12 Fourier Integrals

12.1 From Fourier Series to Fourier Integral

- Extension of the method of Fourier series to nonperiodic functions
- We consider the Fourier series of an arbitrary function f_L of period $2L$ and let $L \to \infty$.

Example 1. Square wave

$$f_L(x) = \begin{cases} 0 & \text{if} \quad -L < x < 1 \\ 1 & \text{if} \quad -1 < x < 1 \\ 0 & \text{if} \quad 1 < x < L \end{cases}$$

- If we let $L \to \infty$,

$$f(x) = \lim_{L\to\infty} f_L(x) = \begin{cases} 1 & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- Since $f_L(x)$ is an even function, $b_n = 0$ for all n.

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-1}^{1} dx = \frac{1}{L} \\ a_n &= \frac{1}{L}\int_{-1}^{1} \cos\frac{n\pi x}{L} dx = \frac{2}{L}\int_{0}^{1} \cos\frac{n\pi x}{L} dx = \frac{2}{L}\cdot\frac{\sin(n\pi/L)}{n\pi/L} \end{aligned}$$

- Amplitude spectrum: $a_n(\omega_n)$, where $\omega_n = n\pi/L$. $\sharp$

Fourier series of any periodic function $f_L(x)$ of period $2L$:

$$\begin{aligned} f_L(x) &= a_0 + \sum_{n=1}^{\infty}(a_n \cos\omega_n x + b_n \sin\omega_n x) \qquad \omega_n = n\pi/L \\ &= \frac{1}{2L}\int_{-L}^{L} f_L(v)dv \\ &\quad + \frac{1}{L}\sum_{n=1}^{\infty}\left[\cos\omega_n x \int_{-L}^{L} f_L(v)\cos\omega_n v\, dv + \sin\omega_n x \int_{-L}^{L} f_L(v)\sin\omega_n v\, dv\right] \end{aligned}$$

Set

$$\Delta\omega = \omega_{n+1} - \omega_n = (n+1)\frac{\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$$

then

$$\begin{aligned} f_L(x) &= \frac{1}{2L}\int_{-L}^{L} f_L(v)dv \\ &+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left[(\cos\omega_n x)\Delta\omega\int_{-L}^{L} f_L(v)\cos\omega_n v dv + (\sin\omega_n x)\Delta\omega\int_{-L}^{L} f_L(v)\sin\omega_n v dv\right] \end{aligned}$$

Now let $L \to \infty$ and assume that the resulting nonperiodic function

$$f(x) = \lim_{L\to\infty} f_L(x).$$

is absolutely integrable on the x-axis; that is, the following limits exist:

$$\int_{-\infty}^{\infty} |f(x)|dx = \lim_{a\to-\infty}\int_a^0 |f(x)|dx + \lim_{b\to\infty}\int_b^0 |f(x)|dx \le M$$

Then, as $L \to \infty$, $1/L \to 0$, and

$$\frac{1}{2L}\int_{-L}^{L} f_L(v)dv \to 0$$

Also, $\Delta w = \pi/L \to 0$. Thus

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\cos\omega x\int_{-\infty}^{\infty} f(v)\cos\omega v dv + \sin\omega x\int_{-\infty}^{\infty} f(v)\sin\omega v dv\right]d\omega$$

If we introduce the notations

$$\underline{A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v dv, \qquad B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\omega v dv,} \tag{1}$$

Fourier integral

$$\underline{f(x) = \int_0^{\infty}[A(\omega)\cos\omega x + B(\omega)\sin\omega x]d\omega} \tag{2}$$

Theorem 1 (Fourier integral)
If $f(x)$ is piecewise continuous in every finite integral and has a right-hand derivative and a left-hand derivative at every point and if absolutely integrable, then $f(x)$ can be represented by a Fourier integral (**??**). At a point where $f(x)$ is discontinuous the value of the Fourier integral equals the average of the left- and right-hand limits of $f(x)$ at that point.

12.2 Applications of the Fourier integral

- Solving differential equations
- Evaluating integrals

Example 2. Single pulse, sine integral

$$f(x)=\begin{cases}1 & \text{if } |x|<1\\ 0 & \text{if } |x|>1\end{cases}$$

From (**??**)

$$\begin{aligned}A(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\omega v dv = \frac{1}{\pi}\int_{-1}^{1}\cos\omega v dv = \frac{\sin\omega v}{\pi\omega}\ |_{-1}^{1} = \frac{\sin\omega}{\pi\omega}+\frac{\sin\omega}{\pi\omega}\\ &= \frac{2\sin\omega}{\pi\omega}\\ B(\omega) &= \frac{1}{\pi}\int_{-1}^{1}\sin\omega v dv = -\frac{\cos\omega v}{\pi\omega}\ |_{-1}^{1} = 0\end{aligned}$$

$$\therefore f(x)=\frac{2}{\pi}\int_0^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega \tag{3}$$

From (**??**), Dirichlet's discontinuous factor is obtained.

$$\frac{2}{\pi}\int_0^{\infty}\frac{\cos\omega x\sin\omega}{\omega}d\omega=\begin{cases}\pi/2 & \text{if } 0\le x\le 1\\ \pi/4 & \text{if } x=1\\ 0 & \text{if } x>1\end{cases}$$

