19 PDEs in Polar Coordinates

19.1 Laplacian in Polar Coordinates

$$x = r\cos\theta, \qquad y = \sin\theta$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

- (x, y, t) to (r, θ, t)

$$u_x = u_r r_x + u_\theta \theta_x, \qquad u_y = u_r r_y + u_\theta \theta_y$$

- By the product rule and the chain rule

$$\begin{aligned} u_{xx} &= (u_r r_x)_x + (u_\theta \theta_x)_x = (u_r)_x r_x + u_r r_{xx} + (u_\theta)_x \theta_x + u_\theta \theta_{xx} \\ &= (u_{rr} r_x + u_{r\theta}\theta_x) r_x + u_r r_{xx} + (u_{\theta r} r_x + u_{\theta\theta}\theta_x)\theta_x + u_\theta \theta_{xx} \\ u_{yy} &= (u_r r_y)_y + (u_\theta \theta_y)_y = (u_r)_y r_y + u_r r_{yy} + (u_\theta)_y \theta_y + u_\theta \theta_{yy} \\ &= (u_{rr} r_y + u_{r\theta}\theta_y) r_y + u_r r_{yy} + (u_{\theta r} r_y + u_{\theta\theta}\theta_y)\theta_y + u_\theta \theta_{yy} \end{aligned}$$

- To determine the partial derivatives r_x and θ_x,

$$r = \sqrt{x^2 + y^2}, \quad \text{and} \quad \theta = \arctan\frac{y}{x}$$

$$r_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}, \qquad r_y = \frac{y}{r}$$

$$\theta_x = \frac{1}{1+(y/x)^2}\cdot\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2} = -\frac{y}{r^2}, \quad \theta_y = \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$

$$r_{xx} = \frac{1}{r} - \frac{x r_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \frac{r^2 - x^2}{r^3} = \frac{y^2}{r^3}, \quad r_{yy} = \frac{x^2}{r^3}$$

$$\theta_{xx} = -y(-\frac{2}{r^3})r_x = \frac{2y}{r^3}\cdot\frac{x}{r} = \frac{2xy}{r^4}, \quad \theta_{yy} = x\cdot\frac{(-2)}{r^3}r_y = -\frac{2xy}{r^4}$$

$$u_{xx} = u_{rr}\frac{x^2}{r^2} + u_{r\theta}\left(-\frac{y}{r^2}\right)\frac{x}{r} + u_r\frac{y^2}{r^3} + u_{\theta r}\frac{x}{r}\left(-\frac{y}{r^2}\right) + u_{\theta\theta}\left(-\frac{y}{r^2}\right)^2 + u_\theta\frac{2xy}{r^4}$$

$$\therefore u_{xx} = \frac{x^2}{r^2}u_{rr} - 2\frac{xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta} + \frac{y^2}{r^3}u_r + 2\frac{xy}{r^4}u_\theta$$

$$u_{yy} = u_{rr}\frac{y^2}{r^2} + u_{r\theta}\frac{x}{r^2}\frac{y}{r} + u_r\frac{x^2}{r^3} + u_{\theta r}\frac{y}{r}\frac{x}{r^2} + u_{\theta\theta}\left(\frac{x}{r^2}\right)^2 + u_\theta\left(-\frac{2xy}{r^4}\right)$$

$$\therefore u_{yy} = \frac{y^2}{r^2}u_{rr} + 2\frac{xy}{r^3}u_{r\theta} + \frac{x^2}{r^4}u_{\theta\theta} + \frac{x^2}{r^3}u_r - 2\frac{xy}{r^4}u_\theta$$

$$u_{xx} + u_{yy} = \frac{x^2+y^2}{r^2} u_{rr} + \frac{x^2+y^2}{r^4} u_{\theta\theta} + \frac{x^2+y^2}{r^3} u_r$$

- Laplacian of u in polar coordinates is

$$\therefore \nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

- Laplacian of u in cylindrical coordinate is

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz}$$

19.2 Circular Membrane: Use of Fourier - Bessel Series

- 2D wave equation for a circular membrane

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$

- Radial symmetry

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right)$$

- Boundary and initial conditions

$$u(R,t) = 0 \qquad \text{for all } t \geq 0. \text{ [fixed along bdry]}$$

$$u(r,0) = f(r) \qquad \text{[initial deflection } f(r)\text{]}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r) \qquad \text{[initial velocity } g(r)\text{]}$$

