School of Mech & Aero Eng
Seoul National University

Multivariable Control
H. Jin Kim

1 Introduction to Linear Multi-Input Multi-Output Systems

1.1 Description of Linear Dynamical System

Let a finite dimensional linear time invariant (LTI) dynamical system be described by the following linear constant coefficient differential equations:

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \tag{1}$$
$$y = Cx + Du \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is called the system state, $x(t_0)$ is called the initial condition of the system, $u(t) \in \mathbb{R}^m$ is called the system input, and $y(t) \in \mathbb{R}^p$ is the system output. The A, B, C, and D are real constant matrices with the appropriate dimension.

A dynamical system with $m = 1$ and $p = 1$ is called a SISO (single-input single-output) system; otherwise a MIMO (multi-input multi-output) system.

If the initial condition is zero, then for any input signal $u(\cdot)$, the system responds with a unique signal $y(\cdot)$ given by

$$y(t) = \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)\ .$$

Hence the system maps the input $u(\cdot)$ to the output $y(\cdot)$. This viewpoint can as well be applied for time-varying or nonlinear systems.

If the input signal has a Laplace transform

$$U(s) = \int_0^\infty e^{-st}u(t)dt$$

then the output signal has a Laplace transform

$$Y(s) = \int_0^\infty e^{-st}y(t)dt = G(s)U(s)$$

where $G(s) = C(sI - A)^{-1}B + D$ is the transfer matrix (TM) of the system. A LTI system can be described in a state-space form using A, B, C, D or by its transfer matrix $G(s)$.

Note that (1) and (2) can be written in the following packed form:

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{3}$$

We'll often use the following notation

$$\left[\begin{array}{cc} A & B \\ C & D \end{array}\right] = C(sI - A)^{-1}B + D$$

Throughout the course, we'll assume that $G(s)$ is a real rational matrix that is proper. Then there exist matrices A, B, C, D such that

$$G(s) = \begin{bmatrix} A|B \\ -|- \\ C|D \end{bmatrix} . \tag{4}$$

Then we call such a state-space model (A, B, C, D) a *realization* of $G(s)$. Clearly, the realization is not unique. For example, think of pole-zero cancellation. We say that (A, B, C, D) is a *minimal realization* of $G(s)$ if A has the smallest possible dimension.

Theorem 1.1 *A state-space realization* (A, B, C, D) *of* $G(s)$ *is minimal iff* (A, B) *is controllable and* (C, A) *is observable.*

The following MATLAB commands can be used to express a system in the packed/unpacked form:

```
G=pck(A,B,C,D)
seesys(G)
[A,B,C,d]=unpck(G)
```

1.2 What's Different in MIMO Systems ?

Let's see some simple difficulties that arise when analyzing multivariable systems.

- Order is important ($G(s)K(s) \neq K(s)G(s)$ in general)!
- Poles and Zeros

Example 1.

$$\begin{aligned} G(s) &= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{2}{s+2} & \frac{1}{s+1} \end{bmatrix} \\ K(s) &= \begin{bmatrix} \frac{s+2}{s-\sqrt{2}} & -\frac{s+1}{s-\sqrt{2}} \\ 0 & 1 \end{bmatrix} \\ H(s) &= \begin{bmatrix} \frac{s+1}{s+3} & \frac{s-1}{s+4} \\ 0 & \frac{1}{(s+1)(s+3)} \end{bmatrix} \end{aligned}$$

MATLAB commands :

```
G = tf({1 1; 2 1},{[1 1],[1 2]; [1 2],[1 1]});
K = tf({[1 2], -[1 1]; 0 1}, {[1 -sqrt(2)], [1 -sqrt(2)]; 1 1});
H = tf({[1 1],[1 -1];0,1}, {[1 3],[1 4];1,conv([1 1],[1 3])});
```

Matlab Commands

```
sys1 = ss(A,B,C,D)
minfo(sys1)
sys2 = tf(num,den)
sys3 = zpk(z,p,k)
sys1 = tf(sys2)
[num,den] = tfdata(sys1)
```

Try these commands, and `madd, msub, mmult, sbs, abv, daug, transp, cjt, minv, mscl`, etc. for interconnecting systems.

1.3 Motivating Example

In this example, we'll see that simple notions such as loop-at-a-time gain and phase margins may become inadequate to describe and assess the robustness of the feedback system.

