8 $\mathcal{H}_\infty$ Control

8.1 $\mathcal{H}_\infty$ Norms and Algebraic Riccati Equations

Consider the LTI sys : $\dot{x} = Ax + Bu$

$y = Cx$

$G(s) = C(sI-A)^{-1}B.$

<u>Thm 1</u> The following statements are equivalent:

1. A is stable, and $\|G\|_\infty < 1$.
2. " , and $\begin{bmatrix} A & BB^T \\ -C^TC & -A^T \end{bmatrix}$ has no Im-axis evals.
3. " , and $\exists\, X \in \mathbb{R}^{n\times n}$ s.t. $X = X^T$, $A + BB^TX$ is stable, and
$$A^TX + XA + XBB^TX + C^TC = 0.$$

<u>Pf</u> $(1 \leftrightarrow 2)$ HW

$(2 \leftrightarrow 3)$ directly from Thm 8. (from the previous chapter)

<u>Thm 2</u> The following statements are equivalent:

1. A is stable, and $\|G\|_\infty < 1$.
2. A is stable, and for some $\varepsilon > 0$, $\left\| \begin{bmatrix} C \\ \varepsilon I \end{bmatrix} (sI-A)^{-1}B \right\|_\infty < 1$.
3. A is stable, and there exists a matrix $X \in \mathbb{R}^{n\times n}$ s.t. $X = X^T$, $A + BB^TX$ is stable, and
$$A^TX + XA + XBB^TX + C^TC < 0.$$

4. A is stable, and there exists a matrix $X \in \mathbb{R}^{n\times n}$ s.t. $X = X^T$ and $A^TX + XA + XBB^TX + C^TC < 0$.

5. $\exists\, X \in \mathbb{R}^{n\times n}$ s.t. $X = X^T > 0$ and
$$A^TX + XA + XBB^TX + C^TC < 0.$$

6. $\exists\, X \in \mathbb{R}^{n\times n}$ s.t. $X = X^T > 0$ and
$$\begin{bmatrix} A^TX + XA + C^TC & XB \\ B^TX & -I \end{bmatrix} < 0$$

7. $\exists\, X \in \mathbb{R}^{n\times n}$ s.t. $X = X^T > 0$ and
$$\begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -I & 0 \\ C & 0 & -I \end{bmatrix} < 0.$$

Pf

$(1\to 2)$ A stable $\Rightarrow \|(sI-A)^{-1}B\|_\infty$ is finite. $\Rightarrow \|\varepsilon(sI-A)^{-1}B\|_\infty$ can be made tiny.

$(2\to 3)$ Use Thm 1, $1 \leftrightarrow 3$, then
$$A^TX + XA + XBB^TX + C^TC = -\varepsilon^2 I < 0.$$

$(3\to 4)$ Obvious ($\because$ 4 is claiming less than 3 does)

$(4\to 5)$ Let $W > 0$ s.t. $A^TX + XA = -XBB^TX - C^TC - W$.
A is stable, and $\left(A, \underbrace{\begin{bmatrix} B^TX \\ C \\ W^{1/2} \end{bmatrix}}_{\text{full rank}}\right)$ is observable.
$\therefore X > 0$. (recall observability condition)

$(5\to 1)$ Completion of the square.

$(5\leftrightarrow 6 \leftrightarrow 7)$ Schur complements.

For systems with a "D" term, the "inequality" thm can be written as the following:

Thm 3 The following are equivelent;

1. $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

   is internally stable, and $\|T_{yu}\|_\infty < 1$.

2. $\bar{\sigma}(D) < 1$, A is stable, and $\exists\, X = X^T$ s.t.

   $$A + BB^TX + BD^T(I - DD^T)^{-1}(C + DB^TX)$$

   is stable and

   $$(A + BD^T(I-DD^T)^{-1}C)^T X + X(A + BD^T(I-DD^T)^{-1}C) + XB[I + D^T(I-DD^T)^{-1}D]B^TX + C^T(I-DD^T)^{-1}C = 0.$$

3. $\bar{\sigma}(D) < 1$ and $\exists\, X = X^T > 0$ s.t.

   $$(A + BD^T(I-DD^T)^{-1}C)^T X + X(A + BD^T(I-DD^T)^{-1}C) + XB[I + D^T(I-DD^T)^{-1}D]B^TX + C^T(I-DD^T)^{-1}C < 0$$

4. $\exists\, X \in \mathbb{R}^{n\times n}$ s.t. $X = X^T > 0$ and

   $$\begin{bmatrix} A^TX + XA & XB & C^T \\ B^TX & -I & D^T \\ C & D & -I \end{bmatrix} < 0.$$

Let the realization of the TM be of the form

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

And suppose the following :

(i) (A, B_1) controllable, (C_1, A) observable

(ii) (A, B_2) stabilizable, (C_2, A) detectable.

