7 Lyapunov and Riccati Equations

Notation:
$\mathbb{F}$ = either $\mathbb{R}$ or $\mathbb{C}$
For $A \in \mathbb{R}^{n\times n}$, $spec(A)$ = the set of eigenvalues of A.
$\mathbb{C}_-^o = \{\lambda \in \mathbb{C} : Re(\lambda) < 0\}$

7.1 Lyapunov Operator

Definition 7.1 *For $A \in \mathbb{F}^{n\times n}$, a linear operator*

$$\mathcal{L}_A(X) := A^*X + XA$$

is called Lyapunov operator.

Suppose A is diagonalizable (the following can be generalized even if this is not the case). Let $\{\lambda_1, \cdots, \lambda_n\}$ be the eigenvalues of A. Then the eigenvector matrix of A^*

$$V = [v_1, \cdots, v_n] \in \mathbb{C}^{n\times n}$$

is invertible, and

$$A^* v_i = \bar{\lambda}_i v_i \ .$$

Lemma 7.2 *For each $1 \le i, j \le n$, define*

$$X_{ij} = v_i v_j^* \in \mathbb{C}^{n\times n}.$$

The set $\{X_{ij}\}_{1\le i,j\le n}$ is a linearly independent set, and this set is a full set of eigenvectors for the linear operator $\mathcal{L}_A$. Moreover, the eigenvalues of $\mathcal{L}_A$ is the set of complex numbers $\{\bar{\lambda}_i + \lambda_j\}_{1\le i,j\le n}$.

Proof. Let

$$V^{-1} = \begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}, \quad w_i \in \mathbb{C}^n \ .$$

Then $V^{-1}V = I$, so

$$w_i^T v_j = \delta_{ij}.$$

Suppose $\{\alpha_{ij}\}_{1\le i,j\le n}$ are scalars such that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} X_{ij} = 0_{n\times n}$$

Then

$$
\begin{aligned}
0 &= w_l^T \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} X_{ij} \right) \bar{w}_k \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} w_l^T v_i v_j^* \bar{w}_k \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \delta_{il} \delta_{jk} \\
&= \alpha_{lk}
\end{aligned}
$$

which holds for any $1 \leq l, k \leq n$. Therefore, $\{X_{ij}\}_{1 \leq i,j \leq n}$ is a liearly independent set.
Also, by definition of $\mathcal{L}_A$, we have

$$
\begin{aligned}
\mathcal{L}_A(X_{ij}) &= A^* X_{ij} + X_{ij} A \\
&= A^* v_i v_j^* + v_i v_j^* A \\
&= (\bar{\lambda}_i v_i) v_j^* + v_i (\lambda_j v_j^*) \\
&= (\bar{\lambda} + \lambda_j) X_{ij}
\end{aligned}
$$

Hence, X_{ij} is an eigenvector of $\mathcal{L}_A$ corresponding to eigenvalue $\{\bar{\lambda}_i + \lambda_j\}$. □

Theorem 7.3 *Let $A \in \mathbb{F}^{n \times n}$ be given and suppose that A is stable. Then $\mathcal{L}_A$ is invertible, and for any $Q \in \mathbb{F}^{n \times n}$, the unique $X \in \mathbb{F}^{n \times n}$ solving $\mathcal{L}_A(X) = -Q$ is given by*

$$
X = \int_0^\infty e^{A^* \tau} Q e^{A \tau} d\tau \,. \tag{1}
$$

Proof. If A is stable (all eigenvalues in $\mathbb{C}_-^\circ$), then $\bar{\lambda}_i + \lambda_j \neq 0$ for all i, j. Thus $\mathcal{L}_A$ is an invertible linear operator. For $t \geq 0$,

$$
S(t) := \int_0^t e^{A^* \tau} Q e^{A \tau} d\tau
$$

has a well-defined limit as $t \to \infty$ because A is stable. And $X := \lim_{t \to \infty} S(t)$.

$$
\begin{aligned}
A^* S(t) + S(t) A &= \int_0^t \left(A^* e^{A^* \tau} Q e^{A \tau} + e^{A^* \tau} Q e^{A \tau} A \right) d\tau \\
&= \int_0^t \frac{d}{d\tau} \left(e^{A^* \tau} Q e^{A \tau} \right) d\tau \\
&= e^{A^* t} Q e^{A t} - Q \,.
\end{aligned}
$$

Taking $\lim_{t \to \infty}$ gives

$$
A^* X + X A = -Q
$$

as desired.