Consider a symmetric spinning body (e.g. a satellite) with torque inputs, T_1 and T_2, along two orthogonal transverse axes, x and y. Assume that the angular velocity of the body with respect to the z axis is constant, ω, and that the inertias of the body with respect to the x, y and z axes are I_1, $I_2 = I_1$, and I_3, respectively. Denote by ω_1 and ω_2 the angular velocities of the body with respect to the x and y axes, respectively. Then the Euler's equations for the body are given by

$$\begin{aligned} I_1\dot{\omega}_1 - \omega_2\omega(I_2 - I_3) &= T_1 \\ I_2\dot{\omega}_2 - \omega\omega_1(I_3 - I_1) &= T_2 \, . \end{aligned}$$

Define

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \triangleq \begin{bmatrix} T_1/I_1 \\ T_2/I_1 \end{bmatrix}, \; \alpha \triangleq (1 - I_3/I_1)\omega \, ,$$

then the system dynamical equations can be written as

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Now suppose that the angular rates ω_1 and ω_2 are measured in scaled and rotated coordinates:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\cos\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix},$$

where $\tan\theta = \alpha$. Then the TM for the spinning body can be computed as

$$Y(s) = G(s)U(s)$$

with

$$G \triangleq \frac{1}{s^2 + \alpha^2} \begin{bmatrix} s - \alpha^2 & \alpha(s+1) \\ -\alpha(s+1) & s - \alpha^2 \end{bmatrix}.$$

Define

$$K_1 \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; K_2 \triangleq \frac{1}{1 + \alpha^2} \begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix},$$

with $\alpha = 10$. A minimal, state-space realization for the plant G is

$$G = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & D_G \end{array} \right] = \left[\begin{array}{cc|cc} 0 & \alpha & 1 & 0 \\ -\alpha & 0 & 0 & 1 \\ \hline 1 & \alpha & 0 & 0 \\ -\alpha & 1 & 0 & 0 \end{array} \right]$$

In Fig. 1, transfer function matrix from r to y is

$$M_{ry} = (I_2 + GK_1)^{-1}GK_2 = \frac{1}{s+1} I_2 \, ,$$

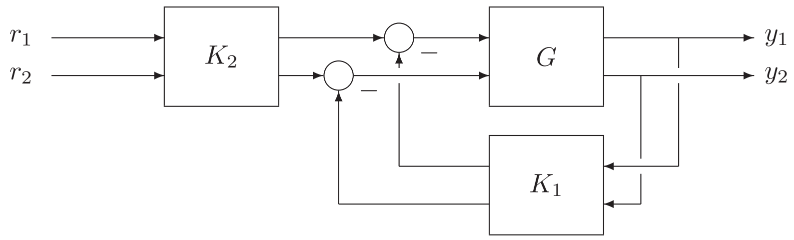

Figure 1: Multi-loop feedback system

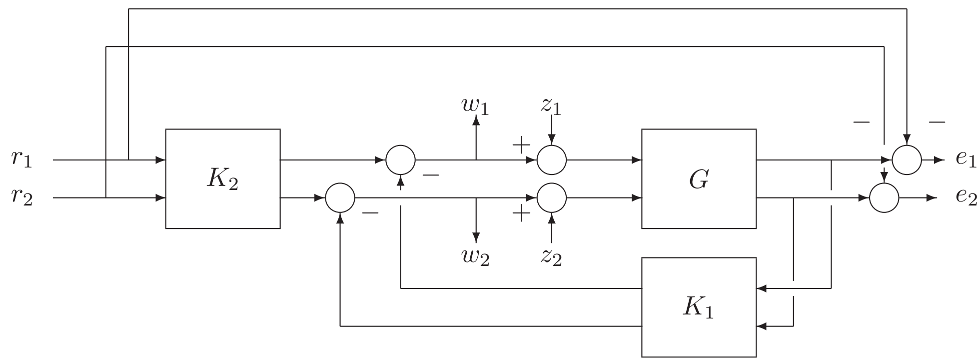

Figure 2: Multi-loop feedback system with uncertainty model

which means that the nominal closed-loop system (c.l.s) has decoupled command response, with a bandwidth of 1 rad/sec.

In order to assess the robustness margins to perturbations in the input channels into the plant, consider Fig. 2. Then

$$M_{wz} = -(I + K_1G)^{-1}K_1G \; = \; -\frac{1}{s+1}\begin{bmatrix} 1 & \alpha \\ -\alpha & 1 \end{bmatrix}$$

Thus,

$$M_{w_1z_1} = M_{w_2z_2} = -\frac{1}{s+1} \; .$$

This means that the crossover frequency in channel 1 is 1 rad/sec, with phase margin of 90 °, and the gain margin in channel 1 is infinite. And the same can be said about channel 2.

These would suggest that the closed-loop response is quite robust to perturbations in each input channel (and that the performance of the c.l.s is somewhat insensitive to perturbations in these loops). Now, consider a 5 % variation in each channel, i.e. set $\delta_1 = 0.05$, and $\delta_2 = -0.05$ in Fig. 3. You will see that as in Fig. 4, the ideal behavior of the nominal system has degraded sharply, despite the seemingly innocuous perturbations and excellent gain/phase margins in the c.l.s.

