6 LQ, LQG, H_2, H_∞ Control System Design

6.1 LQ: Linear systems with Quadratic performance criteria

Consider a linear time-invariant system represented in state space form as[1]

$$\begin{aligned} x(t) &= Ax(t) + Bu(t) \\ z(t) &= Cx(t) \end{aligned} \qquad t \geq 0 \tag{1}$$

For each $t \geq 0$ the state $x(t)$ is an n-dimensional vector, the input $u(t)$ a k-dimensional vector, and the output $z(t)$ an m-dimensional vector. We wish to control the system from any initial state $x(0)$ such that the output z is reduced to a very small value as quickly as possible without making the input u unduly large. To this end we introduce the performance index

$$J = \int_0^\infty [z^T(t)Qz(t) + u^T(t)Ru(t)]dt. \tag{2}$$

Q and R are symmetric weighting matrices. Often it is adequate to let the two matrices simply be diagonal. The two terms $z^T(t)Qz(t)$ and $u^T(t)Ru(t)$ are quadratic forms in the components of the output z and the input u, respectively. The first term in the integral criterion (2) measures the accumulated deviation of the output from zero. The second term measures the accumulated amplitude of the control input. It is most sensible to choose the weighting matrices Q and R such that the two terms are nonnegative, that is, to take Q and R nonnegative-definite. If the matrices are diagonal then this means that their diagonal entries should be nonnegative.

The problem of controlling the system such that the performance index (2) is minimal along all possible trajectories of the system is the optimal linear regulator problem. The reason why the linear regulator problem attracted so much attention is that its solution may be represented in feedback form.

Theorem 6.1 *Suppose that the system* (1) *is stabilizable and detectable. (Sufficient for stabilizability is that the system is controllable. Sufficient for detectability is that it is observable). If the weighting matrices Q and R are positive-definite, then the following facts hold.*

1. *The algebraic Riccati equation (ARE)*

$$A^T X + XA + C^T QC - XBR^{-1}B^T X = 0 \tag{3}$$

[1] This note is based on Design Methods for Control Systems, by Bosgra, Kwakernaak, Meinsma, and Multivariable Control Systems by Megretski.

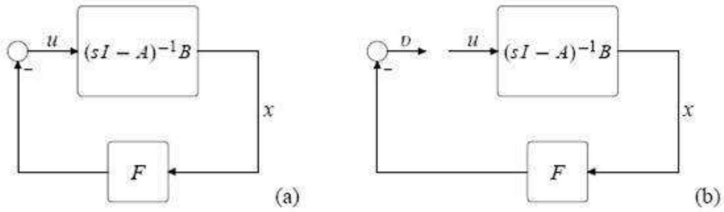

Figure 1: state feedback

has a unique nonnegative-definite symmetric solution X. If the (A, C) pair is observable then X is positive-definite. (There are finitely many other solutions of the ARE.)

2. *The minimal value of the performance index* (2) *is* $J_{min} = x^T(0)Xx(0)$.

3. *The minimal value of the performance index is achieved by the feedback control law*

$$u(t) = -Fx(t), \quad t \geq 0, \text{ where } F = R^{-1}B^TX \tag{4}$$

4. *The closed-loop system* $\dot{x}(t) = (A - BF)x(t), \quad t \geq 0$ *is stable, that is, all the eigenvalues of the matrix* $A - BF$ *have strictly negative real parts.*

□

6.1.1 Return difference equality and inequality

Figure 1(a) shows the feedback connection of the system $\dot{x} = Ax + Bu$ with the state feedback controller $u = -Fx$. If the loop is broken as in Fig. 1(b) then the loop gain is

$$L(s) = F(sI - A)^{-1}B. \tag{5}$$

The quantity

$$J(s) = I + L(s) \tag{6}$$

is known as the return difference, because $J(s)u$ is the difference between the signal u in Fig. 1(b) and the returned signal $v = -L(s)u$. Several properties of the closed-loop system may be related to the return difference. Note

$$\begin{aligned}
det J(s) &= det[I + L(s)] = det[I + F(sI - A)^{-1}B] \\
&= det[I + (sI - A)^{-1}BF] \quad (\because det(I + MN) = det(I + NM)) \\
&= det(sI - A)^{-1}det(sI - A + BF) \\
&= \frac{det(sI - A + BF)}{det(sI - A)} \\
&= \frac{\text{C.L. characteristic poly}}{\text{O.L. characteristic poly}}.
\end{aligned}$$

Suppose that the gain matrix F is optimal. Then, by manipulation of the algebraic Riccati equation (3) that the corresponding return difference satisfies the equality

$$J^T(-s)RJ(s) = R + G^T(-s)QG(s). \tag{7}$$

$G(s) = C(sI-A)^{-1}B$ is the open-loop transfer matrix of the system (1). The relation (7) is known as the return difference equality or as the Kalman-Yakubovic-Popov (KYP) equality, after its discoverers. We can use the return difference equality to study the root loci of the optimal closed-loop poles. By setting $s = j\omega$, we obtain the return difference inequality

$$J^T(-j\omega)RJ(j\omega) \geq R \quad \text{for all } \omega \in \mathbb{R}. \tag{8}$$

Lemma 6.2 (KYP equality) *Consider the linear time-invariant system $\dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t) + Du(t)$, with transfer matrix $G(s) = C(sI - A)^{-1}B + D$, and let Q and R be given symmetric constant matrices. Suppose that the algebraic matrix Riccati equation*

$$0 = A^TX + XA + C^TQC - (XB + C^TQD)(D^TQD + R)^{-1}(B^TX + D^TQC) \tag{9}$$

has a symmetric solution X. Then

$$R + G^{\sim}(s)QG(s) = J^{\sim}(s)R_DJ(s). \tag{10}$$

The constant symmetric matrix R_D and the rational matrix function function J are given by

$$R_D = R + D^TQD, J(s) = I + F(sI - A)^{-1}B, \tag{11}$$

with $F = R_D^{-1}(B^TX + D^TQC)$. The zeros of the numerator of $detJ$ are the eigenvalues of the matrix $A - BF$.

