22 Analytic Functions

22.1 Mapping

$$w = f(z) = u(x,y) + iv(x,y) \qquad (z = x + iy)$$

f : mapping of D into w-plane.
$w_0 = f(z_0)$: the image of z_0 with respect to f.

Example 1. Mapping $w = z^2$
i) polar coordinates $z = r(\cos\theta + i\sin\theta)$

$$w = R(\cos\phi + i\sin\phi) = z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

moduli $R = r^2$, arguments $\phi = 2\theta$.

$$\left(\begin{array}{c} 1 \le |z| \le 3/2 \\ \pi/6 \le \theta \le \pi/3 \end{array}\right. \Rightarrow \left(\begin{array}{c} 1 \le |w| \le 9/4 \\ \pi/3 \le \theta \le 2\pi/3. \end{array}\right.$$

ii) Cartesian coordinates $z = x + iy$

$$u = Re(z^2) = x^2 - y^2, v = Im(z^2) = 2xy.$$
$$x = c = const.u = c^2 - y^2.v = 2cy.v^2 = 4c^2(c^2 - a).$$
$$y = k = const. \Rightarrow v^2 = 4k^2(k^2 + u)$$

22.2 Conformal Mapping.

Definition : Conformal Mapping is a mapping that preserves angles between any oriented curves both in magnitude and in sense.

- Oriented curve :

  $$C : z(t) = x(t) + iy(t) : \text{ parametric representation}$$

  smooth C : $z(t)$ is differentiable and the derivative $\dot{z} = dz/dt$ is continuous and nowhere zero.
  positive sense : increasing values of the parameter t.

- angle of intersection between two curves C_1 and C_2.
  : the angle between the oriented tangents at the intersection point.

Example . circle : $x^2 + y^2 = 1. \quad \Rightarrow x = \cos t, y = \sin t.$

$$\therefore z(t) = \cos t + i \sin t.$$

Theorem 1 (Conformality of mapping by analytic functions).
The mapping defined by an analytic function $f(z)$ is conformal except at critical points, i.e. points at critical points (i.e. points where the derivative $f'(z)$ is zero).
Proof. $\dot{z}(t) = dz/dt = \dot{x}(t) + i\dot{y}(t)$ is tangent to C. $\quad \left(\because \lim_{\Delta t} \frac{z_1 - z_0}{\Delta t}\right)$
Let C^* be the image of C. Then

$$w = f(z(t)) \quad \Rightarrow \dot{w} = \frac{d}{dt}(f(z(t))) = f'(z(t))\dot{z}(t)$$

So tangent direction of C^* is

$$\arg \dot{w} = \arg f' + \arg \dot{z}$$

$\arg \dot{z}$ is the tangent direction of C. Therefore, the mapping rotates all directions at a point z_0 in the domain of analyticity of f through the same angle $\arg f'(z_0)$, which exists as long as $f'(z_0) \neq 0$. $\sharp$

Example 2. Conformality of $w = z^n$, $n = 2, 3, \cdots$?
conformal except at $z = 0$, because .

$$w' = nz^{n-1}, \qquad \text{at } z = 0, w' = 0$$

Example 3. Mapping $x = z + 1/z$. Joukowski airfoil.
In polar coord,

$$\begin{aligned} w &= u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) \\ &\Rightarrow u = a\cos\theta, \quad v = b\sin\theta, \\ &\quad\; a = (r + \frac{1}{r}), \quad b = (r - \frac{1}{r}) \end{aligned}$$

circles in z-plane :

$$|z| = r = const \neq 1 \quad \rightarrow \quad u^2/a^2 + v^2/b^2 = 1 \text{ ellipses in } w\text{-plane}$$

$$|z| = 1 \quad \rightarrow \quad w = 2\cos\theta, \; i.e. \; -2 \leq u \leq 2 \text{ of } u\text{-axis.}$$

Since

$$w' = 1 - \frac{1}{z^2} = \frac{(z-1)(z+1)}{z^2} = 0 \qquad z = \pm 1,$$

the mapping not conformal at $z = \pm 1$. $\sharp$

22.3 Exponential Function.

(1) $e^z = e^x(\cos y + i\sin y)$

properties :
(a) e^z should reduce to the real e^x, when$z = x$ is real.
(b) e^z should be an entire function, i.e. analytic for all z.
(c) its derivative is
(2) $(e^z)' = e^z$

proof)
(a) $y = 0, \cos 0 = 1 \& \sin 0 = 0, e^z = e^x$.
(b) $u = e^x \cos y, v = e^x \sin y$.

