26 Series, Convergence Tests

- Complex Series / Convergence Tests:

Very similar to real series / convergence tests.

- Sequence $\{Z_n\}$ Converges to C, i.e.

$$\lim_{n\to\infty} Z_n = C \qquad (\text{or}, Z_n \to C)$$

if for every $\varepsilon > 0$, We can find N such that,

$$|Z_n - C| < \varepsilon \quad \text{for all } n > N.$$

Theorem

$Z_n = x_n + iy_n$ (n=1,2,3,......) $\rightarrow$ C = a + ib

if $x_n \rightarrow$ a and $y_n \rightarrow$ b

- Series $\displaystyle\sum_{m=1}^{\infty} Z_m$

n_{th} partial Sum $S_n = \sum_{m=1}^{n} Z_m$.
Reminder after the term Z_n : $R_n = Z_{n+1} + Z_{n+2} + \ldots.$

- Convergent Series: Partial sum converges. i.e.

$$\lim_{n\to\infty} S_n = S = \sum_{m=1}^{n} Z_m$$

(then, $R_n \to 0$)

Theorem

$$\text{if } Z_1 + Z_2 + \ldots.. \text{ converges, then } \lim_{m\to\infty} Z_m = 0$$

$$\text{(if } \lim_{m\to\infty} Z_m \neq 0 \text{ , then the series diverges)}$$

Question. Is $Z_m \to 0$, sufficient for convergence of the series ?

Theorem

$$Z_1 + Z_2 + \ldots.. \text{ convergent if for every given } \varepsilon > 0.$$

We can find N (may depend on ε) such that,

$| Z_{n+1} + Z_{n+2} + \ldots\ldots + Z_{n+p} | < \varepsilon$ for every $n > N$ and $p =$ 1,2,3,......

(Cauchy's Convergence principle).

- $Z_1 + Z_2 + \ldots..$ absolutely convergent,

$$\text{if } \sum_{m=1}^{\infty} |Z_m| \text{ Convergent.}$$

- Conditionally Convergent if $Z_1 + Z_2 + \ldots..$ converges,

but $|Z_1| + |Z_2| + \ldots..$ is divergent.

- Absolute Convergent $\Longrightarrow$ Convergent.

Theorem (Comparison Test)

$Z_1 + Z_2 + \ldots..$ given.

$b_1 + b_2 + \ldots..$ Converges, b_n non-negative real,

$|Z_n| \leq b_n$ for n = 1,2,3,......

then $Z_1 + Z_2 + \ldots..$ Converges (absolutely)

Theorem (Geometric Series)

$$\sum_{m=0}^{\infty} q^m \text{ Converges to } \frac{1}{1-q} \text{ if } |q| < 1 \text{ and diverges if } |q| \geq 1.$$

Theorem (Ratio Test)

if $Z_n \neq 0$ (n = 1,2,3,....) and

$$\left|\frac{Z_{n+1}}{Z_n}\right| \leq \text{q} < 1 \text{ (for all } n > N \text{, q is fixed)}$$

then $Z_1 + Z_2 +$ ..... Convergent absolutely.

$$\text{if for every } n > N, \left|\frac{Z_{n+1}}{Z_n}\right| \geq 1. \text{ Series is divergent.}$$

Theorem (Ratio Test)

$$Z_n \neq 0. \quad \lim_{n\to\infty}\left|\frac{Z_{n+1}}{Z_n}\right| = \text{L}$$

(i) $Z_1 + Z_2 +$ ..... Convergent absolutely if L < 1.
(ii) $Z_1 + Z_2 +$ ..... Divergent if L > 1.
(iii) No conclusion if L = 1.

Theorem (Root Test)

$$\text{If} \quad \sqrt[n]{|Z_n|} \leq q < 1. \quad \text{(for every } n > \text{N, } q \text{ is fixed)}$$

then $Z_1 + Z_2 +$ ..... Convergent absolutely.

If for infinitely many n,

$$\sqrt[n]{|Z_n|} \geq 1, \quad \text{then the series is Divergent.}$$

Theorem (Root Test)

$$\text{Same as above with } \lim_{m\to\infty}\sqrt[n]{|Z_n|} = \text{L.}$$