□

We use the notation $G^{\sim}(s) = G^T(-s)$. The KYP equality arises in the study of the regulator problem for the system $\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t) + Du(t)$, with the criterion

$$\int_0^{\infty} [y^T(t)Qy(t) + u^T(t)Ru(t)]dt. \tag{12}$$

The equation (9) is the algebraic Riccati equation associated with this problem, and $u(t) = -Fx(t)$ is the corresponding optimal state feedback law. The KYP equality is best known for the case $D = 0$. It then reduces to the return difference equality

$$J^{\sim}(s)RJ(s) = R + G^{\sim}(s)QG(s). \tag{13}$$

Proof of Kalman-Yakubovic-Popov equality. The algebraic Riccati equation (9) can be written as

$$0 = A^TX + XA + C^TQC - (XB + C^TQD)R_D^{-1}(B^TX + D^TQC), \tag{14}$$

with $R_D = R + D^T QD$. From the relation $F = R_D^{-1}(B^T X + D^T QC)$ we have $B^T X + D^T QC = R_D F$, so that the Riccati equation may be written as

$$0 = A^T X + XA + C^T QC - F^T R_D F. \tag{15}$$

This in turn we rewrite as

$$0 = -(-sI - A^T)X - X(sI - A) + C^T QC - F^T R_D F. \tag{16}$$

Premultiplication by $B^T(-sI - A^T)^{-1}$ and postmultiplication by $(sI - A)^{-1}B$ results in

$$\begin{aligned} 0 \quad = \quad & -B^T X(sI - A)^{-1}B - B^T(-sI - A^T)^{-1}XB \\ & +B^T(-sI - A^T)^{-1}(C^T QC - F^T R_D F)(sI - A)^{-1}B. \end{aligned} \tag{17}$$

Substituting $B^T X = R_D F - D^T QC$ we find

$$\begin{aligned} 0 \quad = \quad & (D^T QC - R_D F)(sI - A)^{-1}B + B^T(-sI - A^T)^{-1}(C^T QD - F^T R_D) \\ & +B^T(-sI - A^T)^{-1}(C^T QC - F^T R_D F)(sI - A)^{-1}B. \end{aligned} \tag{18}$$

Expansion of this expression, substitution of $C(sI - A)^{-1}B = G(s) - D$ and $F(sI - A)^{-1}B = J(s) - I$ and simplification lead to the desired result

$$R + G^{\sim}(s)QG(s) = J^{\sim}(s)R_D J(s). \tag{19}$$

□

6.1.2 Guaranteed gain and phase margins

If the state feedback loop is opened at the plant input then the loop gain is $L(s) = F(sI - A)^{-1}B$. For the single-input case, the return difference inequality (8) takes the form

$$|1 + L(j\omega)| \geq 1 \quad \text{for all } \omega \in \mathbb{R}. \tag{20}$$

This inequality implies that the Nyquist plot of the loop gain stays outside the circle with center at -1 and radius 1.

The SISO results may be generalized to the multi-input case. Suppose that the loop gain satisfies the return difference inequality. Assume that the loop gain $L(s)$ is perturbed to $W(s)L(s)$, with W a stable transfer matrix. It can be proved that the closed-loop remains stable provided

$$RW(j\omega) + W^T(-j\omega)R > R, \quad \omega \in \mathbb{R}. \tag{21}$$

If both R and W are diagonal, then this becomes

$$W(j\omega) + W^T(-j\omega) > I, \quad \omega \in \mathbb{R}.$$

This shows that if the i th diagonal entry W_i of W is real then it may have any value in the interval $(1/2, \infty)$ without destabilizing the closed-loop system. If

the i th diagonal entry is $W_i(j\omega) = e^{j\phi}$ then the closed-loop system remains stable as long as the angle ϕ is less than $\phi/3$.
Proof of (21)**.** From (8), we have

$$(I+L)^{\sim}(I+L) \geq R \text{ on the imaginary axis.}$$

Let's consider the case when $R = I$. Then $L^{\sim} + L + L^{\sim}L \geq 0$ on the imaginary axis, or

$$L^{-1} + (L^{-1})^{\sim} + I \geq 0 \text{ on the imaginary axis} \tag{22}$$

The perturbed system is stable if $I + WL$ has no zeros in the right-half complex plane. Equivalently, the perturbed system is stable if for $0 \leq \epsilon \leq 1$ no zeros of

$$I + [(1-\epsilon)I + \epsilon W]L \text{ on the imaginary axis} \tag{23}$$

cross the imaginary axis. Hence, the perturbed system is stable if only if

$$L^{-1} + (1-\epsilon)I + \epsilon W = M_\epsilon \text{ on the imaginary axis} \tag{24}$$

is nonsingular on the imaginary axis for all $0 \leq \epsilon \leq 1$. Substitution of L^{-1} using (24) into (22) yields

$$M_\epsilon + M_\epsilon^{\sim} \geq 2(1-\epsilon)I + \epsilon(W + W^{\sim}) = (2-\epsilon)I + \epsilon(W + W^{\sim} - I). \text{ on the imaginary axis} \tag{25}$$

Thus if

$$W^{-1} + W^{\sim} > I \text{ on the imaginary axis} \tag{26}$$

then $M_\epsilon + M_\epsilon^{\sim} > 0$ on the imaginary axis for all $0 \leq \epsilon \leq 1$, which means that M_ϵ is nonsingular on the imaginary axis. Therefore, if (26) holds then the perturbed system is stable. □

6.1.3 Cross term in the performance index

The optimal regulator problem for the stabilizable and detectable system (1) with the generalized quadratic performance index

$$\mathcal{J} = \int_0^\infty \begin{bmatrix} z^T(t) & u^T(t) \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} z(t) \\ u(t) \end{bmatrix} dt \tag{27}$$

where $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ is positive definite. With $v(t) = u(t) + R^{-1}S^T z(t)$, minimization of $\mathcal{J}$ is equivalent to minimizing

$$\mathcal{J} = \int_0^\infty [z^T(t)(Q - SR^{-1}S^T)z(t) + v^T(t)Rv(t)]dt \tag{28}$$

for the system

$$\dot{x}(t) = (A - BR^{-1}S^T C)x(t) + Bv(t) \tag{29}$$

The condition that $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ is positive definite is equivalent to the condition that both R and $Q - SR^{-1}S^T$ be positive-definite. Thus we satisfy the conditions of the theorem 6.1, and the Riccati equation now is

$$A^T X + XA + C^T QC - (XB + C^T S)R^{-1}(B^T X + S^T C) = 0. \tag{30}$$

The optimal input for the system (1) is

$$u(t) = -Fx(t), \quad F = R^{-1}(B^T X + S^T C). \tag{31}$$

6.1.4 Solution of the ARE

There are several algorithms for the solutions of ARE (30) is the most general form of the Riccati equation. By redefining $C^T QC$ as Q and $C^T S$ as S, the ARE (30) reduces to

$$A^T X + XA + Q - (XB + S)R^{-1}(B^T X + S^T) = 0. \tag{32}$$

And the solution can be obtained from the Hamiltonian matrix

$$\mathcal{H} \;=\; \begin{bmatrix} A - BR^{-1}S^T & BR^{-1}B^T \\ -Q + SR^{-1}S^T & -(A - BR^{-1}S^T)^T \end{bmatrix} \tag{33}$$