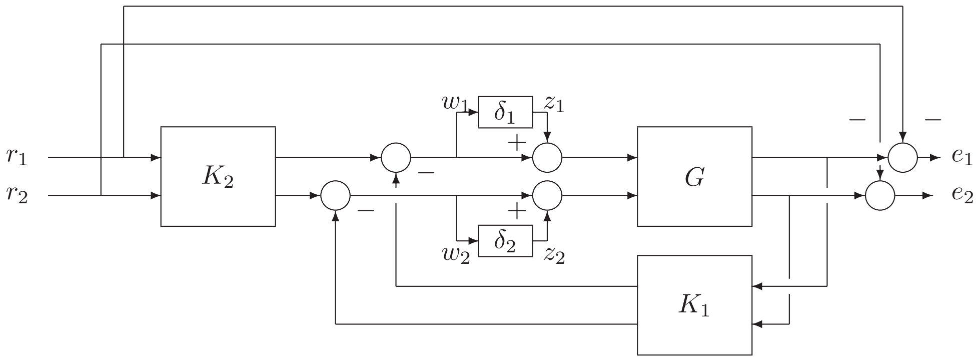

Figure 3: Multi-loop feedback system with uncertain elements

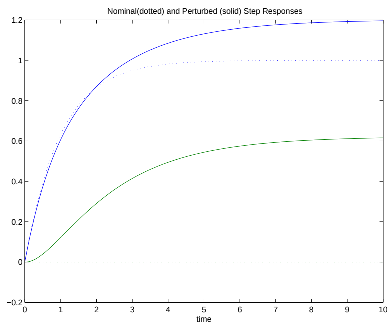

Figure 4: Step Response with 5 % Perturbations

In fact, for a slightly larger perturbation (for ex, $\delta_1 = 0.11$, $\delta_2 = -0.11$), the c.l.s is actually unstable. Why can these small perturbations cause such a significant performance degradation? To see this, let's compute the 4×4 transfer matrix represented in Fig. 2 and 5:

$$\begin{bmatrix} w_1 \\ w_2 \\ e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{s+1} & -\frac{\alpha}{s+1} & \frac{s-\alpha^2}{s+1} & -\alpha \\ \frac{\alpha}{s+1} & -\frac{1}{s+1} & \alpha & \frac{s-\alpha^2}{s+1} \\ \frac{1}{s+1} & \frac{\alpha}{s+1} & -\frac{s}{s+1} & 0 \\ -\frac{\alpha}{s+1} & \frac{1}{s+1} & 0 & -\frac{s}{s+1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ r_1 \\ r_2 \end{bmatrix} \triangleq M \begin{bmatrix} z_1 \\ z_2 \\ r_1 \\ r_2 \end{bmatrix}$$

Note that the preliminary calculations about the closed loop system yielded information only about (1,1), (2,2) and (3:4,3:4) entries of M. In some sense, all these entries are small. The neglected entries, (1,2),(2,1),(1:2,3:4),(3:4,1:2) are all quite large (since $\alpha = 10$). It is these large off-diagonal entries, and the manner in which they enter which causes the extreme sensitivity of the closed-loop system's performance to the perturbations δ_1 and δ_2.

With $\delta_2 \equiv 0$, the perturbation δ_1 can cause instability by making the transfer function $(1 - M_{w_1 z_1}\delta_1)^{-1}$ unstable. Similarly, with $\delta_1 \equiv 0$, perturbation δ_2 can only cause $(1 - M_{w_2 z_2}\delta_2)^{-1}$ unstable. Since both $M_{w_1 z_1}$ and $M_{w_2 z_2}$ are small, this requires large perturbations, and the single-loop gain/phase margins reported earlier are accurate.

However, acting together, the perturbations can cause instability by making

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{s+1} & -\frac{\alpha}{s+1} \\ \frac{\alpha}{s+1} & -\frac{1}{s+1} \end{bmatrix} \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}\right]^{-1}$$

unstable. To see this point, compute the denominator of this multivariable transfer function :

$$s^2 + (2 + \delta_1 + \delta_2)s + [1 + \delta_1 + \delta_2 + (\alpha^2 + 1)\delta_1\delta_2] \; .$$

By choosing $\delta_1 = \frac{1}{\sqrt{\alpha^2+1}} \approx 0.1$ and $\delta_2 = -\delta_1$, the characteristic equation has a root at $s = 0$, which indicates marginal stability. For slightly larger perturbations, a root moves into the right-half-plane. Due to the simultaneous nature of the perturbations, destabilization can be caused by a much smaller perturbation than that predicted by the gain/phase margin calculations.