- **Step I**: ODEs, Bessel's Equation.
  - Separation of variables

$$u(r,t) = W(r)G(t)$$

$$W\ddot{G} = c^2 \left(W''G + \frac{1}{r} W'G \right)$$

$$\frac{\ddot{G}}{c^2 G} = \frac{1}{W}\left(W'' + \frac{W'}{r} \right) = -k^2$$

$$\ddot{G} + \lambda^2 G = 0 \qquad \text{where } \lambda = ck$$

$$W'' + \frac{1}{r} W' + k^2 W = 0$$

  - $s = kr \quad \Rightarrow \quad 1/r = k/s$

$$W' = \frac{dW}{dr} = \frac{dW}{ds}\frac{ds}{dr} = \frac{dW}{ds} k \qquad \text{and} \qquad W'' = \frac{d^2W}{ds^2} k^2$$

  - Bessel's equation with $\nu = 0$:

$$\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + W = 0$$

- **Step II**: Satisfying the Boundary Condition

$$W(r) = c_1 J_0(kr) + c_2 Y_0(kr)$$

Y_0 becomes infinite at 0. $c_2 = 0$

$$W(r) = J_0(s) = J_0(kr) \qquad (s = kr)$$

$$u(R,t) = W(R)G(t) = 0 \quad \Rightarrow \quad W(R) = J_0(kR) = 0$$

- J_0 has infinitely many positive zeros, $s = \alpha_1,\ \alpha_2,\ \cdots$, with numerical values

$$\alpha_1 = 2.4048, \quad \alpha_2 = 5.5201, \quad \alpha_3 = 8.6537, \quad \alpha_4 = 11.7915, \quad \alpha_5 = 14.9309, \cdots.$$

$$kR = \alpha_m \qquad \text{thus} \qquad k = k_m = \frac{\alpha_m}{R}, \qquad m = 1,\ 2,\ \cdots.$$

$$W_m(r) = J_0(k_m r) = J_0\left(\frac{\alpha_m}{R} r\right), \qquad m = 1,\ 2,\ \cdots.$$

$$G_m(t) = a_m \cos\lambda_m t + b_m \sin\lambda_m t$$

- Eigenvalues:

$$\lambda = \lambda_m = c k_m = c\frac{\alpha_m}{R}$$

- Eigenfunctions

$$u_m(r,t) = W_m(r)G_m(t) = (a_m \cos\lambda_m t + b_m \sin\lambda_m t) J_0(k_m r)$$

with $m = 1,\ 2,\ \cdots$

- **Step III**: Solution of the Entire Problem

$$u(r,t) = \sum_{m=1}^{\infty} W_m(r)G(t) = \sum_{m=1}^{\infty} (a_m \cos\lambda_m t + b_m \sin\lambda_m t) J_0\left(\frac{\alpha_m}{R} r\right)$$

$$u(r,0) = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\alpha_m}{R} r\right) = f(r)$$

- Fourier-Bessel series

$$a_m = \frac{2}{R^2 J_1^2(\alpha_m)} \int_0^R r f(r) J_0\left(\frac{\alpha_m}{R} r\right) dr \qquad (m = 1,\ 2,\ \cdots)$$

$\because\ ||J_0(k_m x)||^2 = \int_0^R x J^2(k_m x) dx = R^2 J_1^2(\alpha_m)$
r: weight function

Example 1. Variations of a circular membrane
R=1 ft, ρ=2 slugs/ft^2, T=8 lb/ft, $g(r,t)$=0

$$f(r) = 1 - r^2 \text{ (ft)}$$

$c^2 = T/\rho = 8/2 = 4$ ft^2/sec^2, $b_m = 0$.

$$a_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r(1-r^2) J_0(\alpha_m r) dr = \frac{4J_2(\alpha_m)}{\alpha_m^2 J_1^2(\alpha_m)} = \frac{8}{\alpha_m^3 J_1(\alpha_m)},$$

$$\because J_2(\alpha_m) = \frac{2}{\alpha_m} J_1(\alpha_m) - J_0(\alpha_m) = \frac{2}{\alpha_m} J_1(\alpha_m)$$

$$f(r) = 1.108 J_0(2.4048r) - 0.140 J_0(5.5201r) + 0.045 J_0(8.637r) - \cdots$$

$$\lambda_m = ck_m = c\alpha_m/R = 2\alpha_m$$

$$u(r,t) = 1.108 J_0(2.4048r)\cos 4.8097t - 0.140 J_0(5.5201r)\cos 11.0402t$$
$$+0.045 J_0(8.6537r)\cos 17.3075t - \cdots$$

19.3 Laplace's Equation in Cylindrical and Spherical Coordinates. Potential.

- Laplace's equation

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0 \tag{1}$$

- Application: gravitation, electrostatics, heat flow and fluid flow
- Gravitational potential: $u(x,y,z)$

$$u(x,y,z) = \frac{c}{r} = \frac{c}{\sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2}} \qquad (r > 0) \tag{2}$$

$$u(x,y,z) = k \iiint_T \frac{\rho(X,Y,Z)}{r} dX dY dZ \tag{3}$$

$$\therefore \nabla^2 \left(\frac{1}{r}\right) = 0$$

- Laplace's eq. → boundary value problem
(a) First boundary value problem or Dirichlet problem if u is prescribed on S.
(b) Second boundary value problem or Neumann problem if the normal derivative $u_n = \partial u/\partial n$ is prescribed on S.
(c) Third or mixed boundary value problem if u is prescribed on a portion of S and u_n on the remaining portion of S.

