5 Interconnected Systems

5.1 Internal Stability of Feedback Structure

Consider Fig. 1, which consists of the interconnected plant P and controller K, driven by the external signals w_1 and w_2.

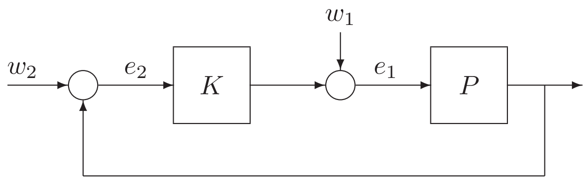


Figure 1: Standard feedback control configuration

Assume that P and K are fixed real rational proper TM with appropriate dimensions. First we should make sure that the feedback interconnection makes sense.

Definition 5.1 *A feedback system is said to be well-posed if all closed-loop TMs are well-defined and proper.*

Two equations at two summing junctions in Fig. 1 can be written as

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} I & -K \\ -P & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{1}$$

which can be inverted to yield

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} (I-KP)^{-1} & K(I-PK)^{-1} \\ P(I-KP)^{-1} & (I-PK)^{-1} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Well-posed means that for any signals w_1, w_2, there exist unique signals e_1, e_2 solving the loop equation (1). If not well-posed, then for given signals w_1, w_2, there are either an infinite number of solutions to the loop equations, or no solution at all. And from the computation above, well-posedness is equivalent to the condition that $(I-KP)^{-1}$ exists and is proper. This is equivalent to the condition that the constant term of $(I-KP)$ is invertible:

Lemma 5.2 *The c.l.s is well-posed iff*

$$I - K(\infty)P(\infty)$$

is invertible.

If we have realizations

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \quad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right],$$

then the well-posed condition is equivalent to the invertibility of $\left[\begin{array}{c|c} I & -D_K \\ \hline D & I \end{array}\right]$. Often we have $D = 0$, so well-posedness will be guaranteed for most control systems.

Definition 5.3 *The c.l.s is internally stable if the TM from w_1, w_2 to e_1, e_2*

$$\begin{aligned} \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} &= \begin{bmatrix} (I-KP)^{-1} & K(I-PK)^{-1} \\ P(I-KP)^{-1} & (I-PK)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I+K(I-PK)^{-1}P & K(I-PK)^{-1} \\ (I-PK)^{-1}P & (I-PK)^{-1} \end{bmatrix} \end{aligned}$$

is proper real rational stable (or, belongs to $\mathcal{RH}_\infty$).

Internal stability guarantees that all signals in a system are bounded provided that the injected signals (at any locations) are bounded.

There are special cases under which system stability can be easily determined.

Corollary 5.4 *Suppose $K \in \mathcal{RH}_\infty$. Then the above system is internally stable iff well-posed and $P(I-KP)^{-1} \in \mathcal{RH}_\infty$.*

Proof. The necessity ($\Rightarrow$) is obvious. To prove $\Leftarrow$, it's enough to show that $(I-PK)^{-1} \in \mathcal{RH}_\infty$. Since

$$(I-PK)^{-1} = I + (I-PK)^{-1}PK$$

and $(I-PK)^{-1}P, K \in \mathcal{RH}_\infty$, we're done. □

Corollary 5.5 *Suppose $P \in \mathcal{RH}_\infty$. Then the above system is internally stable iff well-posed and $K(I-PK)^{-1} \in \mathcal{RH}_\infty$.*

5.2 Representing Uncertainty using LFT

Linear Fractional Transformations (LFT) are a powerful and flexible method to represent uncertainty.

Let M be a complex matrix such that $v = Mr$, and partition M into

$$\begin{aligned} v_1 &= M_{11}r_1 + M_{12}r_2 \\ v_2 &= M_{21}r_1 + M_{22}r_2 \end{aligned}$$

as shown in Fig. 2.

