# 11 Half-Range Expansions, Forced Oscillations

## 11.1 Half-Range Expansions

We often want to employ a Fourier series for a function $f$ that is given only on some interval, say, $0 \leq x \leq L$. Use even-periodic or odd-periodic extension.

**Example .** Triangle on $0 \leq x \leq L$ and its half-range expansions

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} \leq x \leq L \end{cases}$$

(a) Even periodic extension

$$\begin{aligned}
a_0 &= \frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2} x dx + \frac{2k}{L}\int_{L/2}^{L}(L-x)dx\right] = \frac{k}{2} \\
a_n &= \frac{2}{L}\left[\frac{2k}{L}\underbrace{\int_0^{L/2} x\cos\frac{n\pi x}{L}dx}_{(\bigstar)} + \frac{2k}{L}\underbrace{\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx}_{(\bigstar\bigstar)}\right] \\
(\bigstar) &= \frac{Lx}{n\pi}\sin\frac{n\pi x}{L}|_0^{L/2} - \frac{L}{n\pi}\int_o^{L/2}\sin\frac{n\pi x}{L}dx \\
&= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}|_0^{L/2} \\
&= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}(\cos\frac{n\pi}{2} - 1) \\
(\bigstar\bigstar) &= \frac{L}{n\pi}(L-x)\sin\frac{n\pi x}{L}|_{L/2}^{L} + \frac{L}{n\pi}\int_o^{L/2}\sin\frac{n\pi x}{L}dx \\
&= 0 - \frac{L}{n\pi}(L-\frac{L}{2})\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}|_{L/2}^{L} \\
&= -\frac{L^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}(\cos n\pi - \cos\frac{n\pi}{2}) \\
\therefore\ a_n &= \frac{4k}{L^2}[\frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}(\cos\frac{n\pi}{2} - 1) - \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}(\cos n\pi - \cos\frac{n\pi}{2})] \\
&= \frac{4k}{n^2\pi^2}(2\cos\frac{n\pi}{2} - \cos n\pi - 1)
\end{aligned}$$

– When $n$ is odd,

$$2\cos\frac{n\pi}{2} - \cos n\pi - 1 = 0 - (-1) - 1 = 0$$

– When $n = 4, 8, 12, 16, \cdots$,

$$2\cos\frac{n\pi}{2} - \cos n\pi - 1 = 2\cdot 1 - 1 - 1 = 0$$

– And

$$a_2 = -\frac{16k}{2^2\pi^2}, \quad a_6 = -\frac{16k}{6^2\pi^2}, \quad a_{10} = -\frac{16k}{10^2\pi^2}, \cdots$$

– Thus,

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}(\frac{1}{2^2}\cos\frac{2\pi x}{L} + \frac{1}{6^2}\cos\frac{6\pi x}{L} + \cdots)$$

(b) Odd periodic extension

$$\begin{aligned}
b_n &= \frac{2}{L}\left[\underbrace{\int_0^{L/2}\frac{2k}{L}x\sin\frac{n\pi x}{L}dx}_{(\clubsuit)} + \underbrace{\int_{L/2}^{L}\frac{2k}{L}(L-x)\sin\frac{n\pi x}{L}dx}_{(\clubsuit\clubsuit)}\right] \\
(\clubsuit) &= -\frac{Lx}{n\pi}\cos\frac{n\pi x}{L}|_0^{L/2} + \frac{L}{n\pi}\int_o^{L/2}\cos\frac{n\pi x}{L}dx \\
&= -\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}|_0^{L/2} \\
&= -\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2} \\
(\clubsuit\clubsuit) &= -\frac{L}{n\pi}(L-x)\cos\frac{n\pi x}{L}|_{L/2}^{L} - \frac{L}{n\pi}\int_o^{L/2}\cos\frac{n\pi x}{L}dx \\
&= \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}|_{L/2}^{L} \\
&= \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2} \\
\therefore\ b_n &= \frac{4k}{L^2}(-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\cos\frac{n\pi}{2} + \frac{L^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}) \\
&= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}
\end{aligned}$$

Thus,

$$f(x) = \frac{8k}{\pi^2}(\frac{1}{1^2}\sin\frac{\pi x}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L} + \frac{1}{5^2}\sin\frac{5\pi x}{L} - \cdots) \qquad \sharp$$