Under the assumptions of the theorem 6.1, the Hamiltonian matrix $\mathcal{H}$ has no eigenvalues on the imaginary axis. If λ is an eigenvalue of the $2n \times 2n$ matrix $\mathcal{H}$ then $-\lambda$ is also an eigenvalue. Hence, $\mathcal{H}$ has exactly n eigenvalues with negative real part. Let the columns of the real $2n \times n$ matrix E form a basis for the n-dimensional space spanned by the eigenvectors and generalized eigenvectors of $\mathcal{H}$ corresponding to the eigenvalues with strictly negative real parts. Partition $E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$ with E_1 and E_2 both square. Then

$$X = E_2 E_1^{-1} \tag{34}$$

is the desired solution of the ARE.

Hamiltonian matrix and ARE

1. The Hamiltonian matrix is of the form

$$\mathcal{H} \;=\; \begin{bmatrix} A & Q \\ R & -A^T \end{bmatrix}, \tag{35}$$

with all blocks square, and Q and R symmetric. Note that for $J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$,

$$J^2 = -I; \rightarrow \; J^{-1} = -J$$

and

$$J^{-1}\mathcal{H}J = -\mathcal{H}^T \ .$$

Thus, $\mathcal{H}$ and $-\mathcal{H}^T$ are similar. And $\mathcal{H}$ and $\mathcal{H}^T$ have the same eigenvalues. Therefore, if λ is an eigenvalue of $\mathcal{H}$, so is -λ. (When ^{T} is replaced with *, if λ is an eigenvalue of $\mathcal{H}$, so is -$\bar{\lambda}$.)

Therefore, the spectrum of $\mathcal{H}$ is symmetric about the $j\omega$ axis. If $\mathcal{H}$ has no eigenvalues on the $j\omega$ axis, then n of them are in open LHP, and the other n are in open RHP.

2. Using the Riccati equation (30), we obtain

$$\begin{aligned} \mathcal{H}\begin{bmatrix} I \\ X \end{bmatrix} &= \begin{bmatrix} A - BF \\ -Q + SR^{-1}S^T - (A - BR^{-1}S^T)^T X \end{bmatrix} \\ &= \begin{bmatrix} A - BF \\ XA - XBF \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix}(A - BF). \end{aligned} \tag{36}$$

3. If $(A - BF)x = \lambda x$ then

$$\mathcal{H}\begin{bmatrix} I \\ X \end{bmatrix} x = \begin{bmatrix} I \\ X \end{bmatrix}(A - BF)x = \lambda \begin{bmatrix} I \\ X \end{bmatrix} x. \tag{37}$$

Thus, if $A - BF$ has n eigenvalues with negative real parts (such as in the solution of the LQ problem of theorem 6.1, then the eigenvalues of $\mathcal{H}$ consist of these n eigenvalues of $A - BF$ and their negatives.

4. Assume that $\mathcal{H}$ has no eigenvalues with zero real part. Then there is a similarity transformation U that brings H into upper triangular form T such that

$$\mathcal{H} = UTU^{-1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix},$$

where the eigenvalues of the nn diagonal block T_{11} all have negative real parts and those of T_{22} have positive real parts. From $\mathcal{H}U = UT$ we obtain

$$\mathcal{H}\begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} = \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} T_{11}. \tag{38}$$

After multiplying on the right by U_{11}^{-1} it follows that

$$\mathcal{H}\begin{bmatrix} I \\ U_{21}U_{11}^{-1} \end{bmatrix} = \begin{bmatrix} I \\ U_{21}U_{11}^{-1} \end{bmatrix} U_{11}T_{11}U_{11}^{-1}. \tag{39}$$

Comparing with (36), we identify $X = U_{21}U_{11}^{-1}$ and $A - BF = U_{11}T_{11}U_{11}^{-1}$. For the LQ problem the nonsingularity of U_{11} follows by the existence of X such that $A - BF$ is stable.

6.2 LQG: Linear Quadratic Guassian

We consider the system

$$\left.\begin{array}{rcl} x(t) & = & Ax(t) + Bu(t) + Gv(t) \\ y(t) & = & Cx(t) + w(t) \\ z(t) & = & Dx(t) \end{array}\right\} \quad t \in \mathbb{R} \tag{40}$$

The measured output y is available for feedback and the output z is the controlled output. The noise signal v models the plant disturbances and w the measurement noise. The signals v and w are vector-valued Gaussian white noise processes with

$$\left.\begin{array}{rcl} Ev(t)v^T(s) & = & V\delta(t-s) \\ Ev(t)w^T(s) & = & 0 \\ Ew(t)w^T(s) & = & W\delta(t-s) \end{array}\right\} \quad t, s \in \mathbb{R} \tag{41}$$

V and W are nonnegative-definite symmetric constant matrices, representing the intensity of the two white noise processes. The initial state $x(0)$ is assumed to be a random vector. Since the state $x(t), t \in \mathbb{R}$, and the controlled output $z(t), t \in \mathbb{R}$ are random processes, so is the quadratic error expression

$$z^T(t)Qz(t) + u^T(t)Ru(t), t \geq 0, \tag{42}$$

The problem of controlling the system such that the integrated expected value

$$\int_0^T E[zT(t)Qz(t) + uT(t)Ru(t)]dt \tag{43}$$

is minimal is called the *stochastic linear regulator problem.* The time interval $[0, T]$ at this point is taken to be finite but eventually we consider the case that $T \to 1$. At any time t the entire past measurement signal $y(\tau), \tau < t$, is assumed to be available for feedback.