Laplacian in Cylindrical Coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

Laplacian in Spherical Coordinates

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}$$

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\cdot\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\right]$$

19.4 Boundary Value Problem in Spherical Coordinates

$$u(R,\theta,\phi) = f(\phi) \tag{4}$$

- $u(r,\phi) \quad \rightarrow \quad u_{\theta\theta} = 0$ (independent of θ)

$$\nabla^2 u = \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\phi}\cdot\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial u}{\partial \phi}\right) = 0 \tag{5}$$

- At infinity the potential will be zero,

$$\lim_{r\to\infty} u(r,\phi) = 0 \tag{6}$$

Solving the Dirichlet Problem (4)**,** (5)**,** (6)

$$u(r,\phi) = G(r)H(\phi)$$

$$\frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = -\frac{1}{H\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) = k$$

$$\frac{1}{\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dH}{d\phi}\right) + kH = 0 \tag{7}$$

$$\frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = k \tag{8}$$

- From (8), $(r^2G')' = kG$ if $k = n(n+1)$, we obtain Euler-Cauchy equation.

$$r^2G'' + 2rG' - n(n+1)G = 0 \tag{9}$$

- Substituting $G = r^a$ into (9)

$$r^2a(a-1)r^{a-2} + 2rar^{a-1} - n(n+1)r^a = 0$$

$$a^2 - a - n(n+1) = 0 \quad \Rightarrow \quad a = n, \quad \text{and} \quad a = -n-1$$

$$\therefore\ G_n(r) = r^n \quad \text{and} \quad G_n^*(r) = \tfrac{1}{r^{n+1}}$$

- Setting $\cos\phi = w$, $\sin^2\phi = 1 - w^2$.

$$\frac{d}{d\phi} = \frac{d}{dw} \cdot \frac{dw}{d\phi} = -\sin\phi \frac{d}{dw}$$

- Then (7) becomes **Legendre's equation** with $k = n(n+1)$,

$$\frac{d}{dw}\left[(1-w^2)\frac{dH}{dw}\right] + n(n+1)H = 0 \tag{10}$$

$$\text{or, } (1-w^2)\frac{d^2H}{dw^2} - 2w\frac{dH}{dw} + n(n+1)H = 0$$

Solution Using a Fourier-Legendre Series

- For integer $n = 0,\ 1,\ \cdots$, the Legendre polynominals

$$H = P_n(w) = P_n(\cos\phi) \qquad n = 0,\ 1,\ \cdots,$$

are solutions of Legendre's equation (10).

$$u_n(r,\phi) = A_n r^n P_n(\cos\phi), \quad u_n^*(r,\phi) = \frac{B_n}{r^{n+1}} P_n(\cos\phi) \qquad n = 0,\ 1,\ 2,\ \cdots.$$

- Solution of the Interior Problem

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi).$$

Fourier-Legendre series of $f(\phi)$:

$$u(R,\phi) = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\phi) = f(\phi);$$

$$A_n R^n = \frac{2n+1}{2}\int_{-1}^{1} \tilde{f}(w) P_n(w) dw$$

where $\tilde{f}(w)$ denotes $f(\phi)$ as a function of $w = \cos\phi$.

$$\begin{aligned} dw &= -\sin\phi\ d\phi \quad \phi = \pi \to w = -1 \\ dw &= -\sin\phi\ d\phi \quad \phi = 0 \to w = 1 \end{aligned}$$

$$A_n = \frac{2n+1}{2R^n}\int_0^{\pi} f(\phi) P_n(\cos\phi)\sin\phi\ d\phi, \qquad n = 0,\ 1,\ 2,\ \cdots. \tag{11}$$

$\sin\phi$: weight function

- Solution of the Exterior Problem

$$u(r,\phi) = \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\phi) \qquad (r \geq R)$$

with coefficients

$$B_n = \frac{2n+1}{2} R^{n+1} \int_0^{\pi} f(\phi) P_n(\cos\phi)\sin\phi\ d\phi \tag{12}$$

Example . Spherical Capacitor

$$R = 1 \text{ ft}, \qquad f(\phi) = \begin{cases} 110 & \text{if} \quad 0 \le \phi \le \pi/2 \\ 0 & \text{if} \quad \pi/2 \le \phi \le \pi \end{cases}$$

