10 Uncertainty and Robustness

10.1 Model Uncertainty

Most control designs are based on the use of a design model and the quality of a model depends on how closely its responses match those of the true plant. A good model should be simple enough to facilitate the controller design, yet rich enough to give the designer confidence that designs based on the model will work on the true plant.

The *uncertainty* refers to the differences or errors between models and reality, and *representation of uncertainty* vary primarily in terms of the amount of structure they contain.

Examples of uncertainty representation

- LQG: Additive noise
- a set membership statement for the parameters of an otherwise known FDLTI model. e.g. linear models of the same structure are used at various operating points to model aerodynamic coefficients that vary with flight conditions.
- Often, we are forced to use not just a single parameterized model but model sets that allow for plant dynamics that are not explicitly represented in the model structure. e.g. plant might have dynamics that are not represented in the fixed-order model. Then, we need to use less structured representations such as

$$P_\Delta(s) = P(s) + W_1(s)\Delta(s)W_2(s), \quad \bar{\sigma}[\Delta(j\omega)] < 1, \ \forall\omega \geq 0 \ , \tag{1}$$

  where W_1, W_2 are stable TM that characterizes the spatial and frequency structure of the uncertainty. Or, we can use the so-called multiplicative form:

$$P_\Delta(s) = (I + W_1(s)\Delta(s)W_2(s))P(s) \tag{2}$$

  Advantage of this form is that the weighting functions apply to PK as well as P.

Definition 10.1 *Given the description of an uncertainty model set* Π *and a set of performance objectives, suppose* $P \in \Pi$ *is the nominal design model and* K *is the resulting controller. Then the closed-loop feedback system is said to have*

- *Nominal Stability (NS): if* K *internally stabilizes the nominal model* P*.*
- *Robust Stability (RS): if* K *internally stabilizes every plant belonging to* Π*.*

- *Nominal Performance (NP): if the performance objectives are satisfied for the nominal plant P.*
- *Robust Performance (RP): if the performance objectives are satisfied for every plant belonging to Π.*

NS and NP can be easily checked, and we'll study the conditions for which RS and RP are satisfied.

10.2 Small Gain Theorem

Recall the results from Chap 5 about Fig. 1.

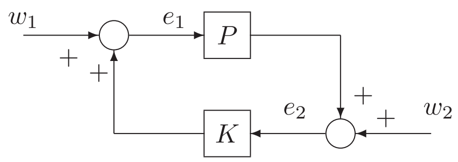


Figure 1: Feedback interconnection

Definition 10.2 *The c.l.s is internally stable if the TM from w_1, w_2 to e_1, e_2*

$$\begin{aligned} \begin{bmatrix} I & -K \\ -P & I \end{bmatrix}^{-1} &= \begin{bmatrix} (I-KP)^{-1} & K(I-PK)^{-1} \\ P(I-KP)^{-1} & (I-PK)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I+K(I-PK)^{-1}P & K(I-PK)^{-1} \\ (I-PK)^{-1}P & (I-PK)^{-1} \end{bmatrix} \end{aligned}$$

is proper real rational stable (or, belongs to $\mathcal{RH}_\infty$).

Corollary 10.3 *Suppose that $P, K \in \mathcal{RH}_\infty$. Then the above system is internally stable if and only if $(I-PK)^{-1} \in \mathcal{RH}_\infty$, or equivalently, $|I-PK|$ has no zeros in the closed RHP.*

Remark. SVD.
Let $A \in \mathbb{F}^{m\times n}$. Then there exist unitary matrices $U \in \mathbb{F}^{m\times m}$,$V \in \mathbb{F}^{n\times n}$ such that

$$\begin{aligned} A &= U\Sigma V^* \\ \Sigma &= \text{diag}[\sigma_1, \cdots, \sigma_p, 0, \cdots, 0] \\ &\quad \text{where } \sigma_1, \cdots, \sigma_p \geq 0, \quad p = \min(m,n) \ . \end{aligned}$$

In this section, we consider the robust stability test of a nominally stable system under unstructured perturbations. Suppose that $M(s)$ is a stable $p \times q$ matrix in Fig. 2.