In terms of robust stability, the loop-at-a-time gain/phase margins only depended on the scalar transfer functions $M_{w_1z_1}$ and $M_{w_2z_2}$, but that the robust stability properties of the closed-loop system to simultaneous perturbations actually depends on the 2×2 transfer function matrix M_{wz}.

Assessing the robust performance characteristics of the closed-loop system involves additional transfer functions which were ignored in the simple-minded analysis. Consider the perturbed closed-loop system in Fig. 3. In terms of the transfer function matrix M, the perturbed transfer function from r to e can be drawn as shown in Fig. 5. Partition M into four 2×2 blocks,

$$M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] \; ,$$

then by combining

$$\begin{bmatrix} w \\ e \end{bmatrix} = M \begin{bmatrix} z \\ r \end{bmatrix}$$

and

$$z = \Delta w \; ,$$

the perturbed closed-loop transfer function from r to e can be written as

$$e = [M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}]r$$

where $\Delta \triangleq \mathrm{diag}[\delta_1, \delta_2]$ is the structured matrix of perturbations.

Our initial consideration of the closed-loop system consisted only of the diagonal entries of M_{11} and the entire matrix M_{22}. We have seen that the large off-diagonal entries of M_{11} created destabilizing interactions between the two perturbations. In the robust performance problem, there are additional relevant transfer functions, M_{21} and M_{12}, which are not analyzed in a pivotal role in the robust performance characteristics of the closed-loop system. Therefore, by calculating a loop-at-a-time robustness test and a nominal performance test, 10 of the 16 elements of the relevant transfer function matrix are ignored. Any test which accounts for simultaneous perturbations along with the subsequent degradation of performance must be performed on the whole matrix.

The point of this example is to show that there are some interesting problems in MIMO system analysis. The standard SISO ideas cannot be made into complete analysis tools simply by applying “loop-at-a-time” analysis.

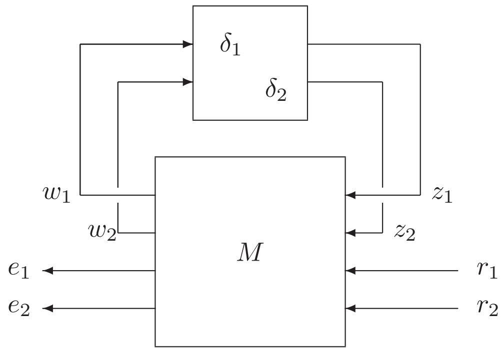

Figure 5: Perturbed system

1.4 Robust Design

Suppose that each δ_i is treated as an unknown, stable transfer function, satisfying bounds

$$|\delta_1(j\omega)| \leq \left| \frac{10\sqrt{\omega^2 + 2^2}}{\sqrt{\omega^2 + 200^2}} \right| , \quad |\delta_2(j\omega)| \leq \left| \frac{10\sqrt{\omega^2 + 24^2}}{3\sqrt{\omega^2 + 200^2}} \right| .$$

Note that at low frequency, the bounds on δ_i are small (0.2 and 0.4) so that the plant model for the actuators has roughly 20 % and 40 % uncertainty at low frequency. As frequency increases, the bounds on the δ_i increase, so that the model of plant uncertainty allows for more deviation at higher frequencies. Mote that at 20 rad/sec, the bound on δ_1 is 1, so that in some sense there is 100 % uncertainty in actuator 1 at this frequency. Similar for actuator 2, though it is not 100 % uncertain until 58 rad/sec (Fig. 6)

These types of uncertainty bounds are common in current multivariable control design theory. We will study them more as the semester progresses.

Using these bounds, along with the tracking performance objective, we can design a controller using the techniques we will cover later in the class.

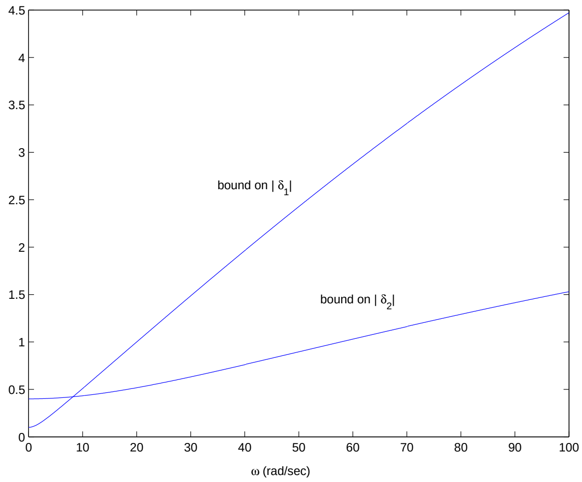

Figure 6: Bound on δ_i