6.2.1 Kalman filter

Suppose that we connect the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)], \quad t \in \mathbb{R}. \tag{44}$$

to the noisy system

$$\left.\begin{array}{rcl} x(t) & = & Ax(t) + Bu(t) + Gv(t), \\ y(t) & = & Cx(t) + w(t), \end{array}\right\} \quad t \in \mathbb{R} \tag{45}$$

Differentiation of $e(t) = \hat{x}(t) - x(t)$ leads to the error differential equation

$$e(t) = (A - KC)e(t) - Gv(t) + Kw(t), \quad t \in \mathbb{R}. \tag{46}$$

Owing to the two noise terms on the right-hand side the error now no longer converges to zero, even if the error system is stable. Suppose that the error system is stable, then it is proved that as $t \to \infty$ the error covariance matrix

$$Ee(t)e^T(t) \tag{47}$$

converges to a constant steady-state value Y that satisfies the linear matrix equation

$$(A - KC)Y + Y(A - KC)^T + GVG^T + KWK^T = 0. \tag{48}$$

This type of matrix equation is known as a Lyapunov equation. It can be shown that as a function of the gain matrix K the steady-state error covariance matrix Y is minimal if K is chosen as

$$K = YC^T W^{-1}. \tag{49}$$

"Minimal" means here that if $\hat{Y}$ is the steady-state error covariance matrix corresponding to any other observer gain K then $\hat{Y} \geq Y$. This inequality is to be taken in the sense that $\hat{Y} - Y$ is nonnegative-definite. A consequence of this result is that the gain (49) minimizes the steady-state mean square state reconstruction error $\lim_{t\to\infty} Ee^T(t)e(t)$. As a matter of fact, the gain minimizes the weighted mean square construction error $lim_{t\to\infty} Ee^T(t)W_e e(t)$ for any nonnegative-definite weighting matrix W_e. Substitution of the optimal gain matrix (49) into the Lyapunov equation (48) yields

$$AY + YA^T + GVG^T - YC^T W^{-1} CY = 0. \tag{50}$$

This is another matrix Riccati equation. The observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)], t \in \mathbb{R}, \tag{51}$$

with the gain chosen as in (49) and the covariance matrix Y the nonnegative-definite solution of the Riccati equation (50) is the famous Kalman filter (Kalman and Bucy, 1961).

Theorem 6.3 *Suppose that*

$$\left.\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Gv(t), \\ y(t) & = & Cx(t), \end{array}\right\} \quad t \in \mathbb{R} \tag{52}$$

is stabilizable and detectable, and the noise intensity matrices V and W are positive-definite. Then the following facts follow from the theorem 6.1 by duality:

1. *The algebraic Riccati equation*

$$AY + YA^T + GVG^T - YC^T W^{-1} CY = 0 \tag{53}$$

has a unique nonnegative-definite symmetric solution Y. If the system (52) is controllable rather than just stabilizable then Y is positive-definite.

2. $\lim_{t\to\infty} Ee^T(t)W_e e(t) = trYW_e$.

3. The minimal value of the mean square reconstruction error is achieved by the observer gain matrix $K = YC^TW^{-1}$.

4. The error system $\dot{e}(t) = (A - KC)e(t)$ *is stable.*

6.2.2 Solution of the stochastic linear regulator problem

The stochastic linear regulator problem consists of minimizing

$$\int_0^T E[z^T(t)Qz(t) + u^T(t)Ru(t)]dt \tag{54}$$

for the system

$$\left.\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) + Gv(t), \\ y(t) & = & Cx(t) + w(t), \\ z(t) & = & Dx(t) \end{array}\right\} \quad t \in \mathbb{R}\,. \tag{55}$$

State feedback.
If the white noise disturbance v is present then the state and input cannot be driven to 0. In this case, the state feedback law $u = -Fx(t)$ with F given in 6.1 minimizes

$$\lim_{t\to\infty} \frac{1}{T}\int_0^T E[z^T(t)Qz(t) + u^T(t)Ru(t)]dt.$$

This limit equals the steady-state mean square error

$$\lim_{t\to\infty} E[z^T(t)Qz(t) + u^T(t)Ru(t)].$$

Output feedback.
If the state cannot be accessed for measurement, then the solution of the stochastic linear regulator problem with output feedback (rather than state feedback) is to replace the state $x(t)$ in the state feedback law (4) with the estimated state $\hat{x}(t)$ as shown in Fig. 2 (a). Using the estimated state as if it were the actual state is known as certainty equivalence.

$$\begin{array}{rcl} \dot{\hat{x}} & = & A\hat{x} + Bu + K(y - C\hat{x}) \\ u & = & -F\hat{x} \end{array} \tag{56}$$

The closed-loop system becomes

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A - BF & -BF \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} Gv(t) \\ -Gv(t) + Kw(t) \end{bmatrix}, \tag{57}$$

and the closed-loop eigenvalues consist of the eigenvalues of $A - BF$ (the regulator poles) together with the eigenvalues of $A - KC$ (the observer poles). It separates state estimation and control input selection. This idea is often referred to as the separation principle.

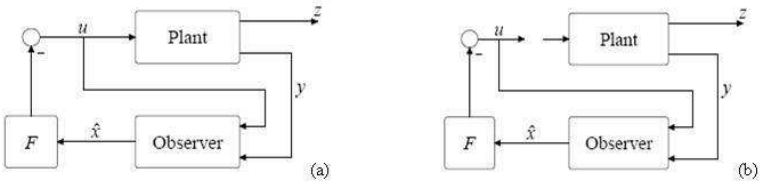

Figure 2: Observer based feedback control

6.2.3 Asymptotic analysis and loop transfer recovery

In order to see the effect of decreasing the intensity W of the measurement noise. Suppose that $W = \sigma W_0$, with W_0 a fixed symmetric positive-definite weighting matrix and σ a positive number. We investigate the asymptotic behavior of the closed-loop system as $\sigma \downarrow 0$.

Suppose that the disturbance v is additive to the plant input u, that is, $G = B$, and the open-loop plant transfer matrix $P(s) = C(sI - A)^{-1}B$ is square, and its zeros all have negative real parts.

