathrm{y} & = & \mathrm{x}^{\mathrm{T}}\mathrm{Ay} & = & \mathrm{x}^{\mathrm{T}}\lambda_2\mathrm{y} \\ \shortparallel & & & & & & & & \shortparallel \\ \lambda_1\mathrm{x}^{\mathrm{T}}\mathrm{y} & & & & & & & & \lambda_2\mathrm{x}^{\mathrm{T}}\mathrm{y}. \end{array}$$

$\lambda_1 \neq \lambda_2 \quad \Rightarrow \mathrm{x}^{\mathrm{T}}\mathrm{y} = 0. \quad \sharp$

Recall Diagonalization $\mathrm{A} = \mathrm{X}\Lambda\mathrm{X}^{-1}$

For a symmetric matrix A, construct X using n orthonormal eigenvectors.

Then

$$\mathrm{X}^{\mathrm{T}}\mathrm{X} = \begin{bmatrix} -- & x_1^{\mathrm{T}} & -- \\ & \vdots & \\ -- & x_n^{\mathrm{T}} & -- \end{bmatrix} \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \cdots & 0 \\ & \ddots & \\ & & 1 \end{bmatrix} = I.$$

$$\therefore \quad \mathrm{X}^{-1} = \mathrm{X}^{\mathrm{T}} .$$

Theorem **[Spectral Theorem or Principal Axis Theorem]**
Every symmetric matrix has the factorization $\mathbf{A} = \mathbf{X}\Lambda\mathbf{X}^{\mathrm{T}}$ with real eigenvalues in Λ and orthonormal eigenvectors in X.

Consider a quadratic form :

$$q = \mathrm{x}^{\mathrm{T}}\mathrm{X}\Lambda\mathrm{X}^{\mathrm{T}}\mathrm{x}$$

$$\text{Set} \quad \mathrm{y} = \mathrm{X}^{\mathrm{T}}\mathrm{x}. \qquad \text{Then} \quad \mathrm{x} = \mathrm{X}\mathrm{y}, \qquad \text{and } \mathrm{x}^{\mathrm{T}}\mathrm{X} = \mathrm{y}^{\mathrm{T}}$$

$$\therefore \ q = \mathrm{y}^{\mathrm{T}}\Lambda\mathrm{y} = \sum_{i=1}^{n} \lambda_i y_i^2 \quad \leftarrow \text{(This is called the \underline{principal axis form})}$$

Example . Find the axes of the tilted ellipse

$$5x_1^2 + 8x_1x_2 + 5x_2^2 = 1$$

$$\shortparallel$$

$$q = \mathrm{x}^{\mathrm{T}}\mathrm{A}\mathrm{x} \qquad \text{where } \mathrm{A} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

The eigenvalues of A :

$$\lambda_1 = 9, \lambda_2 = 1 \qquad \Rightarrow \quad \Lambda = \begin{bmatrix} 9 & \\ & 1 \end{bmatrix}$$

Corresponding eigenvectors :

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \Rightarrow \quad \mathrm{X} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

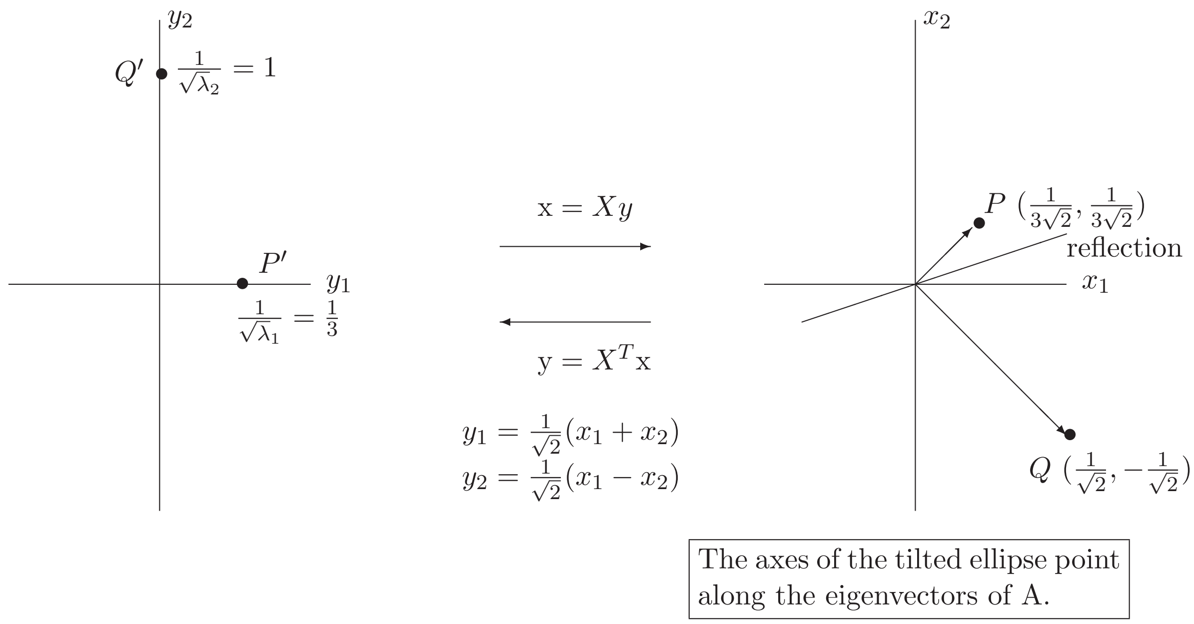
Set

$$\mathrm{y} = \mathrm{X}^{\mathrm{T}}\mathrm{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad y_1 = \frac{1}{\sqrt{2}}(x_1 + x_2), \quad y_2 = \frac{1}{\sqrt{2}}(x_1 - x_2)$$

And

$$\begin{aligned} q &= 9\left(\frac{x_1 + x_2}{\sqrt{2}}\right)^2 + \left(\frac{x_1 - x_2}{\sqrt{2}}\right)^2 \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 \end{aligned}$$



This example shows why the previous theorem is called the principal axis theorem.