$$\frac{\partial u}{\partial x} = e^x \cos y, \frac{\partial v}{\partial y} = e^x \cos y \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -e^x \sin y, \frac{\partial v}{\partial x} = e^x \sin y \Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(c) $(e^z)' = (e^x \cos y)_x + i(e^x \sin y)_x = e^x \cos y + ie^x \sin y = e^z$

Further Properties.
(3) functional relation : $e^{z_1+z_2} = e^{z_1}e^{z_2}$ for $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

$$\begin{aligned} e^{z_1}e^{z_2} &= e^{x_1}(\cos y_1 + i\sin y_1)e^{x^2}(\cos y_2 + i\sin y_2) \\ &= e^{x_1+x_2}[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)] \\ &= e^{x_1+x_2}[\cos(y_1+y_2) + i\sin(y_1+y_2)] = e^{z_1+z_2} \end{aligned}$$

(4) $e^z = e^x \cdot e^{iy}$ by setting $z_1 = x$ & $z_2 = iy$
(5) Euler formula : $e^{iy} = \cos y + i\sin y$
(6) polar form : $x = re^{i\theta}$
(7) $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$
(8)

$$\begin{aligned} &e^{\pi i/2} = \cos \pi/2 + i\sin \pi/2 = i \\ &e^{\pi i} = \cos \pi + i\sin \pi = -1 \\ &e^{-\pi i/2} = \cos(-\pi/2) + i\sin(-\pi/2) = -i \\ &e^{-\pi i} = \cos(-\pi) + i\sin(-\pi) = -1 \end{aligned}$$

From (5)
(9) $|e^{iy}| = |\cos y + i\sin y| = (\cos^2 y + \sin^2 y)^{1/2} = 1$
From (9) and (1)
(10) $|e^z| = e^x$, $\arg e^z = y \pm 2n\pi$ $(n = 0, 1, 2, \cdots .)$
(11) $e^z \neq 0$ for all z.

Periodicity of e^z with period $2\pi i$
(12) $e^{z+2\pi i} = e^z$ for all z
(13) fundamental region of e^z : $-\pi < y \leq \pi$.
: e^z maps a fundamental region bijectively onto the entire plane.

$0 < y \leq \pi$ of the fundamental strip $\rightarrow$ mapped onto the upper half-plane.
$0 < \arg w \leq \pi$

$$\left(\begin{array}{l} \text{left half of the strip goes insede the unit disk } |w| \leq 1, \ (\because e^x \leq 1 \text{ for } x \leq 0) \\ \text{right half goes outside } |x| > 1. \end{array} \right.$$

Example 1. Function values. Solution of equations.
(1) $e^z = e^x(\cos y + i \sin y)$

$$e^{1.4-0.6i} = e^{1.4}(\cos 0.6 - i \sin 0.6) = 4.055(0.825 - 0.565i) = 3.347 - 2.290i.$$
$$|e^{1.4-0.6i}| = e^{1.4} = 4.055 \arg e^{1.4-0.6i} = -0.6$$

(3) $e^{z_1+z_2} = e^{z_1} \cdot z^{z_2}$

$$e^{2+i} = e^2(\cos 1 + i \sin 1) \& e^{4-i} = e^4(\cos 1 - i \sin 1)$$
$$e^2 \cdot e^4(\cos^2 1 + \sin^2 1) = e^6 = e^{(2+i)+(4-i)}$$

Equation $e^z = 3 + 4i = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

$$e^x = |e^z| = 5, x = \ln 5 = 1.609$$
$$\left(\begin{array}{l} e^x \cos y = 3 \\ e^x \sin y = 4 \end{array} \right) \Rightarrow \left(\begin{array}{l} \cos y = 0.6 \\ \sin y = 0.8 \end{array} \right) y = 0.927$$

Ans. $z = 1.609 + 0.927i \pm 2n\pi i \ (n = 0, 1, 2, \cdots)$

22.4 Trigonometric Functions. Euler formulas.

$$e^{ix} = \cos x + i \sin x, \qquad e^{-ix} = \cos x - i \sin x$$
$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \quad \& \quad \sin z = \frac{1}{2i}(e^{ix} - e^{-ix})$$

(1)
$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \qquad \sin \frac{1}{2i}(e^{iz} - e^{-iz})$$

(2)
$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}$$

(3)

$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}$$

(4)

$$(\cos z)' = -\sin z, \qquad (\sin z)' = \cos z, \qquad (\tan z)' = \sec^2 z$$

Euler's formula

(5)

$$e^{iz} = \cos z + i \sin z. \qquad\qquad \text{for all } z.$$

Example 1. Real and Imaginary parts. Absolute value. Periodicity

(6)

$$\text{(a)} \cos \mathrm{z} = \cos \mathrm{x} \cosh \mathrm{y} - \mathrm{i} \sin \mathrm{x} \sinh \mathrm{y}.$$
$$\text{(b)} \sin \mathrm{z} = \sin \mathrm{x} \cosh \mathrm{y} + \mathrm{i} \cos \mathrm{x} \sinh \mathrm{y}.$$