(iii) $D_{12}^T [C_1 \; D_{12}] = [0 \;\; I]$

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$.

The $\mathcal{H}_\infty$ soln involves the following two Hamiltonian matrices :

$$H_\infty := \begin{bmatrix} A & \frac{1}{\gamma^2} B_1 B_1^T - B_2 B_2^T \\ -C_1^T C_1 & -A^T \end{bmatrix}$$

$$J_\infty := \begin{bmatrix} A^T & \frac{1}{\gamma^2} C_1^T C_1 - C_2^T C_2 \\ -B_1 B_1^T & -A \end{bmatrix}.$$

Since (1,2)-blocks are not sign definite, we cannot use Thm 8 of Chap 7 to guarantee that $H_\infty \in \mathrm{dom}_S \mathrm{Ric}$ or $\mathrm{Ric}(H_\infty) \geq 0$. These conditions are related to the existence of $\mathcal{H}_\infty$ suboptimal controllers.

(As $\gamma \to \infty$, these two Hamiltonian matrices become the corresponding $\mathcal{H}_2$ control Hamiltonian matrices.)

Thm $\exists$ an admissible controller s.t. $\|T_{zw}\|_\infty < \gamma$ iff the following three conditions hold:

(i) $H_\infty \in \mathrm{dom}_s \mathrm{Ric}$, and $X_\infty := \mathrm{Ric}(H_\infty) > 0$

(ii) $J_\infty \in \mathrm{dom}_s \mathrm{Ric}$, and $Y_\infty := \mathrm{Ric}(H_\infty) > 0$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

Moreover, when these conditions hold, one such controller is

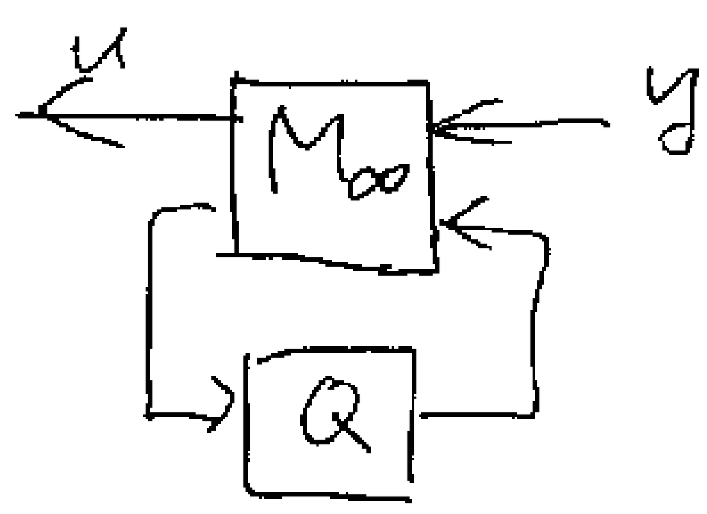
$$K_{sub}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\hat{A}_\infty := A + \gamma^{-2} B_1 B_1^* X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$

$$F_\infty := -B_2^* X_\infty, \quad L_\infty = -Y_\infty C_2^*, \quad Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

Furthermore, the set of all admissible controllers s.t. $\|T_{zw}\|_\infty < \gamma$ equals the set of all TMs from y to u in



$$M_\infty(s) = \left[\begin{array}{c|cc} \hat{A}_\infty & -Z_\infty L_\infty & Z_\infty B_2 \\ \hline F_\infty & 0 & I \\ -C_2 & I & 0 \end{array}\right]$$

where $Q \in \mathcal{RH}_\infty$, $\|Q\|_\infty < \gamma$.