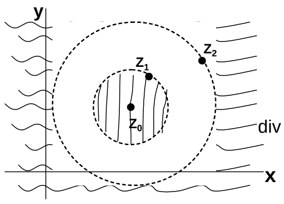
27 Power Series

$$\sum_{n=0}^{\infty} a_n (z - z_n)^n \quad \text{———} \ (1)$$

Theorem

(a) Every power series converges at the center z_0.
(b) If (1) converges at $z = z_1 \neq z_0$, it converges abs for every z closer to z_0 than z_1.
(c) If (1) diverges at $z = z_2$, it diverges for every z farther away from z_0 than z_2.



• Circle of Convergence: the smallest circle with center z_0 that includes all the points where (1) converges. $|z - z_0| = R$ (R = radius of convergence)

R = ∞ if (1) converges for all z
R = 0 if (1) converges only at $z = z_0$
Remark. (1) may converge at some points or none on the circle of convergence.

Theorem (Cauchy - Hadamard Formaula)

$$\text{Suppose that } \left|\frac{a_{n+1}}{a_n}\right| \rightarrow \text{L}^*$$

• If L* = 0, then R = ∞.

• If L* > 0, then $\text{R} = \dfrac{1}{L*} = \lim_{n\to\infty} \left|\dfrac{a_n}{a_{n+1}}\right|$

• If $\left|\dfrac{a_{n+1}}{a_n}\right| \rightarrow \infty$, then R = 0.

14.3 Funtions given by Power Series

For Simplicity, Set $z_0 = 0$.
When the power series has a non-zero radius of convergence $\mathrm{R} > 0$, its sum is a function of z. i.e.

$$f(z){=}\sum_{n=0}^{\infty} a_n z^n \qquad\qquad |z| < \mathrm{R}. \text{ —— } (2)$$

Q. Given $f(z)$ and a center, is the power series representation unique?

Theorem 1 $f(z)$ in (z) with $\mathrm{R} > 0$ is continues at $z = 0$.

Proof. From (z), $f(0) = a_0$.

$$\text{We need to show } \lim_{z \to 0} f(z) = f(0) = a_0$$

i.e. for a given $\varepsilon > 0$ there exists a $\delta > 0$.
such that $|z| < \delta \Longrightarrow |f(z) - a_0| < \varepsilon$.

Theorem in Section 14.2 $\Longrightarrow$ (2) Converges abs for $|z| \le r$ with any $0 < r < \mathrm{R}$.

$$\Longrightarrow \sum_{n=1}^{\infty} |a_n| r^{n-1} = \frac{1}{r} \sum_{n=1}^{\infty} |a_n| r^n = \mathrm{S} \text{ (converges, S} \neq 0), \text{ (S} = 0, \text{ is trivial)}$$

Then for $0 < |z| \le r$,

$$|f(z) - a_0| = \left| \sum_{n=1}^{\infty} a_n z^n \right| \le |z| \sum_{n=1}^{\infty} |a_n| |z|^{n-1} \le |z| \sum_{n=1}^{\infty} |a_n| r^{n-1} = |z| \mathrm{S}.$$

Given $\varepsilon > 0$, choose δ such that,

$0 < \delta < r$ and $\delta < \frac{\varepsilon}{\delta}$.

Then $|f(z) - a_0| < \varepsilon$.

Theorem 2 (Uniqueness of Representation)

Suppose,

$$\sum_{n=0}^{\infty} a_n z^n \quad and \quad \sum_{n=0}^{\infty} b_n z^n$$

both converge for $|z| <$ R (R > 0) and have the same sum for all these z.
Then $a_n = b_n$ (n=0,1,2,.......).

Proof. $a_0 + a_1 z + ... = b_0 + b_1 z + ...$ $\qquad |z| <$ R.
Both series are continues at z = 0 (by Theorem 1).
$\Rightarrow$ Let $z \to 0$ $\qquad \Rightarrow a_0 = b_0$.
For $z \neq 0$, $\quad a_1 + a_2 z + ... = b_1 + b_2 z + ...$
$\Rightarrow Let z \to 0$ $\qquad \Rightarrow a_1 = b_1$.

Operation on Power Series

$$f(z) = \sum_{k=0}^{\infty} a_k (z)^k \qquad |z| < R_1$$

$$g(z) = \sum_{m=0}^{\infty} b_m (z)^m \qquad |z| < R_2$$

- Term-wise addition / subtraction

$\Rightarrow R =$ min (R_1, R_2)

- Term-wise multiplication (Cauchy product)

$$f(z)g(z) = \sum_{n=0}^{\infty} (a_0 b_n + a_1 b_{n-1} + ... + a_n b_0)(z)^n$$

Converges absolutely for each z within $R_1 \& R_2$

- Term-wise differentiation / multiplication

$$\sum_{n=1}^{\infty} n a_n z^{n-1}$$ has the same rad. of conv. as the original series.

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1}$$ has the same rad. of conv. as the original series.

Theorem (Power Series represent analytic funtions)

A power series with $R > 0$ represents an analytic funtion at every point interior to its circle of convergence. The derivatives are obtained by differentiating term by term.

All the series thus obtained have the same R as the original series (Thus each of them represents an analytic funtion).