- If $x=0$,

$$\int_0^{\infty}\frac{\sin\omega}{\omega}d\omega=\frac{\pi}{2}$$

- **Sine integral**

$$\mathrm{Si}(u)=\int_0^{u}\frac{\sin\omega}{\omega}d\omega$$

- **Gibbs phenomenon**: With increasing a, the oscillations near $x=\pm1$ are shifted closer to the points $x=\pm1$.

$$\frac{2}{\pi}\int_0^{a}\frac{\cos\omega x\sin\omega}{\omega}d\omega=\frac{1}{\pi}\int_0^{a}\frac{\sin(\omega+\omega x)}{\omega}+\frac{1}{\pi}\int_0^{a}\frac{\sin(\omega-\omega x)}{\omega}d\omega$$

- 1st integral: $\omega+\omega x=t \Rightarrow d\omega/\omega=dt/t$ and $0\le\omega\le a\Rightarrow 0\le t\le(x+1)a$.
- 2nd integral: $\omega-\omega x=-t \Rightarrow d\omega/\omega=dt/t$ and $0\le\omega\le a\Rightarrow 0\le t\le(x-1)a$.

$$\begin{aligned}\frac{2}{\pi}\int_0^{a}\frac{\cos\omega x\sin\omega}{\omega}d\omega &= \frac{1}{\pi}\int_0^{(x+1)a}\frac{\sin t}{t}dt-\frac{1}{\pi}\int_0^{(x-1)a}\frac{\sin t}{t}dt\\ &= \frac{1}{\pi}\mathrm{Si}[a(x+1)]-\frac{1}{\pi}\mathrm{Si}[a(x-1)]\end{aligned}$$

12.3 Fourier Cosine and Sine Integrals

- If $f(x)$ is even,

$$A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos wv\,dv, \tag{4}$$
$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin\omega v\,dv = 0$$

  Fourier cosine integral

$$f(x) = \int_0^\infty A(\omega)\cos\omega x d\omega \qquad (f \text{ even}) \tag{5}$$

- If $f(x)$ is odd,

$$A(\omega) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\cos wv\,dv = 0,$$
$$B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin\omega v\,dv \tag{6}$$

  Fourier sine integral

$$f(x) = \int_0^\infty B(\omega)\sin\omega x d\omega \tag{7}$$

12.4 Evaluation of Integrals

Example 3. (Laplace integrals)
Find the Fourier cosine and sine integrals of

$$f(x) = e^{-kx} \qquad (x > 0,\ k > 0)$$

(a) From (**??**),

$$A(\omega) = \frac{2}{\pi}\int_0^\infty e^{-kv}\cos\omega v dv$$

Integration by parts,

$$\begin{aligned}
\int e^{-kv}\cos\omega v\,dv &= \frac{1}{\omega}e^{-kv}\sin\omega v + \frac{k}{\omega}\int e^{-kv}\sin\omega v\,dv \\
\int e^{-kv}\sin\omega v\,dv &= -\frac{1}{\omega}e^{-kv}\cos\omega v - \frac{k}{\omega}\int e^{-kv}\cos\omega v\,dv \\
\therefore \int e^{-kv}\cos\omega v\,dv &= \frac{1}{\omega}e^{-kv}\sin\omega v - \frac{k}{\omega^2}e^{-kv}\cos\omega v - \frac{k^2}{\omega^2}\int e^{-kv}\cos\omega v\,dv \\
\Rightarrow \qquad \int e^{-kv}\cos\omega v dv &= \frac{e^{-kv}}{\omega^2+k^2}(\omega\sin\omega v - k\cos\omega v)
\end{aligned}$$

$$A(\omega) = \frac{2}{\pi} \cdot \frac{e^{-kv}}{\omega^2 + k^2}(\omega \sin \omega v - k \cos \omega v) \mid_0^\infty = \frac{2k/\pi}{\omega^2 + k^2}$$

$$\therefore\ f(x) = e^{-kx} = \frac{2k}{\pi} \int_0^\infty \frac{\cos \omega x}{k^2 + \omega^2} d\omega \qquad (x > 0,\ k > 0)$$

$$\Rightarrow \int_0^\infty \frac{\cos \omega x}{k^2 + \omega^2} d\omega = \frac{\pi}{2k} e^{-kx} \qquad (x > 0,\ k > 0) \tag{8}$$

(b) From (**??**),

$$B(\omega) = \frac{2}{\pi} \int_0^\infty e^{-kv} \sin \omega v dv$$

Integration by parts,

$$\int e^{-kv} \sin \omega v dv = -\frac{\omega}{k^2 + \omega^2} e^{-kv} (\frac{k}{\omega} \sin \omega v + \cos \omega v)$$

$$B(\omega) = -\frac{2}{\pi} \cdot \frac{\omega}{k^2 + \omega^2} e^{-kv} (\frac{k}{\omega} \sin \omega v + \cos \omega v) \mid_0^\infty = \frac{2\omega/\pi}{k^2 + \omega^2}$$

$$\therefore\ f(x) = e^{-kv} = \frac{2}{\pi} \int_0^\infty \frac{\omega \sin \omega x}{k^2 + \omega^2} d\omega$$

$$\Rightarrow \int_0^\infty \frac{\omega \sin \omega x}{k^2 + \omega^2} d\omega = \frac{\pi}{2} e^{-kx} \qquad (x > 0,\ k > 0) \tag{9}$$

(**??**) and (**??**): Laplace integrals ! $\sharp$