□

Theorem 7.4 *Let $A, Q \in \mathbb{F}^{n\times n}$ be given and suppose that A is stable and $Q = Q^* \geq 0$. Then (A, Q) is observable iff*

$$X := \int_0^\infty e^{A^*\tau} Q e^{A\tau} d\tau \; > 0 \, .$$

Proof.
$(\Leftarrow)$ Suppose that (A, Q) is not observable. Then $\exists x_0 \in \mathbb{F}^n, x_0 \neq 0$ such that

$$Qe^{At}x_0 = 0$$

for all $t \geq 0$. Thus $x_0^* e^{A^*t} Q e^{At} x_0 = 0$ for all $t \geq 0$. Integration gives

$$\begin{aligned} 0 &= \int_0^t \left(x_0^* e^{A^*t} Q e^{At} x_0 \right) d\tau \\ &= x_0^* \left(\int_0^t e^{A^*\tau} Q e^{A\tau} d\tau \right) x_0 \\ &= e^{A^*t} X e^{At} \, . \end{aligned}$$

Thus X is not pos def.
$(\Rightarrow)$ Suppose that X is not pos def (note X is at least pos. semidef.). Then $\exists x_0 \in \mathbb{F}^n, x_0 \neq 0$ such that $x_0^* X x_0 = 0$. Using (1) and the fact that $Q \geq 0$ gives

$$\int_0^\infty ||Q^{1/2} e^{A\tau} x_0|| d\tau = 0$$

The integrand is conti and nonneg for all $\tau \geq 0$, this it must be 0 for all $\tau \geq 0$. Since $x_0 \neq 0$, (A, Q) is not observable.

□

Theorem 7.5 *Let (A, C) be detectable and suppose X is any solution to $\mathcal{L}_A = -C^*C$. Then $X \geq 0$ iff A is stable.*
(Note that there is no assumption that $\mathcal{L}_A$ is invertible, hence there could be multiple solutions.)

Proof.
$(\Leftarrow)$ Since A is stable, $\mathcal{L}_A$ is invertible, and the unique $X \in \mathbb{F}^{n\times n}$ solving $\mathcal{L}_A(X) = -C^*C$ is given by

$$X = \int_0^\infty e^{A^*\tau} C^* C e^{A\tau} d\tau \, ,$$

which is clearly pos semidef.

($\Rightarrow$) Suppose that A is not stable. Then $\exists v \in \mathbb{C}^n, v \neq 0, \lambda \in \mathbb{F}, Re(\lambda) \geq 0$ such that $Av = \lambda v$. Since (A, C) is detectable, $Cv \neq 0$. Note that

$$\begin{aligned} 0 \quad ||Cv||^2 &= v^* C^* C v \\ &= -v^*(A^* X + XA)v \\ &= -(\bar{\lambda} + \lambda) v^* X v \\ &= -2Re(\lambda) v^* X v \end{aligned}$$

Since $Re(\lambda) \geq 0$ we must have $Re(\lambda) > 0$ and $v^* X v < 0$. Hence X is not pos def.

□

Let A be stable, so $\mathcal{L}_A$ is invertible. Use $\mathcal{L}_A^{-1}(-Q)$ to denote the unique solution to $\mathcal{L}_A = -Q$ where $Q = Q^*$. Then we can obtain the following results about the ordering of the solutions.

Lemma 7.6 *If symmetric matrices $Q_1 \geq Q_2$, then*

$$\mathcal{L}_A^{-1}(-Q_1) \geq \mathcal{L}_A^{-1}(-Q_2) \, .$$

Proof. See below.