Theorem 10.4 (Small Gain Theorem) *Suppose $M \in \mathcal{R}H_\infty$ (i.e. proper, real rational stable) and $\gamma > 0$. Then the interconnected system shown in Fig. 2 is well-posed and internally stable for all $\Delta(s) \in \mathcal{R}H_\infty$ with*

(a) $||\Delta||_\infty \leq 1/\gamma$ *iff* $||M(s)||_\infty < \gamma$

(b) $||\Delta||_\infty < 1/\gamma$ *iff* $||M(s)||_\infty \leq \gamma$

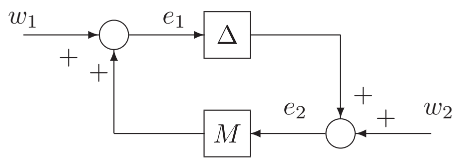


Figure 2: $M - \Delta$ interconnection

Proof. (Will just prove (a). Pf for (b) is similar. WOLG assume $\gamma = 1$.)
$(\Longleftarrow)$
Suppose that $||M||_\infty < 1$.
M, Δ both stable.
$\longrightarrow$ The CL sys is stable if $(I - M\Delta)^{-1} \in \mathcal{RH}_\infty$ (or, if $\det(I - M\Delta)$ has no zero in the closed RHP, $\forall\, \Delta \in \mathcal{RH}_\infty,\ ||\Delta||_\infty \leq 1$).
$\longrightarrow$ The CL sys is stable if

$$\inf_{s \in \bar{\mathbb{C}}_+} \underline{\sigma}(I - M(s)\Delta(s)) \neq 0 \quad \forall\, \Delta \in \mathcal{RH}_\infty,\ ||\Delta||_\infty \leq 1\ .$$

In fact, this holds because

$$\begin{aligned} \inf_{s \in \bar{\mathbb{C}}_+} \underline{\sigma}(I - M(s)\Delta(s)) &\geq 1 - \sup_{s \in \bar{\mathbb{C}}_+} \bar{\sigma}(M(s)\Delta(s)) \ = \ 1 - ||M(s)\Delta(s)||_\infty \\ &\geq 1 - ||M(s)||_\infty ||\Delta(s)||_\infty \ \geq \ 1 - ||M(s)||_\infty \ > 0\ . \end{aligned}$$

$(\Longrightarrow)$
We will show by contradiction. i.e. we will suppose that $||M(s)||_\infty \geq 1$ and then there exists a $\Delta \in \mathcal{RH}_\infty,\ ||\Delta||_\infty \leq 1$ such that the system is unstable.
Suppose that $\omega_0 \in \mathbb{R}_+ \cup \{\infty\}$ is such that $\bar{\sigma}(M(j\omega_0)) \geq 1$. Let's perform a SVD $M(j\omega_0) = U\Sigma V^*$ such that

$$U = [u_1\ \cdots\ u_p] V = [v_1\ \cdots\ v_p] U = \mathrm{diag}[\sigma_1\ \sigma_2 \cdots]$$

Now we'll construct a $\Delta \in \mathcal{RH}_\infty$ s.t. $\Delta(j\omega_0) = \frac{1}{\sigma_1} v_1 u_1^*$ and $||\Delta||_\infty \leq 1$. For such Δ, we have

$$\det(I - M(j\omega_0)\Delta(j\omega_0)) \ = \ \det(I - U\Sigma V^* v_1 u_1^*/\sigma_1) \ = \ 1 - u_1^* U\Sigma V^* v_1/\sigma_1 = 0,$$

and thus the CL sys is either not well-posed (if $\omega_0 = \infty$) or unstable (if $\omega_0 \in \mathbb{R}$). So we consider two different cases:

(1) $\omega_0 = 0$ or ∞: then $M(j\omega_0)$ is real and so are U and V. In this case, we can choose

$$\Delta(s) = \Delta = \frac{1}{\sigma_1} v_1 u_1^* \in \mathbb{R}^{q \times p} .$$

(2) $0 < \omega_0 < \infty$: write u_1, v_1 in the following form:

$$u_1^* = [u_{11} e^{j\theta_1} \ \cdots \ u_{1p} e^{j\theta_p}], \ v_1 = \begin{bmatrix} v_{11} e^{j\phi_1} \\ \vdots \\ v_{1p} e^{j\phi_q} \end{bmatrix}$$

where $u_{1i}, v_{1k} \in \mathbb{R}$ and $\theta_i, \phi_k \in [-\pi, 0)$ for all i, j. Choose $\beta_i, \alpha_k \geq 0$ s.t.

$$\angle \left(\frac{\beta_i - j\omega_0}{\beta + j\omega_0} \right) = \theta_i, \quad \angle \left(\frac{\alpha_k - j\omega_0}{\alpha_k + j\omega_0} \right) = \phi_k$$

for $i = 1, \cdots, p$, $j = 1, \cdots, q$. Define

$$\Delta(s) = \frac{1}{\sigma_1} \begin{bmatrix} v_{11} \frac{\alpha_1 - s}{\alpha_1 + s} \\ \vdots \\ v_{1q} \frac{\alpha_q - s}{\alpha_q + s} \end{bmatrix} \left[u_{11} \frac{\beta_1 - s}{\beta_1 + s} \ \cdots \ u_{1p} \frac{\beta_p - s}{\beta_p + s} \right]$$

then $\Delta \in \mathcal{RH}_\infty$, $||\Delta||_\infty = \frac{1}{\sigma_1} \leq 1$, and $\Delta(j\omega_0) = \frac{1}{\sigma_1} v_1 u_1^*$. □

Remark. Similar small gain theorems exist when Δ is a nonlinear or time-varying stable operator.