Let Δ_L be a complex matrix such that

$$r_2 = \Delta_L v_2 \ ,$$

then LFT defines the interconnection of these two elements:

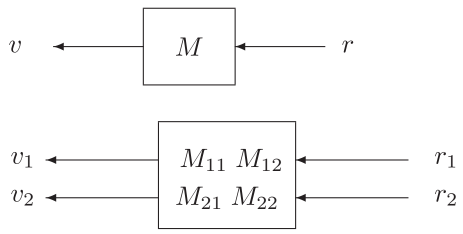


Figure 2: Partitioned system

Definition 5.6 (Lower LFT) *The lower LFT of M by Δ_L is :*

$$F_L(M, \Delta_L) \triangleq M_{11} + M_{12}\Delta_L(I - M_{22}\Delta_L)^{-1}M_{21}$$

provided that $(I - M_{22}\Delta_L)^{-1}$ exists.

On the other hand, if we suppose that Δ_U is a matrix such that

$$r_1 = \Delta_U v_1,$$

then we can define

Definition 5.7 (Upper LFT) *The upper LFT of M by Δ_U is :*

$$F_U(M, \Delta_U) \triangleq M_{22} + M_{21}\Delta_U(I - M_{11}\Delta_U)^{-1}M_{12}$$

provided that $(I - M_{11}\Delta_U)^{-1}$ exists.

The reason for the name "lower" and "upper" should be clear from the Fig. 3. Eliminating v_2 and r_2 from the left figure, we can easily verify that

$$v_1 = F_L(M, \Delta_L)\ r_1\ ,$$

i.e. $F_L(M, \Delta_L)$ is a transformation obtained from closing the *lower* loop on the left diagram. And similarly, the right figure represents

$$v_2 = F_U(M, \Delta_U)\ r_2\ ,$$

i.e. $F_U(M, \Delta_U)$ is a transformation obtained from closing the *upper* loop on the right diagram.

5.3 Interpretations

Let M be a proper transfer matrix. Then we can think of LFTs as the closed-loop TFs from r_1 to v_1 and r_2 to v_2, respectively, where M may be the controlled

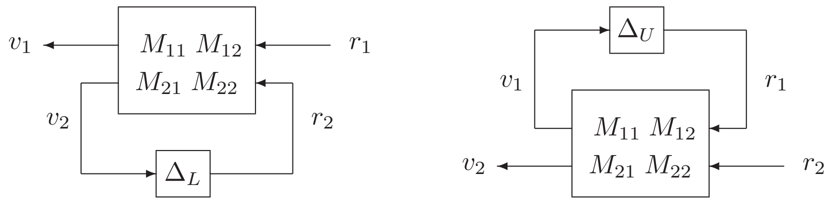


Figure 3: LFT

plant and Δ may be either the model uncertainties, perturbation, or the controllers.

Then, how do we use LFTs to represent an uncertain parameter?

Example. Suppose c is an uncertain parameter in your plant, and it is known that

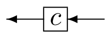
$$2.0 \leq c \leq 2.8\ .$$

We can write this as

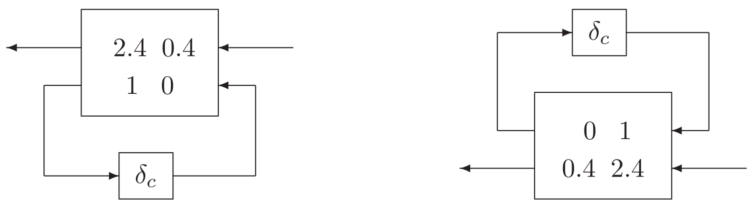
$$c = 2.4 + 0.4\delta_c \quad \delta_c \in [-1, 1]\ ,$$

and this is a LFT !

$$c = F_L\left(\begin{bmatrix} 2.4 & 0.4 \\ 1 & 0 \end{bmatrix}, \delta_c\right) = F_U\left(\begin{bmatrix} 0 & 1 \\ 0.4 & 2.4 \end{bmatrix}, \delta_c\right)$$

Thus, whenever 

appears in a block diagram, simply replace it with either of the following:



It is clear that LFT representation is not unique. In fact, $F_U(N, \Delta) = F_L(M, \Delta)$ with $N = \begin{bmatrix} M_{22} & M_{21} \\ M_{12} & M_{11} \end{bmatrix}$. But in overall, they represent the same system.

