25 Cauchy's Integral Formula, Derivatives of Analytic Functions

25.1 Independence of Path

Theorem 2 (Independence of path)
If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D.

Principle of Deformation of Path.
Hence we may impose a continuous deformation of the path of an integral, keeping the ends fixed. As long as our deforming path always contains only points at which $f(z)$ is analytic, the integral retains the same value. This is called the principle of path.

Theorem 3 (Existence of an infinite integral)
If $f(z)$ is analytic in a simply connected domain D, then there exists an indefinite integral $F(z)$ of $f(z)$ in D-thus, $F'(z) = f(z)$- which is analytic in D, and for al paths in D joining any two points z_0 and z_1 in D, the integral of $f(z)$ from z_0 to z_1 can be evaluated by

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0). \quad [F'(z) = f(z)]$$

proof) If $f(z)$ is analytic in a simply connected domain D, then the integral of $f(z)$ is independent of path in D

$$F(z) = \int_{z_0}^{z_1} f(z^*)dz^*,$$

which is uniquely determined. We show that this $F(z)$ is analytic in D and $F'(z) = f(z)$

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(z^*)dz^* - \int_{z_0}^{z} f(z^*)dz^*\right] = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^*)dz^*$$

$$(4) - f(z) : \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z^*)dz^* - f(z) \qquad ---a)$$

Show that R.H.S approaches zero as $\Delta z \to 0$
$f(z)$ is a constant because z is kept fixed

$$\int_{z}^{z+\Delta z} f(z)dz^* = f(z)\int_{z}^{z+\Delta z} dz^* = f(z)\Delta z.$$

$$\text{Thus} \qquad f(z) = \frac{1}{\Delta z}\int_z^{z+\Delta z} f(z)dz^* \qquad ---b)$$

$$b) \to a) \qquad \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_z^{z+\Delta z}[f(z^*) - f(z)]dz^*$$

Since $f(z)$ is analytic, it is continuous. An $\varepsilon > 0$ being given, we can thus find a $\delta > 0$ such that $|f(z^*) - f(z)| < \varepsilon$ when $|z^* - z| < \delta$. Hence, letting $|\Delta z| < \delta$, we see that the ML-inequality yields

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| = \frac{1}{\Delta z}\left|\int_z^{z+\Delta z}[f(z^*) - f(z)]dz^*\right| \le \frac{1}{|\Delta z|}\varepsilon|\Delta z| = \varepsilon.$$

By the definition of limit and derivative,

$$F'(z) = \lim_{\Delta z \to 0}\frac{F(+\Delta z) - f(z)}{\Delta z} = f(z)$$

Since z is any point in D, this implies that $F(z)$ is analytic in D and is an indefinite integral on antiderivative of $f(z)$ in D, written

$$F(z) = \int f(z)dz$$

Also of $G'(z) = f(z)$, then $F'(z) - G'(z) \equiv 0$ in D : hence $F(z) - G(z)$ is constant in D. Two indefinite integrals of $f(z)$ can differ only by a constant. This proves theorem.

Cauchy's Theorem for Multiply Connected Domains.
For a doubly connected domain D

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$$

Proof)

$$D_1 : \int_{c_{1_0}} f(z)dz + \int_{\tilde{c_2}} f(z)dz + \int_{c_2^*} f(z)dz + \int_{\tilde{c_2}} f(z)dz = 0 \qquad ---1)$$

since $f(z)$ is analytic in D_1

$$D_2 : \int_1^* f(z)dz - \int_{\tilde{c_2}} f(z)dz + \int_{c_{2_0}} f(z)dz - \int_{\tilde{c_1}} f(z)dz = 0 \qquad ---2)$$

$$1)+2); \quad \int_{z_{1_0}} f(z)dz + \int_{c_{1^*}} f(z)dz + \int_{c_{2^*}} f(z)dz + \int_{c_{2_0}} f(z)dz = 0$$

$$C_{1_0} + C_{1^*} = C_1(ccw), \qquad C_{2^*} + C_{2_0} = C_2(cw)$$

$$\int_{c_1} f(z)dz - \int_{c_2} f(z)dz = 0 \qquad \text{in both ccw.}$$

$$\therefore \int_{c_1} f(z)dz = \int_{c_2} f(z)dz.$$

Example A basic result : Integral of integer power.

$$\oint (z - z_0)^m dz = \begin{cases} 2\pi i & (m = -1) \\ 0 & (m \neq -1 \text{ and integer}) \end{cases}$$

for ccw integration around any simple closed path containing z_0 in its interior.