Remark. Robust stability and Nyquist criterion.
In the figure below, *if M, Δ stable, then feedback loop is stable for all stable Δ such that*

$$||\Delta||_\infty < 1/||M||_\infty.$$

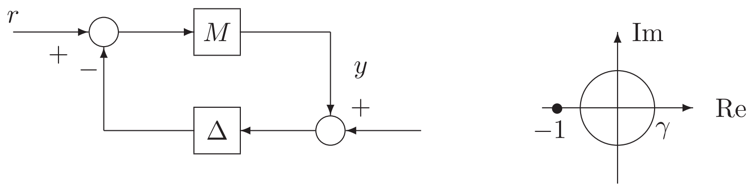
For SISO, this statement can be proven by the Nyquist approach.

$$\frac{y}{r} = \frac{M}{1 + M\Delta}$$

and

$$N = Z - P$$

$N =$ no. of CE encirclement of -1 by Nyquist contour $M\Delta(j\omega)$
$Z =$ no. of RHP zeros of $1 + M\Delta =$ no. of unstable CL poles
$P =$ no. of RHP poles of $M\Delta$, which is 0 in this case where M, Δ are stable.
Therefore, CL sys is stable iff the Nyquist plot of $M\Delta(j\omega)$ does not pass through -1 and encircles it 0 times. Thus, if $\gamma := ||\Delta||_\infty ||M||_\infty < 1$, the feedback loop is stable.



10.3 Stability under unstructured uncertainties

Consider the standard setup shown in Fig. 3, where K is the internally stabilizing controller for the nominal plant P and $\tilde{P}$ denotes the perturbed plant.

Recall the definition:

$$S_0 = (I + PK)^{-1},\ T_0 = I - S_0$$

$$S_i = (I + KP)^{-1},\ T_i = I - S_i\ .$$

and that the c.l.s is well-posed and internally stable iff the TM from $[w_1, w_2] = [d_i, r]$ to $[e_1, e_2] = [u_p, e]$:

$$\begin{bmatrix} I & K \\ -\tilde{P} & I \end{bmatrix}^{-1} = \begin{bmatrix} (I + K\tilde{P})^{-1} & -K(I + \tilde{P}K)^{-1} \\ \tilde{P}(I + K\tilde{P})^{-1} & (I + \tilde{P}K)^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} I + K(I + \tilde{P}K)^{-1}\tilde{P} & -K(I + \tilde{P}K)^{-1} \\ (I + \tilde{P}K)^{-1}P & (I + \tilde{P}K)^{-1} \end{bmatrix}$$

is proper real rational stable (or, belongs to $\mathcal{RH}_\infty$).

Assume that the uncertainty model is described by

$$\tilde{P} = (I + W_1 \Delta W_2)P\ , \quad W_1, W_2, \Delta \in \mathcal{RH}_\infty$$

and consider the feedback system shown in Fig. 4. The transfer ftn $M(s)$ from w to z is given by $W_2 PK(I + PK)^{-1}W_1 = W_2 T_o W_1$. Think of Fig. 4 as Fig. 5, and compare it with Fig. 2.

Theorem 10.5 *Let*

$$\Pi = \{(I + W_1 \Delta W_2)P : \quad W_1, W_2, \Delta \in \mathcal{RH}_\infty\}$$

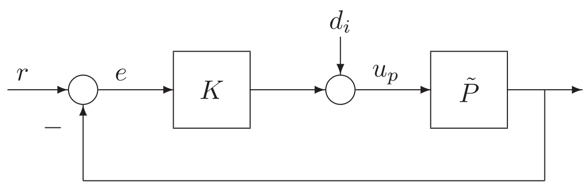


Figure 3: Standard feedback control configuration

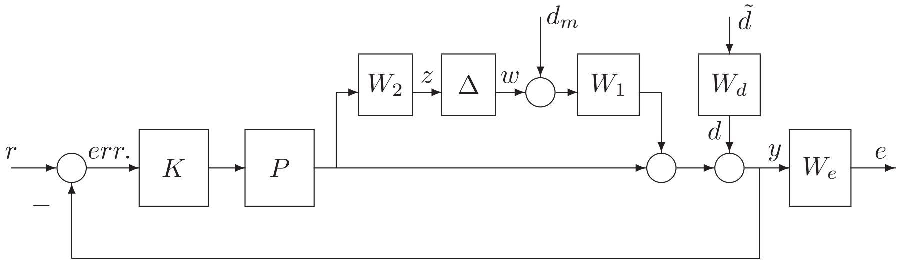


Figure 4: Feedback control for multiplicative perturbed systems

and let K be a stabilizing controller for P. Then the CL sys is well-posed and internally stable for all $\Delta \in \mathcal{RH}_\infty$ with $||\Delta||_\infty < 1$ iff $||W_2 T_o W_1||_\infty \leq 1$.