In (56), $u = -F(sI - A + BF + K_\sigma C)^{-1}K_\sigma y$. Thus breaking the loop at the plant input as in Fig. 2(b), we obtain the loop gain

$$L_\sigma(s) = K(s)P(s) = F(sI - A + BF + K_\sigma C)^{-1}K_\sigma C(sI - A)^{-1}B. \tag{58}$$

As $\sigma \downarrow 0$, the error covariance matrix Y_σ approaches the zero matrix, which indicates that in the limit the observer reconstructs the state completely. With $G = B$, the Riccati equation for the optimal observer is

$$AY_\sigma + Y_\sigma A^T + BVB^T - Y_\sigma C^T W^{-1} C Y_\sigma = 0.$$

i.e.

$$AY_\sigma + Y_\sigma A^T + BVB^T - \sigma K_\sigma W_0 K_\sigma^T = 0, \quad K_\sigma = Y_\sigma C^T W^{-1}.$$

From this, if $Y_\sigma \downarrow 0$, then $K_\sigma \approx \frac{1}{\sqrt{\sigma}} BU_\sigma$, where U_σ is a square nonsingular matrix (which may depend on σ) such that $U_\sigma W_0 U_\sigma^T = V$.

As $\sigma \downarrow 0$, (58) becomes

$$\begin{aligned}
L_\sigma(s) &\approx F(sI - A + BF + \frac{1}{\sqrt{\sigma}} BU_\sigma C)^{-1} \frac{1}{\sqrt{\sigma}} BU_\sigma C(sI - A)^{-1}B \\
&\approx F(sI - A + \frac{1}{\sqrt{\sigma}} BU_\sigma C)^{-1} \frac{1}{\sqrt{\sigma}} BU_\sigma C(sI - A)^{-1}B \\
&= F(sI - A)^{-1} \left(I + \frac{1}{\sqrt{\sigma}} BU_\sigma C(sI - A)^{-1} \right)^{-1} \frac{1}{\sqrt{\sigma}} BU_\sigma C(sI - A)^{-1}B \\
&=^{(\dagger)} F(sI - A)^{-1} \frac{1}{\sqrt{\sigma}} BU_\sigma \left(I + \frac{1}{\sqrt{\sigma}} C(sI - A)^{-1}BU_\sigma \right)^{-1} C(sI - A)^{-1}B \\
&= F(sI - A)^{-1}BU_\sigma \left(\sqrt{\sigma} I + C(sI - A)^{-1}BU_\sigma \right)^{-1} C(sI - A)^{-1}B \\
&\to F(sI - A)^{-1}B \text{ as } \sigma \downarrow 0.
\end{aligned} \tag{59}$$

In (†), we used the matrix identity $(I + AB)^{-1}A = A(I + BA)^{-1}$. Thus, L_σ approaches the expression $L_0(s) = F(sI-A)^{-1}B$, i.e. the loop gain for full state feedback. Accordingly, the guaranteed gain and phase margins are recouped. This is called loop transfer recovery (LTR).

6.3 $\mathcal{H}_2$ optimization

In this section we define the LQG problem as a special case of a larger class of problems known as H2 optimization. In many applications it is difficult to establish the precise stochastic properties of disturbances and noise signals. Very often in the application of the LQG problem to control system design the noise intensities V and W play the role of design parameters rather than that they model reality.

6.3.1 $\mathcal{H}_2$ norm

For the system

$$\left.\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bv(t), \\ y(t) & = & Cx(t) \end{array}\right\} \quad t \in \mathbb{R}\,. \tag{60}$$

with the transfer matrix $H(s) = C(sI - A)^{-1}B$, suppose that the signal v is white noise with covariance function $Ev(t)v^T(t') = V\delta(t-t')$. Then the output y of the system is a stationary stochastic process with spectral density matrix

$$S_y(f) = H(j2\pi f)VH^*(j2\pi f), \quad f \in \mathbb{R}.$$

As a result, the mean square output is

$$Ey^T(t)y(t) = tr\int_{-\infty}^{\infty} S_y(f)df = tr\int_{-\infty}^{\infty} H(j2\pi f)VH^*(j2\pi f)df$$

Recall that the H2-norm of the system

$$||H||_2^2 = \frac{1}{2\pi}tr\int_{-\infty}^{\infty} H(j\omega)H^*(j\omega)d\omega = tr\int_{-\infty}^{\infty} H(j2\pi f)H^*(j2\pi f)df\ .$$

Thus, for the unit white noise $V = I$, the mean square output $Ey^T(t)y(t)$ equals precisely the square of the H2-norm of the system.

6.3.2 $\mathcal{H}_2$ norm

In this subsection we rewrite the time domain LQG problem into an equivalent frequency domain H2 optimization problem. To simplify the expressions to come we assume that $Q = I$ and $R = I$, that is, the LQG performance index is

$$\lim_{t\to\infty} E[zT(t)z(t) + uT(t)u(t)]. \tag{61}$$

And the open-loop system is

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) + Gv(t), \\ y(t) & = & Cx(t) + w(t) \end{array} \qquad t \in \mathbb{R}\ . \tag{62}$$

Consider Fig. 3, which consists of the interconnected plant $P(s)$ and controller $u = K(s)y$, driven by the external signals v and w.

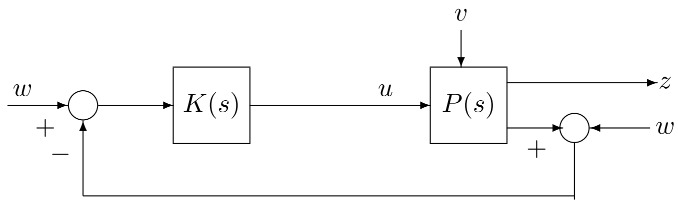

Figure 3: Feedback control configuration

We can compute the transfer matrix $H(s)$ such that

$$\begin{bmatrix} z \\ u \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \tag{63}$$

and the steady-state mean square error (61) becomes

$$\begin{array}{rcl} \lim_{t\to\infty} E[zT(t)z(t) + uT(t)u(t)] & = & \lim_{t\to\infty} E\left(\begin{bmatrix} z(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} z(t) \\ u(t) \end{bmatrix} \right) \\ & = & tr \int_{-\infty}^{\infty} H(j2\pi f)H^*(j2\pi f)df \\ & = & ||H||_2^2. \end{array}$$

Thus, solving the LQG problem amounts to minimizing the H2 norm of the closed-loop system of Fig. 3 with (v, w) as input and (z, u) as output.

6.3.3 The standard H2 problem and its solution

Configuration of Fig. 3 is a special case of a standard LTI feedback configuration of Figure 4. In Figure 4, w is the external input (v and w in Fig. 3). It is typically used to describe external noises and internal perturbations caused by nonlinearity and uncertainty. The signal z is called cost output, and represents signals which the designer wants to be small (z and u in Fig. 3). The second output y is the observed output, which represents input of the controller K (to be designed). ideally should be zero Furthermore, u is the control input, and y the observed output. The block G is the generalized plant, and K the compensator.