- Since R=1, from (11)

$$A_n = \frac{2n+1}{2} \cdot 110 \int_0^{\pi/2} P_n(\cos\phi) \sin\phi d\phi$$

- Set $w = \cos\phi$, $P_n(\cos\phi)\sin\phi d\phi = -P_n(w)dw$ and we integrate from 1 to 0.

$$A_n = 55(2n+1) \int_1^0 -P_n(w)dw = 55(2n+1) \int_0^1 P_n(w)dw$$

- From (11) in Sec. 4.3

$$A_n = 55(2n+1) \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m!)} \int_0^1 w^{n-2m} dw$$

where $M = n/2$ and for even n and $M = (n-1)/2$ for odd n.

$$\int_0^1 w^{n-2m} dw = \frac{1}{(n-2m+1)}$$

$$A_n = \frac{55(2n+1)}{2^n} \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{m!(n-m)!(n-2m+1)!} \tag{13}$$

- For $n = 1, \ 2, \ 3, \ \cdot,$

$$\begin{aligned} A_0 &= 55 \\ A_1 &= \frac{165}{2} \cdot \frac{2!}{0!1!2!} = \frac{165}{2} \\ A_2 &= \frac{275}{4} \left(\frac{4!}{0!2!3!} - \frac{2!}{1!1!1!} \right) = 0 \\ A_3 &= \frac{385}{8} \left(\frac{6!}{0!3!4!} - \frac{4!}{1!2!2!} \right) = -\frac{385}{8} \end{aligned}$$

- Potential inside the sphere,

$$u(r,\phi) = 55 + \frac{165}{2} r P_1(\cos\phi) - \frac{385}{8} r^3 P_3(\cos\phi) + \cdots \tag{14}$$

- Since R=1, from (11) and (12) $B_n = A_n$. Outside the sphere,

$$u(r,\phi) = \frac{55}{r} + \frac{165}{2r^2} P_1(\cos\phi) - \frac{385}{8r^4} P_3(\cos\phi) + \cdots \tag{15}$$

19.5 Solution by Laplace Transforms

Example . Semi-infinite string
(i) The string is initially at rest on the x-axis from $x = 0$ to ∞.
(ii) For time $t > 0$,

$$w(0,t) = f(t) = \begin{cases} \sin t & \text{if } 0 \le t \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$

(iii) Furthermore,

$$\lim_{x\to\infty} w(x,t) = 0 \qquad \text{for } t \ge 0.$$

Solution.
- Wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \qquad c^2 = \frac{T}{\rho}$$

- Boundary conditions

$$w(0,t) = f(t), \qquad \lim_{x\to\infty} w(x,t) = 0$$

- Initial condtions

$$w(x,0) = 0, \qquad \left.\frac{\partial w}{\partial t}\right|_{t=0} = 0$$

- Take the Laplace transform with respect to t,

$$Ł\left(\frac{\partial^2 w}{\partial t^2}\right) = s^2 Ł(w) - sw(x,0) - \left.\frac{\partial w}{\partial t}\right|_{t=0} = c^2 Ł\left(\frac{\partial^2 w}{\partial x^2}\right)$$

$$Ł\left(\frac{\partial^2 w}{\partial x^2}\right) = \int_0^\infty e^{-st}\frac{\partial^2 w}{\partial x^2}dt = \frac{\partial^2}{\partial x^2}\int_0^\infty e^{-st}w(x,t)dt = \frac{\partial^2}{\partial x^2}Ł\{w(x,t)\}$$

- Writing $W(x,s) = Ł\{w(x,t)\}$,

$$s^2 W = c^2\frac{\partial^2 W}{\partial x^2}, \qquad \text{thus} \qquad \frac{\partial^2 W}{\partial x^2} - \frac{s^2}{c^2}W = 0.$$

$$\begin{aligned} W(x,s) &= A(s)e^{sx/c} + B(s)e^{-sx/c} \\ W(0,s) &= Ł\{w(o,t)\} = Ł\{f(t)\} = F(s) \end{aligned}$$

$$\lim_{x\to\infty} W(x,s) = \lim_{x\to\infty}\int_0^\infty e^{-st}w(x,t)dt = \int_0^\infty e^{-st}\lim_{x\to\infty} w(x,t)dt = 0$$

$$\therefore A(s) = 0$$

$$\begin{aligned} W(0,s) &= B(s) = F(s) \\ W(x,s) &= F(s)e^{-sx/c} \\ w(x,t) &= f\left(t - \frac{x}{c}\right)u\left(t - \frac{x}{c}\right) \end{aligned}$$

$$w(x,t) = \sin\left(t - \frac{x}{c}\right) \quad \text{if} \quad \frac{x}{c} < t < \frac{x}{c} + 2\pi \quad \text{or } ct > x > (t - 2\pi)c \quad \sharp.$$