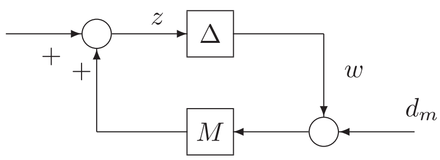


Figure 5: $M - \Delta$ interconnection

10.4 Robust performance under unstructured uncertainties

Suppose that the performance criterion is to keep the worst-case energy of the error as small as possible, i.e.

$$\sup_{||\tilde{d}||_2 \leq 1} ||e||_2 \leq \epsilon$$

for some small ϵ. WOLG, we can assume that $\epsilon = 1$, by scaling W_e. Since

$$T_{\tilde{d} \to e} = W_e (I + \tilde{P} K)^{-1} W_d, \ \tilde{P} \in \Pi \ ,$$

the robust performance criterion in this case can be described as requiring that the CL sys be robustly stable, and that

$$||T_{\tilde{d} \to e}||_\infty \leq 1, \ \forall \tilde{P} \in \Pi \ .$$

For the multiplicative perturbed system

$$\Pi = \{(I + W_1 \Delta W_2) P : \ \Delta \in \mathcal{RH}_\infty, ||\Delta||_\infty < 1\}$$

with $W_1, W_2 \in \mathcal{RH}_\infty$ shown in Fig. 4, we have

$$\begin{aligned} T_{\tilde{d}\to e} &= W_e(I + (I + W_1\Delta W_2 W_2)PK)^{-1}Wd \\ &= W_e\left[(I + W_1\Delta W_2 PK(I+PK)^{-1})(I+PK)\right]^{-1}Wd \\ &= W_e S_o (I + W_1\Delta W_2 T_o)^{-1} W_d \end{aligned}$$

In this setting, the robust performance is satisfied iff

$$||W_2 T_o W_1||_\infty \leq 1 \quad (i.e.\ \text{RS})$$

and

$$||T_{\tilde{d}\to e}|| \leq 1,\ \forall \Delta \in \mathcal{RH}_\infty,\ ||\Delta||_\infty < 1\ .$$

The exact robust performance analysis is not trivial and will be deferred to a few weeks from now. Here, we'll just take a look at a sufficient condition:

Theorem 10.6 *Suppose*

$$\tilde{P} \in \{(I + W_1\Delta W_2)P :\ \Delta \in \mathcal{RH}_\infty, ||\Delta||_\infty < 1\}$$

and K internally stabilizes P. Then the system robust performance is guaranteed if for every freq,

$$\bar{\sigma}(W_d)\bar{\sigma}(W_e S_o) + \bar{\sigma}(W_1)\bar{\sigma}(W_2 T_o) \leq 1\ .$$

Proof. It's obvious that the above condition guarantees that $||W_2 T_o W_1||_\infty \leq 1$. So it's sufficient to show that $||T_{\tilde{d}\to e}|| \leq 1,\ \forall \Delta \in \mathcal{RH}_\infty,\ ||\Delta||_\infty < 1$.

For every freq,

$$\begin{aligned} \bar{\sigma}(T_{\tilde{d}\to e}) &\leq \bar{\sigma}(W_e S_o)\ \bar{\sigma}((I + W_1\Delta W_2 T_o)^{-1})\ \bar{\sigma}(W_d) \\ &= \frac{\bar{\sigma}(W_e S_o)\bar{\sigma}(W_d)}{\underline{\sigma}((I + W_1\Delta W_2 T_o)} \leq \frac{\bar{\sigma}(W_e S_o)\bar{\sigma}(W_d)}{1 - \bar{\sigma}(W_1\Delta W_2 T_o)} \\ &\leq \frac{\bar{\sigma}(W_e S_o)\bar{\sigma}(W_d)}{1 - \bar{\sigma}(W_1)\bar{\sigma}(W_2 T_o)\bar{\sigma}(\Delta)} \\ &\leq \frac{1 - \bar{\sigma}(W_1)\bar{\sigma}(W_2 T_o)}{1 - \bar{\sigma}(W_1)\bar{\sigma}(W_2 T_o)\bar{\sigma}(\Delta)} \leq 1\ . \end{aligned}$$