In order to define a standard LTI feedback optimization problem, sketched on Figure 4 (note that this is an LFT), one has to specify the plant G, and a performance measure. The performance measure specifies a particular qualitative measure of smallness for the cost output z. The optimization process will aim at finding an LTI feedback system K which makes the feedback system on Figure 4 stable, and minimizes the closed loop system from w to z. Two most popular measures (norms) of how large a stable LTI system in feedback optimization are: H2 norm and H-Infinity norm. As previously discussed, the H2 norm measures the size of an LTI system as an integral of square of amplitude of its frequency response, while the H-Infinity norm uses the maximal (over all frequencies) amplitude. There are plants G for which the standard H2 and H-Infinity optimization algorithms are guaranteed to fail. Certain well-posedness conditions have to be satisfied to avoid such failures.

When a CT LTI system describes the plant in a standard feedback optimization setup, its input is partitioned into the disturbance and actuator components. Similarly, the output is partitioned into the cost and measurement components. Consequently, it is natural to decompose the corresponding B, C, D matrices:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u && (64) \\ z &= C_1 x + D_{11} w + D_{12} u && (65) \\ y &= C_2 x + D_{21} w + D_{22} u && (66) \end{aligned}$$

or

$$G = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right) \tag{67}$$

The standard H2 optimization problem is the problem of choosing the compensator K in the block diagram of Fig. 4 such that it

1. stabilizes the closed-loop system, and
2. minimizes the H2-norm of the closed-loop system (with w as input and z as output).

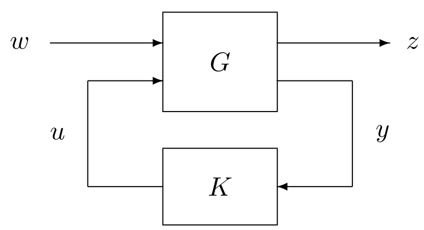

Figure 4: Standard feedback optimization setup

The H2 problem may be solved by reducing it to an LQG problem. The derivation necessitates the introduction of some assumptions as below. They are natural assumptions for LQG problems. First, if $D_{11} \neq 0$ below then the output z has a white noise component that may well make the mean square output $Ez^T(t)z(t)$ infinite. We therefore assume that $D_{11} = 0$.

Theorem 6.4 *Consider the standard H2 optimization problem for the generalized plant*

$$\begin{array}{rcllll} \dot{x} & = & Ax & +B_1 w & +B_2 u \\ z & = & C_1 x & & +D_{12} u \\ y & = & C_2 x & +D_{21} w & +D_{22} u \end{array} \tag{68}$$

with the following assumptions:

1. *The pair (A, B_2) to be stabilizable, and the pair (C_2, A) to be detectable*
2. *The matrix*

$$M_y(s) = \begin{bmatrix} A - sI & B_1 \\ C_2 & D_{21} \end{bmatrix} \tag{69}$$

   must be right invertible (i.e. its rows must be linearly independent) for all s on the imaginary axis, including the case $s = \infty$, when the condition is that matrix D_{21} must be right invertible.

3. *The matrix*

$$M_u(s) = \begin{bmatrix} A - sI & B_2 \\ C_1 & D_{12} \end{bmatrix} \tag{70}$$

   must be left invertible (i.e. its columns must be linearly independent) for all s on the imaginary axis, including the case $s = \infty$, when the condition is that matrix D_{12} must be left invertible.

Under these assumptions the optimal output feedback controller $u = K(s)y$ is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_2 u(t) + L[y(t) - C_2\hat{x}(t) - D_{22}u(t)] \tag{71}$$

$$u(t) = -F\hat{x}(t). \tag{72}$$

The observer and state feedback gain matrices are

$$F = (D_{12}^T D_{12})^{-1}(B_2^T X + D_{12}^T C_1), \tag{73}$$

$$L = (YC_2^T + B_1 D_{21}^T)(D_{21}D_{21}^T)^{-1}. \tag{74}$$

The symmetric matrices X and Y are the unique positive-definite solutions of the algebraic Riccati equations

$$\begin{array}{rcl} A^T X + XA + C_1^T C_1 - (XB_2 + C_1^T D_{12})(D_{12}^T D_{12})^{-1}(B_2^T X + D_{12}^T C_1) & = & 0 \\ AY + AY^T + B_1 B_1^T - (YC_2^T + B_1 D_{21}^T)(D_{21}D_{21}^T)^{-1}(C_2 Y + D_{21}B_1^T) & = & 0. \end{array}$$

$\square$

The first set of constraints guarantees **existence of a stabilizing feedback** (so that the set of feasible decision parameters is not empty). If this is not the case, the physical feedback control setup should be modified by adding extra actuators (to make the pair (A, B_2) stabilizable) and/or sensors (to make the pair (C_2, A) detectable).

The second set of constraints guarantees **existence of an optimal controller** (and is also related to numerical well-posedness of the optimization problem). Informally speaking, it requires that

- $w \to y$: every component of the measurement output of the plant be dependent on the disturbance input at every frequency, and
- $u \to z$: every component of the actuator input affect the cost output at every frequency.

When A has no eigenvalues on the imaginary axis, the condition can be rewritten as left invertibility of transfer matrix

$$P_{12}(s) = P_{u \to z}(s) = C_1(sI - A)^{-1}B_2 + D_{12} ,$$

and right invertibility of

$$P_{21}(s) = P_{w \to y}(s) = C_2(sI - A)^{-1}B_1 + D_{21} .$$

The case when $M_u(s)$ is not left invertible at some $s = j\omega$ will be referred to as control singularity at frequency ω. Similarly, the case when $M_y(s)$ is not right invertible at some $s = j\omega$. will be referred to as sensor singularity at frequency ω.

6.4 State space solution of the standard $\mathcal{H}_\infty$ problem

Among the various solutions of the suboptimal standard $\mathcal{H}_\infty$ problem, the one based on state space realizations is the most popular [2]. In these approaches it is assumed that the generalized plant G is proper. Hence it has a realization of the form

$$\begin{aligned} x &= Ax + B1w + B2u, && (75)\\ z &= C_1x + D_{11}w + D_{12}u, && (76)\\ y &= C_2x + D_{21}w + D_{22}u. && (77) \end{aligned}$$

A solution of the corresponding $\mathcal{H}_\infty$ problem based on Riccati equations is implemented that requires the following conditions to be satisfied:

1. (A, B_2) is stabilizable and (C_2, A) is detectable.