Lemma 7.7 *If symmetric matrices $Q_1 > Q_2$, then*

$$\mathcal{L}_A^{-1}(-Q_1) > \mathcal{L}_A^{-1}(-Q_2) \, .$$

Proof. Let X_1, X_2 be the solutions, $A^* X_i + X_i A = -Q_i$. Then

$$A^*(X_1 - X_2) + (X_1 - X_2)A = -(Q_1 - Q_2)$$

Since A is stable,

$$(X_1 - X_2) = \int_0^\infty e^{A^* \tau}(Q_1 - Q_2) e^{A\tau} d\tau \, ,$$

If $(Q_1 - Q_2) \geq 0$ then $(X_1 - X_2) \geq 0$, which proves the previous lemma. If $(Q_1 - Q_2) > 0$ then $(A, (Q_1 - Q_2))$ is observable, so by the theorem 7.4, $(X_1 - X_2) > 0$.

□

But converses of the lemmas 7.6 and 7.7 are NOT true.

7.2 Algebraic Riccati Equation

- $A^T X + XA + XRX - Q = 0$: ARE (1)

  $A, R, Q \in \mathbb{R}^{n \times n}$ given.

  $R = R^T$, $Q = Q^T$

  solutions $X \in \mathbb{C}^{n \times n}$.

- multiple solns in general,

  but at most one soln w/ $A+RX$ Hurwitz.

- We're primarily interested in solns X s.t.

  $\mathrm{spec}(A+RX) \subset \mathbb{C}^{\circ}_{-}$,

  called stabilizing solns of the Riccati eqn

- Hamiltonian Matrix

  $$H := \begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix} \quad (2)$$

  is used to obtain the solns to ARE.

H is real, we know that the eigvalues of H are symmetric about the real axis. Now, we'll see additional symmetry in the eigvalues.

Also, there is 1-1 correspondence between n--dim invariant subspaces of H with the invertibility property and solutions of the Riccati equation. And $A + RX$ is similar to the restriction of H on the inv subspace ν, $H|_{\nu}$. So the eigvalues of $A + RX$ willl always be a subset of the eigenvalues of H.

Note λ is an eval of H iff $-\bar{\lambda}$ is.

Pf $J := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$. $\Rightarrow J^{-1}HJ = -H^{*}$ ($= -JHJ$ ($\because J^2 = -I$)).

$\therefore H$ and $-H^{*}$ are similar.

$\Rightarrow$ the spectrum of H is symmetric about the Im axis.

If H has no evals on the Im axis, then n evals in OLHP, and n evals in ORHP.

Thm Let $\mathcal{V} \subset \mathbb{C}^{2n}$ denote the n-dimensional invariant subspace of H, and let $X_1 \in \mathbb{C}^{n\times n}$, $X_2 \in \mathbb{C}^{n\times n}$ be two cpx matrices st.

$$\mathcal{V} = \operatorname{span}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

If X_1 is invertible, then $X := X_2 X_1^{-1}$ is a stabilizing soln to ARE, and $\operatorname{spec}(A+RX) = \operatorname{spec}(H|_{\mathcal{V}})$.

Pf Since $\mathcal{V}$ is H-invariant, $\exists \Lambda \in \mathbb{C}^{n\times n}$ s.t.

$$\begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\Lambda \quad (5)$$

Λ: matrix representation of the map $H|_{\mathcal{V}}$

$$\Rightarrow \begin{cases} AX_1 + RX_2 = X_1\Lambda & (3) \\ QX_1 - A^T X_2 = X_2\Lambda & (4) \end{cases}$$

$X_2X_1^{-1} \times (3) \times X_1^{-1} \Rightarrow X_2X_1^{-1}A + (X_2X_1^{-1})R(X_2X_1^{-1}) = X_2\Lambda X_1^{-1}$.

$(4) \times X_1^{-1} \Rightarrow Q - A^T X_2X_1^{-1} = X_2\Lambda X_1^{-1}$

$\Rightarrow A^T X_2X_1^{-1} + X_2X_1^{-1}A + X_2X_1^{-1}RX_2X_1^{-1} - Q = 0$

$\therefore X_2X_1^{-1}$ is a soln to ARE.