## 11.2 Forced Oscillations

Equation of motion

$$r(t) - ky - cy' = my''$$
$$\therefore my'' + cy' + ky = r(t)$$

- RLC-circuit

$$LI'' + RI' + \frac{1}{C}I = E'(t)$$

**Example .** Forced oscillations under a *nonsinusoidal* periodic driving force. Find the steady-state solution $y(t)$.
Given : $m = 1 \times 10^{-3}$ kg, $c = 0.02$ g/sec, $k = 25$ g/sec$^2$, and $r(t)$ measures in $g$·cm/sec$^2$.

$$y'' + 0.02y' + 25y = r(t) = \begin{cases} t + \pi/2 & \text{if } -\pi < t < 0 \\ -t + \pi/2 & \text{if } 0 < t < \pi \end{cases}$$

**Solution.** - $r(t)$: even function
- The steady-state solution is the particular solution of the equation.

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^\pi r(t)dt = \frac{1}{\pi}\int_0^\pi (-t + \frac{\pi}{2})dt = \frac{1}{\pi}[-\frac{t^2}{2} + \frac{\pi}{2}t]|_0^\pi \\ &= \frac{1}{\pi}(-\frac{\pi^2}{2} + \frac{\pi^2}{2}) = 0 \\ a_n &= \frac{2}{\pi}\int_0^\pi (-t + \frac{\pi}{2})\cos ntdt = -\frac{2}{\pi}\int_0^\pi t\cos ntdt + \int_0^\pi \cos ntdt \\ &= -\frac{2}{\pi}\cdot\frac{1}{n}t\cdot\sin nt|_0^\pi + \frac{2}{\pi n}\int_0^\pi \sin ntdt = -\frac{2}{\pi n^2}\cos nt|_0^\pi \\ &= \frac{2}{\pi n^2}(1 - \cos n\pi) \\ &= \begin{cases} 0 & \text{even } n \\ \frac{4}{\pi n^2} & \text{odd } n \end{cases} \\ \therefore\ r(t) &= \frac{4}{\pi}(\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots) \end{aligned}$$

-Consider

$$y_n'' + 0.02y_n' + 25y_n = \frac{4}{n^2\pi}\cos nt \qquad (n = 1, 3, 5, \cdots) \tag{1}$$

- Steady-state solution $y_n(t)$ of (**??**):

$$\begin{aligned} y_n(t) &= A_n\cos nt + B_n\sin nt \\ y_n'(t) &= -nA_n\sin nt + nB_n\cos nt \\ y_n''(t) &= -n^2A_n\cos nt - n^2B_n\sin nt \end{aligned} \tag{2}$$

- By substitution (**??**) into (**??**),

$$\begin{gathered} -n^2A_n\cos nt - n^2B_n\sin nt - 0.02nA_n\sin nt + 0.02nB_n\cos nt \\ +25A_n\cos nt + 25B_n\sin nt = \tfrac{4}{n^2\pi}\cos nt \end{gathered}$$

$$(25A_n - n^2A_n + 0.02nB_n)\cos nt + (25B_n - n^2B_n - 0.02nA_n)\sin nt = \frac{4}{n^2\pi}\cos nt$$

$$25A_n - n^2A_n + 0.02nB_n = \frac{4}{n^2\pi} \tag{3}$$
$$-0.02nA_n + 25B_n - n^2B_n = 0 \tag{4}$$

- From (**??**)

$$B_n = \frac{0.02n}{25-n^2}A_n$$

- Substituting this into (**??**)

$$(25-n^2)A_n + \frac{0.02n}{25-n^2}A_n = \frac{4}{n^2\pi}$$

$$\therefore A_n = \frac{4(25-n^2)}{n^2\pi[(25-n^2)^2+(0.02n)^2]}$$
$$\therefore B_n = \frac{0.08}{n\pi[(25-n^2)^2+(0.02n)^2]}$$

- Amplitude of $y_n$

$$C_n = \sqrt{A_n^2+B_n^2}, \qquad \text{Let } D = (25-n^2)^2+(0.02n)^2$$
$$\therefore C_n = \frac{4}{n^2\pi D}[(25-n^2)^2+(0.02n)^2]^{1/2} = \frac{4}{n^2\pi\sqrt{D}}$$

- Numerical values: $C_1 = 0.0011$, $C_3 = 0.0088$, $C_5 = 0.5100$, $C_7 = -.0011$, $C_9 = 0.0003$, $\cdots$.
- In conclusion, the steady state motion is almost a harmonic oscillation whose frequency equals five times that of the exciting force. $\sharp$

## 11.3 Approximations by Trigonometric Polynomials

**Q.** (Trigonometric Approximation)
$f(x)$: a periodic function of period $2\pi$ that can be represented by a Fourier series.

$$f(x) \approx a_0 + \sum_{n=1}^{N}(a_n\cos nx + b_n\sin nx) \tag{5}$$

Best approximation to $f$ by a trigonometric polynomial of deg $N$:

$$F(x) = A_0 + \sum_{n=1}^{N}(A_n\cos nx + B_n\sin nx) \tag{6}$$

Total square error of $F$ relative to the function $f$ on the interval $-\pi \le x \le \pi$

$$E = \int_{-\pi}^{\pi}(f-F)^2dx \ge 0$$

N fixed. Determine (**??**) s.t. $E$ is minimum.