[2] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis. State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems. IEEE Trans. Aut. Control, 34:831.847, 1989.

2. $\begin{bmatrix} A - j\omega & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.

3. $\begin{bmatrix} A - j\omega & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega \in \mathbb{R}$.

4. D_{12} and D_{21} have full rank.

With these assumptions the formulae for *suboptimal* controllers are as follows:

Theorem 6.5 (solution of the standard $\mathcal{H}_\infty$ problem) *Consider the configuration of Fig. 4 and assume the above four assumptions are satisfied, and for simplicity, that also*

$$\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad D_{12}^T \begin{bmatrix} C_1 \\ D_{12} \end{bmatrix} \begin{bmatrix} 0 & I \end{bmatrix} \tag{78}$$

Then there exists a stabilizing controller for which $||H||_\infty < \gamma$ *iff the following three conditions hold.*

1. $AQ + QA^T + Q(\frac{1}{\gamma^2} C_1^T C_1 - C_2^T C_2)Q + B_1 B_1^T = 0$ *has a stabilizing solution* $Q \geq 0$,

2. $PA + A^T P + P(\frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T)P + C_1^T C_1 = 0$ *has a stabilizing solution* $P \geq 0$.

3. *All eigenvalues of* QP *have magnitude less than* γ^2.

And this controller can be realized by

$$\begin{array}{rcl} x & = & (A + [\frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T]P)\hat{x} + (I - \frac{1}{\gamma^2} QP)^{-1} Q C_2^T (y - C_2 \hat{x}) \\ u & = & -B_2^T P \hat{x} \end{array} \tag{79}$$

□

The formulae for K are rather cumbersome if the assumptions (78) do not hold, computationally it makes no difference. The solution, as we see, involves two algebraic Riccati equations whose solutions define an observer and state feedback law. [3] The problem can also be solved using linear matrix inequalities (LMIs).

The above $\mathcal{H}_\infty$ solution have the following properties:

[3]The full solution is documented in a paper by Glover and Doyle, State-space formulae for all stabilizing controllers that satisfy an H1-norm bound and relations to risk sensitivity. Systems & Control Letters, 11:167.172, 1988, and K. Glover and J. C. Doyle. A state space approach to $\mathcal{H}_\infty$ optimal control. In H. Nijmeijer and J. M. Schumacher, editors, Three Decades of Mathematical System Theory, volume 135 of Lecture Notes in Control and Information Sciences. Springer-Verlag, Heidelberg, etc., 1989.

- For the two Riccati equations to have a solution it is required that the associated Hamiltonian matrices

$$\left[\begin{array}{cc} A & B_1 B_1^T \\ -\frac{1}{\gamma^2} C_1^T C_1 + C_2^T C_2 & -A^T \end{array}\right], \quad \left[\begin{array}{cc} A & \frac{1}{\gamma^2} B_1 B_1^T - B_2^T B_2 \\ -C_1^T C_1 & -A^T \end{array}\right]$$

  have no imaginary eigenvalues. Stated differently, if γ_0 is the largest value of γ for which one or both of the above two Hamiltonian matrices has an imaginary eigenvalue, then $\gamma_{opt} \geq \gamma_0$.

- The controller in (79) is stabilizing iff

$$Q \geq 0, \; P \geq 0, \; \lambda_{max}(QP) < \gamma^2$$

- The controller (79) is of the same order as the generalized plant.

- The transfer matrix of the controller (79) is strictly proper.

6.5 MATLAB format for H2 and H-Infinity optimization

This subsection describes the use of μ-Analysis and Synthesis Toolbox, the recommended set of routines for H2 and H-Infinity optimization. To define a plant state space model P in this toolbox, use
`P=pck(A,B,C,D);`
where A, B, C, D are the matrices defining

$$P = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{80}$$

To call an H2 optimization algorithm with a minimal set of input and output arguments, use
`K=h2syn(P,`n_y`,`n_u`);`
where P is the plant model in the packed format, n_y is the number of sensors (i.e. the dimension of vector y), n_u is the number of actuators (the dimension of u), K is the optimal controller model in the packed format. If
`[K,CL,GAM,INFO] = h2syn(P,NMEAS,NCON);`
is used, then K= LTI controller, CL= lft(P,K) is a closed-loop system $T_{w \to z}$, GAM = norm(CL) is H2 norm of $T_{w \to z}$, and INFO contains additional information.

To get the coefficient matrices of the controller, unpack K with
`[Af,Bf,Cf,Df]=unpck(K);`

To call an H∞ optimization algorithm with a minimal set of input and output arguments, use
`K=hinfsyn(P,nmeas,ncon,gmin,gmax,tol);`
Here the output argument and the first three input arguments have same meaning as in H2 optimization. The presence of the last three input arguments is caused by the fact that function `hinfsyn.m` is not capable of finding the optimal H-Infinity controller. Instead it searches for a controller which yields closed

loop H∞ norm γ, such that $(\gamma - \gamma_{\min})/\gamma$ is not larger than the relative tolerance parameter `tol`. Here $\gamma_{\min}$ is the minimal achievable H-Infinity norm, `gmin` is a known lower bound for $\gamma_{\min}$ (one can safely use `gmin`=0), and `gmax` is a known upper bound for $\gamma_{\min}$. While it is not always easy to find an upper bound, `hinfsyn.m` will tell you if your current guess is too low. Similarly, you can use
`[K,CL,GAM,INFO] = hinfsyn(...);`
and also specify 'METHOD'. Riccati solution is default, and 'lmi' (LMI solution) or 'maxe'(Maximum entropy solution) can be used alternatively.

6.6 A simple design example

Consider a simple feedback design task shown on Figure 5, where $G = G(s) = 1/s^2$ is a given open loop plant model, and $F = F(s)$ is the feedback controller to be designed to provide a desired closed loop response $T = T(s)$ from reference input r to controlled output q. Assume that the ideal desired closed loop response is $T(s) = T_0(s) = 1/(s+1)$. This response cannot be achieved by using a proper controller transfer function $F(s)$. However, one can try to approximate the ideal response $T_0(s)$ by choosing an appropriate stabilizing proper controller F, while checking the trade-off between the quality of approximation and the power utilized by the controller. Both H2 and H-Infinity optimization frameworks are easy to use for this purpose.