$(3) \Rightarrow A + R(X_2X_1^{-1}) = X_1\Lambda X_1^{-1}$

$\therefore A + R(X_2X_1^{-1})$ and Λ have the same evals.

Thm 2 Let $\mathcal{V}, X_1, X_2$ be as above.
Suppose there are two other matrices $\tilde{X}_1, \tilde{X}_2 \in \mathbb{C}^{n\times n}$ s.t. $\mathcal{V} = \text{span}\begin{bmatrix}\tilde{X}_1\\ \tilde{X}_2\end{bmatrix}$. Then X_1 is invertible iff $\tilde{X}_1$ is invertible. And in fact, $X_2X_1^{-1} = \tilde{X}_2\tilde{X}_1^{-1}$.

Pf Since both sets of columns span the same n-dim subspace $\exists$ an invertible $K \in \mathbb{C}^{n\times n}$ s.t.

$$\begin{bmatrix}X_1\\ X_2\end{bmatrix} = \begin{bmatrix}\tilde{X}_1\\ \tilde{X}_2\end{bmatrix} K.$$

$\therefore$ X_1 is invertible iff $\tilde{X}_1$ is invertible,

$$X_2X_1^{-1} = (\tilde{X}_2K)(\tilde{X}_1K)^{-1} = \tilde{X}_2\tilde{X}_1^{-1}.$$

Rmk This thm shows that the specific choice of matrices X_1, X_2 is not important. And the question of whether or not X_1 is invertible is a property of the subspace, not of the particular basis choice for the subspace.

Thm3 (Converse of thm1.)
If $X \in \mathbb{C}^{n\times n}$ is a soln to the ARE, then $\exists X_1, X_2 \in \mathbb{C}^{n\times n}$ with X_1 invertible, such that $X = X_2X_1^{-1}$ and the columns of $\begin{bmatrix}X_1\\ X_2\end{bmatrix}$ span an n-dim invariant subspace of H.

Pf $\Lambda := A + RX \in \mathbb{C}^{n\times n}$ ①

$\Rightarrow X\Lambda = XA + XRX = Q - A^TX$ ②

① & ② $\Rightarrow$ $\begin{bmatrix}A & R\\ Q & -A^T\end{bmatrix}\begin{bmatrix}I\\ X\end{bmatrix} = \begin{bmatrix}I\\ X\end{bmatrix}\Lambda.$

$\therefore$ The columns of $\begin{bmatrix}I\\ X\end{bmatrix}$ span an n-dim invariant subspace of H. Defining $X_1 = I$, $X_2 = X$ completes the proof. #

Thm 4 Suppose $\mathcal{V}$ is an n-dim invariant subspace of H, and $X_1, X_2 \in \mathbb{C}^n$ form a matrix whose columns span $\mathcal{V}$ as before. Let Λ be a matrix representation of $H|_{\mathcal{V}}$. If $\lambda_i + \overline{\lambda_j} \neq 0$ for all evals $\lambda_i, \lambda_j \in \operatorname{spec}(\Lambda)$, then $X_1^* X_2 = X_2^* X_1$.

Pf Using $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ defined before,

$$[X_1^* \; X_2^*]\, J H \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \text{is Hermitian.} \quad (\because JJ^* = I,\ H^* = JHJ)$$

$$\| $$

$$[X_1^* \; X_2^*] \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Lambda = \underbrace{(-X_1^* X_2 + X_2^* X_1)}_{:= W} \Lambda .$$

Then
$$W^* = X_1^* X_2 - X_2^* X_1 = -W,$$
$$W\Lambda + \Lambda^* W = W\Lambda + (-W\Lambda)^* = 0$$
$$(\because W\Lambda \text{ is Hermitian from above}).$$

By the eigvalue assumption on Λ, the linear map $\mathcal{L}_\Lambda : \mathbb{C}^{n\times n} \to \mathbb{C}^{n \times n}$ defined by
$$\mathcal{L}_\Lambda(X) = X\Lambda + \Lambda^* X$$
is invertible.

$\therefore\ W = 0$. #

Corollary Under the same assumptions as in thm 4, if X_1 is invertible then $X_2 X_1^{-1}$ is Hermitian.