**A.**

$$E = \int_{-\pi}^{\pi} f^2 dx - 2\int_{-\pi}^{\pi} fF dx + \int_{-\pi}^{\pi} F^2 dx \tag{7}$$

- Use the orthogonality of trigonometric functions.

$$\int_{-\pi}^{\pi} \cos mx \cdot \sin nx dx = 0 \qquad \text{for all } m, n$$

$$\begin{aligned} \int_{-\pi}^{\pi} \cos^2 nx dx &= \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2nx) dx = \pi \\ \int_{-\pi}^{\pi} \sin^2 nx dx &= \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2nx) dx = \pi \end{aligned}$$

- 3rd integral in (**??**):

$$\begin{aligned} \int_{-\pi}^{\pi} F^2 dx &= \int_{-\pi}^{\pi} (A_0^2 + A_1^2 \cos^2 x + B_1^2 \sin^2 x + \cdots + A_N^2 \cos^2 Nx + B_N^2 \sin^2 Nx + \cdots \\ &+ 2A_0 A_1 \cos x + \cdots 2A_1 B_1 \cos x \sin x + \cdots + 2A_N B_N \cos Nx \sin Nx) dx \\ &= \pi(2A_0^2 + A_1^2 + \cdots + A_N^2 + B_1^2 + \cdots + B_N^2) \end{aligned}$$

- 2nd integral in (**??**):

$$\begin{aligned} \int_{-\pi}^{\pi} fF dx &= \int_{-\pi}^{\pi} (fA_0 + fA_1 \cos x + fB_1 \sin x + \cdots + fA_N \cos Nx + fB_N \sin Nx) dx \\ &= \pi(2A_0 a_0 + A_1 a_1 + \cdots + A_N a_N + B_1 b_1 + \cdots + B_N b_N) \\ &\because\ a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \cos nx dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \sin nx dx \end{aligned}$$

- Thus,

$$E = \int_{-\pi}^{\pi} f^2 dx - 2\pi \left[2A_0 a_0 + \sum_{n=1}^{N} (A_n a_n + B_n b_n)\right] + \pi \left[2A_0^2 + \sum_{n=1}^{N} (A_n^2 + B_n^2)\right] \tag{8}$$

- If we take $A_n = a_n$ and $B_n = b_n$ assuming that $F$ is a Fourier series,

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi \left[2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2)\right] \tag{9}$$

- (**??**)-(**??**): using $A_n^2 - 2A_n a_n + a_n^2 = (A_n - a_n)^2$,

$$E - E^* = \pi \left[2(A_0 - a_0)^2 + \sum_{n=1}^{N} \left\{(A_n - a_n)^2 + (B_n - b_n)^2\right\}\right] \geq 0$$

Thus $E \geq E^*$

$E = E^*$ if and only if $A_0 = a_0, \cdots, B_n = b_n$ $\quad\sharp$

**Theorem** (Minimum square error)
The total square error of $F$ (with fixed $N$) relative to $f$ on $-\pi \leq x \leq \pi$ is minimum if and only if the coefficients of $F$ are the Fourier coefficients of $f$. This minimum value $E^*$ is given by (**??**).

**Q.** What happens to $E^*$ as $N$ increases?
**A.** With increasing $N$ the partial sums of the Fourier series of $f$ yield better and better approximations to $f$.

- **Bessel inequality** Since $E^* \geq 0$,

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

- **Parseval's identity**

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

**Example .** Square error for the sawtooth wave

$$f(x) = x + \pi \qquad (-\pi < x < \pi)$$

$$F(x) = \pi + 2\sin x - \sin 2x + \frac{2}{3}\sin 3x$$

$$\begin{aligned} E^* &= \int_{-\pi}^{\pi}(x+\pi)^2 dx - \pi\left[2\pi^2 + 2^2 + 1^2 + \left(\frac{2}{3}\right)^2\right] \\ &= \frac{8}{3}\pi^3 - \pi\left(2\pi^2 + \frac{49}{9}\right) \approx 3.567 \qquad \sharp \end{aligned}$$