What if the gain $1/c$ appears? The LFT representation can still be used, because inverses of an LFT is still an LFT (on the same δ) (ref. Lemma 9.3 in Zhou, if you want to know the details). Observe that

$$c^{-1} = \frac{1}{2.4 + 0.4\delta_c} = \frac{1}{2.4} + \frac{-\frac{1}{6\times 2.4}\delta_c}{1 - (-\frac{1}{6})\delta_c} = F_L\left(\begin{bmatrix} \frac{1}{2.4} & -\frac{1}{6} \\ \frac{1}{2.4} & -\frac{1}{6} \end{bmatrix}, \delta_c\right)$$

Example. $mx+cx+kx=u$, $\pm 50\%$ uncertainty in $m, \pm 30\%$ in $c, \pm 40\%$ in k

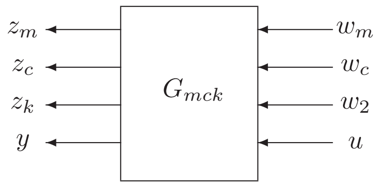


Figure 4: Known part of the system

$$y = F_U(G_{mck}, diag[\delta_m, \delta_c, \delta_k])$$

We can compute G_{mck} using `sysic` command as shown in Fig. 5.

5.4 Star Product

Star product is a generalization of the LFT shown in Fig. 6.

Suppose that P and K are partitioned as below

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \quad K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

and that $P_{22}K_{11}$ is well-defined and square, and that $I - P_{22}K_{11}$ is invertible. Then the *star product* of p and K with respect to this partition is defined as

$$P \star K := \begin{bmatrix} F_L(P, K_{11}) & P_{12}(I - K_{11}P_{22})^{-1}K_{12} \\ K_{21}(I - P_{22}K_{11})^{-1}P_{21} & F_U(K, P_{22}) \end{bmatrix} \tag{2}$$

Note that this definition depends on the partitioning of the matrices.

In order to compute $P \star K$, use the MATLAB command:
`starPK=starp(P,K,dimy, dimu);`
where dimy and dimu are the dimensions of y and u, respectively. In particular, when dim($\hat{z}$)=0 and dim($\hat{w}$) =0, we have
`starp(P,K)`$=F_L(P, K)$.

```
mbar=3; cbar=1; kbar=2;
matmass=[-.5 1/mbar; -0.5 1/mbar];
matc=[cbar 0.3*cbar; 1 0];
matk=[kbar .4*kbar; 1 0];
int1=nd2sys([1],[1 0]);
int2=nd2sys([1],[1 0]);
systemnames='matmass matc matk int1 int2';
sysoutname = 'Gmck';
inputvar='[wm; wc;wk;u]';
input_to_matmass='[wm;u-matc(1)-matk(1)]';
input_to_matc='[int1; wc]';
input_to_matk='[int2; wk]';
input_to_int1='[matmass(2)]';
input_to_int2='[int1]';
outputvar='[matmass(1);matc(2);matk(2);int2]';
sysic;

Gmck_g=frsp(Gmck, logspace(-1,1,100));

delnom=diag([0;0;0]);
rifd(spoles(starp(delnom,Gmck)))

delnpn=diag([-1;1;-1]);
rifd(spoles(starp(delnpn,Gmck)))

delpnp=diag([1;-1;1]);
rifd(spoles(starp(delpnp,Gmck)))

vplot('bode',starp(delnom,Gmck_g),'-',...
  starp(delnpn,Gmck_g),'-.', starp(delpnp,Gmck_g),'--')
```

Figure 5: `sysic` command

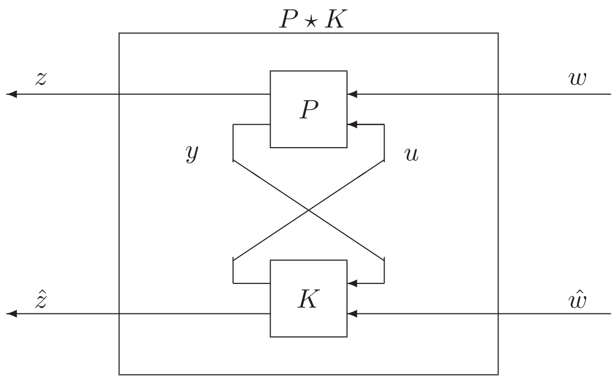


Figure 6: Star product