15 Solving PDEs

15.1 Separation of Variables. Use of Fourier Series.

Governing equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \ , \tag{1}$$

Q. how many conditions do we need ?
- Boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for all } t \tag{2}$$

- Initial conditions

$$u(x,0) = f(x) \tag{3}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \tag{4}$$

- Solution procedures

Step I: Method of separating variables or product method $\rightarrow$ two ODEs
Step II: Determination of the solutions of those two ODEs satisfying the BCs (**??**)
Step III: Using Fourier series, acquirement of a solution of (**??**) satisfying the ICs (**??**) and (**??**).

15.1.1 Step I : Two Ordinary Differential Equations

- Method of separating variables

$$u(x,t) = F(x) \cdot G(t) \tag{5}$$

$$\Rightarrow \ \frac{\partial^2 u}{\partial t^2} = F\ddot{G} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = F''G$$

- By inserting into the PDE (**??**),

$$F\ddot{G} = c^2 F'' G$$

$$\Rightarrow \quad \frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k = \text{constant}$$

$$\Rightarrow \quad \boxed{\text{2 ODEs}}$$

$$F'' - kF = 0 \tag{6}$$

$$\ddot{G} - c^2 k G = 0 \tag{7}$$

15.1.2 Step II: Satisfying the Boundary Conditions (??)

$$u(0,t) = F(0)G(t) = 0, \quad u(L,t) = F(L)G(t) = 0 \qquad \text{for all } t$$

- **Solving** (??)
  - If $G(t) \equiv 0$, then trivial soln. Thus

$$\text{(a) } F(0) = 0, \qquad \text{(b) } F(L) = 0. \tag{8}$$

  - For $k = 0$, (??)$\Rightarrow$ $F(x) = ax + b$, and from (??), $a = b = 0$. *trivial*
  - For positive $k = \mu^2$, (??)$\Rightarrow$ $F = Ae^{\mu x} + Be^{-\mu x}$, and from (??), $F \equiv 0$. *trivial*
  - For negative $k = -p^2$, (??)$\Rightarrow$ $F(x) = A\cos px + B\sin px$

$$F(0) = A = (0), \qquad F(L) = B\sin pL = 0$$

    - $\sin pL = 0 \;\Rightarrow\; pL = n\pi \qquad p = \frac{n\pi}{L} \quad (n: \text{ integer})$
    - Setting $B = 1$,

$$F_n(x) = \sin\frac{n\pi}{L}x \qquad (n = 1,\ 2,\ \cdots).$$

- **Solving** (??)
  Since $k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$,

$$\ddot{G} + \lambda_n^2 G = 0 \quad \text{where} \quad \lambda_n = \frac{cn\pi}{L}$$

$$G_n(t) = B_n\cos\lambda_n t + B_n^*\sin\lambda_n t$$

$$\therefore\ u_n(x,t) = F_n(x)G_n(t) = (B_n\cos\lambda_n + B_n^*\sin\lambda_n)\sin\frac{n\pi}{L}x \qquad (n = 1,\ 2,\cdots) \tag{9}$$

- The function in (??) are the eigenfunctions or characteristic functions.
- $\lambda_n = cn\pi/L$ are the eigenvalues, or characteristic values. The set $\{\lambda_1,\ \lambda_2,\cdots\}$ is the spectrum.
- Each u_n represents a harmonic motion with freq $\frac{\lambda_n}{2\pi} = \frac{cn}{2L}$: called n^{th} normal mode. $n = 1$: fundamental mode.
- $\sin\frac{n\pi x}{L} = 0 \quad \Rightarrow\ x = \frac{L}{n}, \frac{2L}{n}, cdpts, \frac{n-1}{n}$. Thus n^{th} normal mode has $n-1$ points that do not move (called node).
- tuning ? $\lambda_n = cn\pi/L,\ c = T/\rho$

15.1.3 Step III: Solution of the Entire Problem. Fourier Series

Since we have a lin. homo. PDE,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* sin \lambda_n t) \sin \frac{n\pi x}{L} \tag{10}$$

- **Satisfying IC** (**??**) (given Initial Displacement)

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} = f(x) \tag{11}$$

Fourier series !

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx \qquad n = 1, 2, \cdots \tag{12}$$

- **Satisfying IC** (**??**) (given Initial Velocity)

$$\begin{aligned} \left.\frac{\partial u}{\partial t}\right|_{t=0} &= \left[\sum_{n=1}^{\infty} (-B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t) \sin \frac{n\pi x}{L} \right]_{t=0} \\ &= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x) \end{aligned}$$

$$B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

Fourier series ! Since $\lambda_n = cn\pi/L$,

$$B_n^* = \frac{2}{cn\pi} \int_0^L g(x) \cdot \sin \frac{n\pi x}{L} dx \qquad n = 1, 2, \cdots \tag{13}$$

Solution (**??**) is established!

When $g(x) = 0$, $B_n^* = 0$.

$$u(x,t) = \sum_{n=1}^{\infty} B_n \cos \lambda_n t \sin \frac{n\pi x}{L}, \qquad \lambda_n = \frac{cn\pi}{L} \tag{14}$$

Since

$$\cos \frac{cn\pi t}{L} \sin \frac{n\pi x}{L} = \frac{1}{2} \left[\sin \left\{ \frac{n\pi}{L}(x - ct) \right\} + \sin \left\{ \frac{n\pi}{L}(x + ct) \right\} \right] ,$$

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} B_n \sin \left\{ \frac{n\pi}{L}(x - ct) \right\} + \frac{1}{2} \sum_{n=1}^{\infty} B_n \sin \left\{ \frac{n\pi}{L}(x + ct) \right\}$$

Recall B_n's are Fourier sine series coefficients. These two series are those obtained by substituting $x - ct$ and $x + ct$, respectively, for the variable x in the Fourier sine series (**??**) for $f(x)$.

$$\therefore\ u(x,t) = \frac{1}{2}[f^*(x-ct) + f^*(x+ct)] \tag{15}$$

where f^* is the odd periodic extension of f with the period $2L$.
Physical Interpretation of the Solution:
$f^*(x-ct)$: a wave traveling to the right as t increases ($c > 0$).
$f^*(x+ct)$: a wave traveling to the left as t increases.
$\rightarrow\ u(x,t)$ is the superposition of these two waves.

Example . Vibrating string if the initial deflection is triangular

$$f(x) = \begin{cases} \frac{2k}{L}x & \text{if } 0 < x < \frac{L}{2} \\ \frac{2k}{L}(L-x) & \text{if } \frac{L}{2} < x < L \end{cases}$$

with thei nitial velocity

$$u_t(x,0) = 0$$

Solution.

$$g(x) \equiv 0 \quad \Longrightarrow \quad B_n^* \equiv 0$$

- From Example 3. in sec. 10. 4

$$B_n = \frac{8k}{n^2\pi^2} \sin\frac{n\pi}{2}$$

$$\therefore u(x,t) = \frac{8k}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{L} \cdot \cos\frac{\pi ct}{L} - \frac{1}{3^2}\sin\frac{3\pi x}{L} \cdot \cos\frac{3\pi ct}{L} \cdots\right]$$