25.2 Cauchy's Integral Formula

Theorem 1 (Cauchy's integral formula).
Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0,
(1)

$$\oint_c \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) \quad \text{(Cauchy's integral formula)}$$

the integration being taken ccw.
(1*)

$$f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz \quad \text{(Cauchy's integral formula)}$$

Proof)

$$f(z) = f(z_0) + [f(z) - f(z_0)]$$

(2)

$$\oint_c \frac{f(z)}{z - z_0} dz = f(z_0) \oint_c \frac{dz}{z - z_0} + \oint_c \frac{f(z) - f(z_0)}{z - z_0} dz$$

$$\oint \frac{dz}{z - z_0} = 2\pi i \quad \text{(Example 6 in sec.13.2)}$$

1^{st} term on the R.H.S

$$\therefore f(z_0) \oint_c \frac{dz}{z - z_0} = 2\pi i f(z_0)$$

C is replaced by a small circle k of radius ρ by the principle of deformation of path . Hence an $\varepsilon > 0$ being given, we can find a $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ for all z in the disk $|z - z_0| < \delta$.

$$\therefore \left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \frac{\varepsilon}{\rho}$$

By the ML-inequality,

$$\left| \oint_k \frac{f(z) - f(z_0)}{z - z_0} dz \right| < \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon$$

Since $\varepsilon (> 0)$ can be chosen arbitrarily small, it follows that the above integral must have the value zero.

Example 1 Cauchy's integral formula.

$$\oint_c \frac{e^z}{z-2}dz = 2\pi i e^z|_{z=2} = 2\pi i e^2 \approx 46.4268i$$

for any contour enclosing $z_0 = 2$.

Example 2 Cauchy's integral formula.

$$\oint_c \frac{z^3-6}{2z-i}dz = \oint_c \frac{\frac{1}{2}z^3-3}{z-\frac{i}{2}}dz - 2\pi i(\frac{1}{2}z^3-3)|_{z=i/2} = \pi/8 - 6\pi i \quad (z_0 = i/2 \text{ inside } C)$$

Example 3 Integration around different contours.

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z-1)(z+1)}$$

Solution)
(a) circle $|z-1| = 1$, encloses $z_0 = 1$

$$g(z) = \frac{z^2+1}{z^2-1} = \frac{z^2+1}{z+1}\cdot\frac{1}{z-1} \;;\; f(z) = \frac{z^2+1}{z+1}$$

(b) gives the same as (a) by the principle of deformation of path
(c) $z_0 = -1$

$$g(z) = \frac{z^2+1}{z-1}\cdot\frac{1}{z+1} \quad : \quad \text{thus } f(z) = \frac{z^2+1}{z-1}$$

(d) 0. $g(z)$ is analytic

Example 4 use of partial fractions.

$$g(z) = \frac{\tan z}{z^2-1} \qquad : \text{ the circle } C : |z| = 3/2 \text{ (ccw)}$$

Solution) $\tan z$ is not analytic at $\pm\pi/2, \pm 3\pi/2, \cdots$, but all these points lie outside the contour.

$$(z^2-1)^{-1} = 1/(z-1)(z+1) \text{ is not analytic at 1 and -1}$$

$$\frac{1}{z^2-1} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$

$$\begin{aligned}\oint \frac{\tan z}{z^2-1}dz &= \frac{1}{2}\left[\oint \frac{\tan z}{z-1}dz - \oint \frac{\tan z}{z+1}dz\right] \\ &= \frac{1\pi i}{2}[\tan 1 - \tan(-1)] = 2\pi i \tan 1 \approx 9.785i\end{aligned}$$

Multiply connected domain.
For instance, if $f(z)$ is analytic on C_1 and C_2 and in the ring-shaped domain bounded by C_1 and C_2 and z_0 is any point in that domain, then
(3)

$$f(z_0) = \frac{1}{2\pi i}\oint_{c_1} \frac{f(z)}{z-z_0}dz + \frac{1}{2\pi i}\oint_{c_2} \frac{f(z)}{z-z_0}dz,$$

where the outer integral C_1 is taken ccw and the inner clockwise.

25.3 Derivatives of Analytic Functions.

Theorem 1 (Derivatives of an analytic function)
If $f(z)$ is analytic in a domain D, then it has derivatives of all orders in D, which are then also analytic functions in D. The values of derivatives at a point z_0 in D are given by the formulas
(1')

$$f'(z_0) = \frac{1}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^2}dz,$$

(1")

$$f''(z_0) = \frac{2}{2\pi i}\oint_c \frac{f(z)}{(z-z_0}^{3}dz,$$

and in general
(1)

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_c \frac{f(z)}{(z-z_0)^{n+1}}dz \qquad (n = 1, 2, \cdots);$$

here C is any simple closed path in D that encloses z_0 and whose full interior belongs to D.
proof)

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

By Cauchy's integral formula ;

$$\begin{aligned}\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} &= \frac{1}{2\pi i \Delta z}\left[\oint \frac{f(z)}{z-(z_0+\Delta z)}dz - \oint \frac{f(z)}{z-z_0}dz\right] \\ &= \frac{1}{2\pi i \Delta z}\oint \frac{f(z)\{z-z_0-[z-(z_0+\Delta z)]\}}{[z-(z_0+\Delta z)][z-z_0]}dz\end{aligned}$$

$$\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}dz$$

We consider the difference between these two integrals.