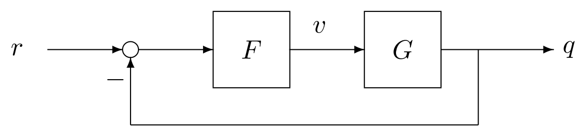

Figure 5: A simple feedback design example

6.6.1 Reduction to a standard optimization setup

To rewrite the design specifications as a standard feedback optimization setup, introduce the ideal response transfer function to the block diagram, and define e as the difference between the actual and the desired response (Fig. 6).

Our objective is to make the closed loop transfer function from r to e small by selecting a controller with input $r - q$ and output v. In terms of the standard setup, this calls for selecting r as the disturbance input w, v as the actuator input u, $r - q$ as the sensor output y, and e as the cost output z (Fig. 7). Then

$$\begin{aligned} z &= T_0 r - q = \frac{1}{s+1} r - \frac{1}{s^2} v \\ y &= r - \frac{1}{s^2} v \, . \end{aligned}$$

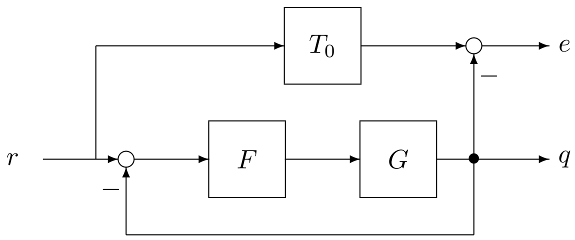

Figure 6: Reduction to standard optimization setup

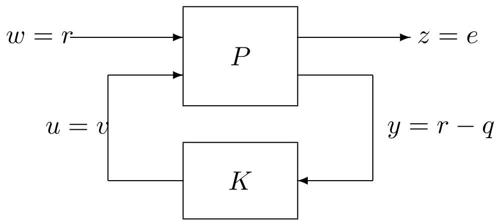

Figure 7: Modification to the standard feedback optimization setup

The corresponding plant transfer matrix P is given by

$$P(s) = \begin{bmatrix} 1/(s+1) & -1/s^2 \\ 1 & -1/s^2 \end{bmatrix} ,$$

which corresponds to a minimal state space model with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} , \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} , \tag{81}$$

$$C_1 = [1\ 0\ -1] , \quad C_2 = [0\ 0\ -1] , \quad D_{11} = D_{22} = D_{12} = 0 , \quad D_{21} = 1 \tag{82}$$

6.6.2 Modification for well-posedness

An attempt to use `h2syn.m` or `hinfsyn.m` on this setup will produce an error message, because the setup is not well-posed. One obvious reason for this is absence of a control penalty (causing a control singularity at $\omega = \infty$ – recall $P_{u \to z}(s) = C_1(sI - A)^{-1}B_2 + D_{12}$, and $D_{12} = 0$ here.). This can be fixed by adding $\epsilon_u v$ as an extra component of the cost z, where ϵ_u will become a tuning parameter for the designer (the larger ϵ_u is, the less power the optimal controller will use, at the expense of providing a poorer approximation of the desired closed loop response). A less obvious problem with the setup is a sensor singularity at $\omega = 0$, which is not as easy to spot since the open loop plant has a pole at $s = 0$ (note A in (81) has eigenvalues on $j\omega$ axis). Actually, this is a double sensor singularity at $\omega = 0$, since the determinant of $M_y(s)$ has a double root at $s = 0$ (note $det(sI - A) = s^2(s+1)$). This singularity can be fixed by having an extra disturbance signal $f = \epsilon_y w_2$ added to the input of the double integrator. Here parameter ϵ_y will quantify sensitivity of the closed loop system with respect to the plant disturbances (the smaller ϵ_y, the larger the sensitivity).

Fig. 8 shows this modification. The resulting standard setup will have two-component w, two-component z, and two tuning parameters ϵ_u and ϵ_y.

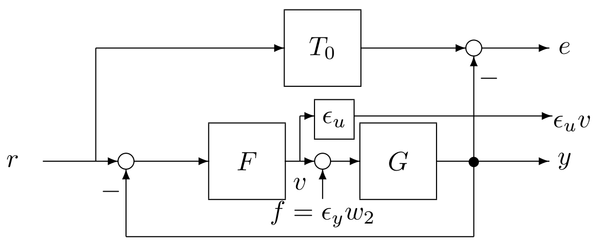

Figure 8: Adding extra disturbance

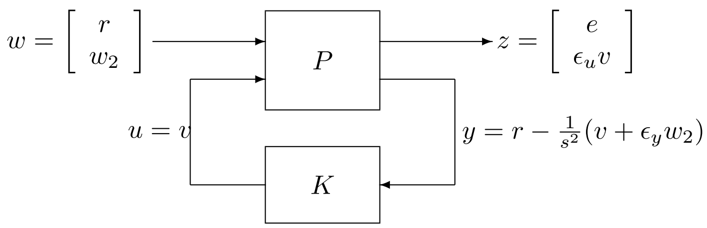

Figure 9: Modification to the standard feedback optimization setup

The plant transfer matrix in Fig. 9 will have the form

$$P(s) = \begin{bmatrix} 1/(s+1) & -\epsilon_y/s^2 & -1/s^2 \\ 0 & 0 & \epsilon_u \\ 1 & -\epsilon_y/s^2 & -1/s^2 \end{bmatrix} \tag{83}$$

and a minimal state space model given by

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + w_1(t) \ , \\ \dot{x}_2(t) &= u(t) + \epsilon_y w_2(t) \ , \\ \dot{x}_3(t) &= x_2(t) \ , \\ z_1(t) &= x_1(t) - x_3(t) \ , \\ z_2(t) &= \epsilon_u u(t) \ , \\ y(t) &= -x_3(t) + w1(t) \ . \end{aligned}$$

A sample MATLAB code for H2 optimization is given by

```
eu=0.01;
ey=0.01;
A=[-1 0 0;0 0 0;0 1 0];
B1=[1 0;0 ey;0 0];
B2=[0;1;0];
C1=[1 0 -1;0 0 0];
C2=[0 0 -1];
D11=zeros(2);
D12=[0;eu];
D21=[1 0];
D22=0;
P=pck(A,[B1 B2],[C1;C2],[D11 D12;D21 D22]);
K=h2syn(P,1,1);
[A,Bf,Cf,Df]=unpck(K);
```

To modify this for H-Infinity optimization, simply replace the `h2syn` line with `K=hinfsyn(P,1,1,0,1,0.01);`.