(7)

$$\text{(a)} |\cos \mathrm{z}|^2 = \cos^2 \mathrm{x} + \sinh^2 \mathrm{y}.$$
$$\text{(b)} |\sin \mathrm{z}|^2 = \sin^2 \mathrm{x} + \sinh^2 \mathrm{y}.$$

Solution)

$$\begin{aligned}
\cos z &= \frac{1}{2}[e^{i(x+iy)} + e^{-i(x+iy)}] \\
&= \frac{1}{2}e^{-y}(\cos x + i \sin x) + \frac{1}{2}e^{y}(\cos x - i \sin x) \\
&= \frac{1}{2}(e^y + e^{-y}) \cos x - \frac{1}{2}i(e^y - e^{-y}) \sin x.
\end{aligned}$$

(8)

$$\cosh y = \frac{1}{2}(e^y + e^{-y} \qquad\qquad \sinh y = \frac{1}{2}(e^y - e^{-y}$$
$$\therefore \cos z = \cos x \cosh y - i \sin x \sinh y$$

$$\begin{aligned}
\therefore \sin z &= \frac{1}{2i}[e^{i(x+iy)} - e^{-i(x+iy)}] \\
&= \frac{1}{2i}e^{-y}(\cos x + i \sin x) - \frac{1}{2i}e^{y}(\cos z - i \sin x) \\
&= \frac{1}{2}(e^y + e^{-y}) \sin x + \frac{1}{2}i(e^y - e^{-y}) \cos x
\end{aligned}$$

$$\therefore \sin z = \sin x \cosh y + i \cos x \sinh y$$

$$\text{since} \therefore \cosh^2 \mathrm{y} - \sinh^2 \mathrm{y} = \frac{1}{4}(\mathrm{e}^{2\mathrm{y}} + \mathrm{e}^{-2\mathrm{y}} + 2 - \mathrm{e}^{-2\mathrm{y}} + 2) = 1$$

From(6a)

$$\begin{aligned}
|\cos z|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\
&= \cos^2 x(1 + \sinh^2 y) + \sin^2 x \sinh^2 y \\
&= \cos^2 x + \sinh^2 y
\end{aligned}$$

From(6b)

$$\begin{aligned}
|\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
&= \sin^2 x(1+\sinh^2 y) + \cos^2 x \sin^2 y \\
&= \sin^2 x + \sinh^2 y
\end{aligned}$$

For instance $\quad \cos(2+3i) = \cos 2 \cdot \cosh 3 - i \sin 2 \cdot \sinh 3 = -4.190 - 9.109i$

$$\cos z \quad \& \quad \sin z \text{ are periodic with period } 2\pi.$$

$$\tan z \quad \& \quad \cot z \text{ are periodic with period } 2\pi.$$

Example 2. Solution of equations. zeros of $\cos z$ and $\sin z$.

(a) $\cos z = 5$

$$5 = \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$

$$\therefore e^{2iz} - 10e^{iz} + 1 = 0$$

$$e^{iz} = e^{-y+ix} = 5 \pm \sqrt{25-1} = 9.899 \text{ and } 0.101$$

$$\therefore e^{-y} = 9.899 \text{ or } 0.101, \qquad e^{ix} = 1$$

$$x = 2n\pi \qquad y = \pm 2.292i.$$

$$\text{Ans. } z = \pm 2n\pi \pm 2.292i$$

(b) $\cos z = 0$
$\Rightarrow \cos x = 0 \quad \& \quad \sinh y = 0$

$$\text{Ans. } z = \pm\frac{1}{2}(2n+1)\pi \qquad (n = 0, 1, 2, \cdots)$$

(c) $\sin z = 0$
$\Rightarrow \sin x = 0 \quad \& \quad \sinh y = 0$

$$\text{Ans. } z = \pm 2n\pi \qquad (n = 0, 1, 2, \cdots)$$

General formulas
(9)

$$\cos(z_1 \pm z_2) = \cos z_1 \cdot \cos z_2 \mp \sin z_1 \cdot \sin z_2$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cdot \cos z_2 \pm \sin z_2 \cdot \cos z_1$$

(10)

$$\cos^2 z + \sin^2 z = 1$$

$$\text{pf) } \cos^2 z + \sin^2 z = \frac{1}{4}(e^{2iz} + 2 + e^{-2iz}) - \frac{1}{4}(e^{2iz} - 2 + e^{-2iz}) = 1$$