<u>Thm5</u> Let $\mathcal{V}$ be an n-dim invariant subspace of H with the invertibility property (i.e. in any basis, the top portion is invertible), and let $X_1, X_2 \in \mathbb{C}^n$ form a matrix whose columns span $\mathcal{V}$. Then $\mathcal{V}$ is conjugate symmetric (i.e. $\{\bar{v} : v \in \mathcal{V}\} = \mathcal{V}$) iff $X_2 X_1^{-1} \in \mathbb{R}^{n\times n}$.

<u>Rmk</u> This says that real (rather than cpx) solns arise when a symmetry condition is imposed on the invariant subspace.

<u>Pf</u> ($\Leftarrow$) $X := X_2 X_1^{-1}$. By assumption $X \in \mathbb{R}^{n\times n}$ and $\text{Span}\begin{bmatrix} I \\ X \end{bmatrix} = \mathcal{V}$. $\therefore$ $\mathcal{V}$ is conjugate symmetric.

($\Rightarrow$) Since $\mathcal{V}$ is conjugate symmetric, $\exists$ an invertible matrix $K \in \mathbb{C}^{n\times n}$ s.t.

$$\begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} K .$$

$$\Rightarrow \overline{(X_2 X_1^{-1})} = \bar{X}_2 \bar{X}_1^{-1} = X_2 K (X_1 K)^{-1} = X_2 X_1^{-1}. \quad \therefore \in \mathbb{R}^{n\times n}. \#$$

<u>Thm6</u> There is at most one soln X to the Riccati eqn s.t. A+RX is a stable matrix, and if it does exist, then it is real & symmetric.

<u>Rmk</u> For any square matrix M, there is a 'largest' stable inv subspaces, i.e. a stable inv subspace that contains every other stable inv subspace of the linear map. It is simply the span of all the eigvecs and gen eigvecs associated with the OLHP eigvals.

Pf If X is such a stabilizing soln, then by Thm 3, it can be constructed from some n-dim invariant subspace of H (call it $\mathcal{V}$).

If $A+RX$ is stable, then $\operatorname{spec}(H|_{\mathcal{V}}) = \operatorname{spec}(A+RX)$ are stable. But the spectrum of H is symmetric about the Im axis, so H has at most n eigvals in $\mathbb{C}_-^\circ$.

$\Rightarrow$ The only possible choice for $\mathcal{V}$ is the stable eigenspace of H.

By Thm 2, every soln (if there are any) constructed from $\mathcal{V}$ will be the same. Therefore, there is at most one stabilizing soln.

Associated with the stable eigenspace, any Λ s.t.

$$H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Lambda$$ is stable.

If the stable eigenspace has the invertibility property, then we have that $X := X_2 X_1^{-1}$ is Hermitian (by Corollary).

Since H is real, the stable eigenspace of H is conj sym, so the associated soln $X = X_2 X_1^{-1}$ is real & symmetric. #

If H has no Im-axis evals, then H has a unique n-dim stable invariant subspace $\mathcal{V}_s \subset \mathbb{C}^{2n}$, and since H is real, $\exists\, X_1, X_2 \in \mathbb{R}^{n\times n}$ s.t. $$\operatorname{span}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathcal{V}_s .$$

More concretely, $\exists\, \Lambda \in \mathbb{R}^{n\times n}$ s.t.

$$H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Lambda , \quad \operatorname{spec}(\Lambda) \subset \mathbb{C}_-^\circ .$$

Def (dom$_S$ Ric)

If $H := \begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$ has no Im-axis evals,

and the matrices $X_1, X_2 \in \mathbb{R}^{n\times n}$ for which $\operatorname{span}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is the stable, n-dim invariant subspace of H satisfy $\det(X_1) \neq 0$,

then $\underline{H \in \operatorname{dom}_S \operatorname{Ric}}$.

For $H \in \operatorname{dom}_S \operatorname{Ric}$, define $\underline{\operatorname{Ric}(H) := X_2 X_1^{-1}}$

(i.e. the unique matrix $X \in \mathbb{R}^{n\times n}$ s.t. $A+RX$ is stable, $A^T X + XA + XRX - Q = 0$, and $X = X^T$).