$$\begin{aligned}\oint_c \frac{f(z)}{(z-z_0-\Delta z)(z-z_0)}dz - \oint_c \frac{f(z)}{(z-z_0)^2}dz &= \oint_c \frac{f(z)[z-z_0-(z-z_0-\Delta z)]}{(z-z_0-\Delta z)(z-z_0)^2}dz \\ &= \oint \frac{f(z)\Delta z}{(z-z_0-\Delta z)(z-z_0)^2}dz\end{aligned}$$

Being analytic, the function $f(z)$ is continuous on C, hence bounded in absolute value, $|f(z) \leq K$, Let d be the smallest distance from z_0 to the points of C.

$$|z - z_0|^2 \geq d^2, \qquad \text{hence} \qquad \frac{1}{|z - z_0|^2} \leq \frac{1}{d^2}$$

By the triangle inequality,

$$d \leq |z - z_0| = |z - z_0 - \Delta z + \Delta z| \leq |z - z_0 - \Delta z| + |\Delta z|$$

let $|\Delta z| \leq d/2$, so that $-|\Delta z| \geq -d/2$

$$\left| \oint_c \frac{f(z)\Delta z}{(z - z_0 - \Delta z)(z - z_0)^2} dz \right| \leq KL|\Delta z| \cdot \frac{1}{d} \cdot \frac{1}{d^2}$$

This approaches zero as $\Delta z \to 0$

Example 1 Evaluation of line integrals.
for any contour enclosing the point πi (ccw)

$$\oint_c \frac{\cos z}{(z - \pi i)^2} dz = 2\pi i (\cos z)'|_{z=\pi i} = -2\pi i \sin \pi i = 2\pi \sinh \pi$$

Example 2 for any contour enclosing the point $-i$ (ccw)

$$\begin{aligned} \oint_c \frac{z^4 - 3z^2 + 6}{(z+i)^3} dz &= \pi i (z^4 - 3z^2 + 6)''|_{z=-i} = \pi i (4z^3 - 6z)'|_{z=-i} \\ &= \pi i (12z^2 - 6)|_{z=-i} = \pi i(-12 - 6) = -18\pi i \end{aligned}$$

Example 3 for any contour for which 1 lies inside and $\pm 2i$ lie outside (ccw)

$$\begin{aligned} \oint \frac{e^z}{(z-1)^2(z^2+4)} dz &= 2\pi i \left(\frac{e^z}{z^2+4} \right)' \bigg|_{z=1} = 2\pi i \frac{e^z(z^2+4) - e^z(2z)}{(z^2+4)^2} \bigg|_{z=1} \\ &= 2\pi i \frac{e(5) - e(2)}{25} = \frac{6e\pi}{25} i \approx 2.050i \end{aligned}$$

25.4 Cauchy's Inequality. Liouville's and Morera's Theorems.

Choose for C a circle of radius r and center z_0 with $|f(z)| \leq M$ on C

$$|f^{(n)}(z_0)| = \frac{n!}{2\pi} \left| \oint \frac{f(z)}{(z - z_0)^{n+1}} dz \right| \leq \frac{n!}{2\pi} M \cdot \frac{1}{r^{n+1}} 2\pi r$$

(2)

$$|f^{(n)}(z_0)| \leq \frac{n! M}{r^n} \qquad : \text{Cauchy's inequality}$$

Theorem 2 Liouville's theorem

If an entire function $f(z)$ is bounded in absolute value for all z, then $f(z)$ must be a constant.
proof) By assumption, $|f(z)|$ is bounded, say, $|f(z)| < k$ for all z. Using Cauchy's inequality, $|f'(z_0)| < k/r$. Since $f(z)$ is entire, this is true for every r, so that we can take r as large as we please and conclude that $f'(z_0) = 0$. Since z_0 is arbitrary, $f'(z) = 0$ for all z, and $f(z)$ is constant.

Theorem 3 Morera's theorem (Converse of Cauchy's integral theorem)
If $f(z)$ is continuous in a simply connected domain D and if

$$\oint_c f(z)dz = 0$$

for every closed path in D, then $f(z)$ is analytic in D.
proof) If $f(z)$ is analytic in D, then

$$F(z) = \int_{z_0}^{z} f(z^*)dz^*$$

is analytic in D and $F'(z) = f(z)$. In the proof we used only the continuity of $f(z)$ and the property that its integral around every closed path in D is zero ; from these assumptions we conclude that $F(z)$ is analytic. By theorem 1, the derivative of $F(z)$ is analytic, that is $f(z)$ is analytic in D.