8.2 Positive Definite Matrices

Note In the above example, for any nonzero vector $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$$q = \mathrm{x}^{\mathrm{T}}\mathrm{A}\mathrm{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 \quad > \ 0$$

Such a matrix A is called positive definite. (Strang, page331)

Definition **A symmetric matrix A is <u>positive definite</u> if $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} > 0$ for every nonzero vector x.**

Recall $q = \mathrm{x}^{\mathrm{T}}\mathrm{Ax} = \sum_{i=1}^{n} y_i^2 \lambda_i$ where $\lambda_1, \cdots, \lambda_n$ are eigenvalues of A.

· Suppose that $\lambda_k \le 0$. Then for $\mathrm{y} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k^{th}$,

$q = \lambda_k \le 0$. Thus, there exists a nonzero vector $\mathrm{x} = \mathrm{Xy}$ s.t. $q \le 0$.

· If all $\lambda_i >$, then $q > 0$ for every nonzero x.

Therefore we have the following theorem :

Theorem **A : n×n symmetric matrix. Then,**

All n eigenvalues are positive

$\Updownarrow$

$\mathbf{x}^{\mathrm{T}}\mathbf{Ax} > \mathbf{0}$ **except at** $\mathbf{x} = \mathbf{0}$ **(A is positive definite).**

2×2 case $\quad \mathrm{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, when is A positive definite?

$$|\mathrm{A} - \lambda \mathrm{I}| = (a-\lambda)(c-\lambda) - b^2 = \lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

· If $\lambda_1, \lambda_2 > 0$, $\quad \begin{aligned} \lambda_1 + \lambda_2 &= a + c > 0 \\ \lambda_1 \lambda_2 &= ac - b^2 > 0 \end{aligned}$

If $a > 0$ and $c \le 0$, then $ac - b^2 \le 0$
If $a \le 0$ and $c > 0$, then $ac - b^2 \le 0$
Therefore, we have $a > 0$, $c > 0$ and $ac - b^2 > 0$

· Now, suppose $\underbrace{a > 0}_{\text{1×1 upperleft determinant}}$ and $\underbrace{ac - b^2 > 0}_{\text{2×2 determinant}}$

This forces $c > 0$

$$\Rightarrow \quad \lambda_1 + \lambda_2 > 0, \quad \lambda_1 \lambda_2 > 0$$
$$\therefore \ \lambda_1, \lambda_2 > 0$$

$$\begin{aligned} \cdot\ \mathrm{x}^{\mathrm{T}}\mathrm{Ax} &= [x_1 \quad x_2] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= a x_1^2 + 2b x_1 x_2 + c x_2^2 \\ &= a\left(x_1 + \frac{b}{a} x_2\right)^2 + \left(\frac{ac - b^2}{a}\right) x_2^2 \end{aligned}$$

$$= \begin{bmatrix} x_1 + \frac{b}{a}x_2 & x_2 \end{bmatrix} \begin{bmatrix} a & \\ & \frac{ac-b^2}{a} \end{bmatrix} \begin{bmatrix} x_1 + \frac{b}{a}x_2 \\ x_2 \end{bmatrix}$$

$$= [x_1 \quad x_2] \begin{bmatrix} 1 & 0 \\ \frac{b}{a} & 1 \end{bmatrix} \begin{bmatrix} a & \\ & \frac{ac-b^2}{a} \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \mathrm{x}^{\mathrm{T}}\mathrm{LDL}^{\mathrm{T}}\mathrm{x}$$

Recall the factorization of a symmetric matrix $\mathrm{A} = \mathrm{LDL}^{\mathrm{T}}$
D contains the diagonal elements of the upper triangular matrix, and they are pivots!

first pivot (if $a > 0$)

$$\begin{bmatrix} \underline{a} & b \\ b & c \end{bmatrix} \longrightarrow \begin{bmatrix} a & b \\ 0 & \underline{c - \frac{b}{a}b} \end{bmatrix}$$

second pivot

Thus, $\mathrm{x}^{\mathrm{T}}\mathrm{Ax} > 0$ except at $\mathrm{x} = 0$ mean positive pivots and vice versa.

The above analysis holds for $\mathrm{n} \times \mathrm{n}$ symmetric matrices.

Theorem For an $\mathrm{n}\times\mathrm{n}$ symmetric matrix A, the following are equivalent.

1. All n eigenvalues are positive.
2. All n upperleft determinants are positive.
3. All n pivots are positive.
4. $\mathrm{x}^{\mathrm{T}}\mathrm{Ax} > 0$ except at $\mathrm{x} = 0$. (A is positive definite)

· Suppose A is positive definite. Then,
(i) $\mathrm{x}^{\mathrm{T}}\mathrm{Ax} = 1$ is an ellipse. $(\mathrm{x}^{\mathrm{T}}\mathrm{Ax} = \mathrm{y}^{\mathrm{T}}\Lambda\mathrm{y} = 1)$
(ii) the quadratic function $f(x) = \mathrm{x}^{\mathrm{T}}\mathrm{Ax}$ has a minimum at $\mathrm{x} = 0$.