Thm 7 Suppose H does not have any Im-axis evals, R is either ≥ 0 or ≤ 0, and (A, R) is stabilizable. Then $H \in \operatorname{dom}_S \operatorname{Ric}$.

Pf Let $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ form a basis for the stable, n-dim invariant subspace of H (this exists, due to the eigenvalue assumption about H). Then

$$\begin{bmatrix} A & R \\ Q & -A^T \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \Lambda \qquad (5)$$

where $\Lambda \in \mathbb{C}^{n\times n}$ is a stable matrix.

Then $X_1^* X_2 = X_2^* X_1$ ($\because$ Thm 4).

Now we just need to show that X_1 is invertible.

Step 1 We prove that $\operatorname{Ker} X_1$ is Λ-invariant, i.e.

$$x \in \operatorname{Ker} X_1 \Rightarrow \Lambda x \in \operatorname{Ker} X_1.$$

Let $x \in \operatorname{Ker} X_1$.

The top row of (5) : $RX_2 = X_1\Lambda - AX_1 \qquad (6)$

$$\Rightarrow \quad x^* X_2^* R X_2 x = x^* X_2^* (X_1\Lambda - A\underbrace{X_1) x}$$
$$= x^* X_2^* X_1 \Lambda x$$
$$= \underbrace{x^* X_1^*} X_2 \Lambda x$$
$$= 0.$$

Since $R \geq 0$ or $R \leq 0$, we have $RX_2 x = 0$.

Now (6) gives $X_1 \Lambda x = 0$. $\quad \therefore \Lambda x \in \operatorname{Ker} X_1$.

<u>Step 2</u> Now, if $\operatorname{Ker} X_1 \neq \{0\}$, then $\exists$ a vector $v \neq 0$ and $\lambda \in \mathbb{C}$, $\operatorname{Re}(\lambda) < 0$ s.t.

$$X_1 v = 0$$
$$\Lambda v = \lambda v$$
$$R X_2 v = 0 \qquad (7)$$

The bottom row of (5) : $QX_1 - A^T X_2 = X_2 \Lambda$.

$$\Rightarrow \quad \cancel{QX_1 v}_0 - A^T X_2 v = X_2 \Lambda v = \lambda X_2 v$$
$$\Rightarrow (A^T + \lambda I) X_2 v = 0 \qquad (8)$$

(7) & (8) $\Rightarrow$ $v^* X_2^* \left[(A + \bar{\lambda} I) \quad R \right] = 0$

Since $\operatorname{Re}(\bar{\lambda}) < 0$ and (A, R) stabilizable, it must be that $X_2 v = 0$.

But $X_1 v = 0$ and $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ has full column rank.

$\therefore v = 0 \quad \Rightarrow \operatorname{Ker} X_1 = \{0\}$ trivial.

$\therefore X_1$ is invertible, and $H \in \operatorname{dom} Ric$. #

Main Thm A

A, B, C given. (A, B) stabilizable
(A, C) detectable.

Then

$$H := \begin{bmatrix} A & -BB^T \\ -C^TC & -A^T \end{bmatrix} \in \mathrm{dom}_S\, \mathrm{Ric}.$$

Moreover,

$$0 \le X := \mathrm{Ric}(H),$$

and

$$\mathrm{Ker}\, X \subset \mathrm{Ker} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^m \end{bmatrix}.$$

Sketch of Pf

Stab. detectable conditions can be used to show that H has no Im axis eigvals.
Use psuedo-inverse of B to show that if (A, B) stab then $(A, -BB^T)$ is also stab.
Then apply Thm 7 to conclude $H \in \mathrm{dom}_S\, \mathrm{Ric}$.
Then, rearrange the Ric eqn into a Lyap eqn and use Thm 7.3 from the previous section to show that $X \ge 0$.
Finally, show that $\mathrm{Ker}\, X \subset \mathrm{Ker}\, C$
and that $\mathrm{Ker}\, X$ is A inveriant.
That shows $\mathrm{Ker}\, X \subset \mathrm{Ker}\, CA^j$ for any $j \ge 0$. #