13 Balanced realization

13.1 Hankel Singular Values

Consider a stable linear system

$$G(s) = \left[\begin{array}{c|c} \mathrm{A} & \mathrm{B} \\ \hline \mathrm{C} & \mathrm{D} \end{array}\right].$$

Let W_c and W_o denote the controllability and observability gramians, respectively. Then

$$\begin{aligned} AW_c + W_cA^* + BB^* &= 0 \quad (1) \\ A^*W_o + W_oA + C^*C &= 0 \quad (2) \end{aligned}$$

and $W_c, W_o \geq 0$.

Suppose the state is transformed by $\bar{x} = Tx$ (T: non singular) to yield the realization

$$G(s) = \left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array}\right] = \left[\begin{array}{c|c} TAT^{-1} & \mathrm{TB} \\ \hline CT^{-1} & \mathrm{D} \end{array}\right]$$

Then,

$$\begin{aligned} T(1)T^* &\Rightarrow TAT^{-1}TW_cT^* + TW_cTT^{-1}A^*T^* + TBB^*T^* = 0 \\ (T^{-1})^*(2)T^{-1} &\Rightarrow (T^{-1})^*A^*T(T^{-1})^*W_oT^{-1} + (T^{-1})^*W_oT^{-1}TAT^{-1} + (T^{-1})^*C^*CT^{-1} = 0 \end{aligned}$$

So the gramians are transformed to

$$\overline{W_c} = \mathrm{T}W_cT^*, \;\; \overline{W_o} = (T^{-1})^*W_oT^{-1}$$

Note that,

$$\overline{W_c}\,\overline{W_o} = \mathrm{T}W_cW_oT^{-1}$$

So the eigenvalues of the product of the gramians are invariant under state transformation.

Consider the similarity transformation (where the columns of T^{-1} are eigenvectors of W_cW_o corresponding to λ_i) which gives the eigen vector decomposition.

$$W_c\ W_o = T^{-1}\Lambda T, \;\; \Lambda = \text{diag}\ (\lambda_1 I_{s1}, ..., \lambda_N I_{sN})$$

(Because $W_c, W_0 \geq 0$, W_cW_o has a real diagonal Jordan form and that $\Lambda \geq 0$.)

Although eigenvectors are not unique, in the case of minimal realization they can be always chosen, such that

$$\overline{W_c} = TW_cT^* = \Sigma$$

$$\overline{W_o} = (T^{-1})^*W_oT^{-1} = \Sigma$$

where,

$$\Sigma = \text{diag}\ (\sigma_1 I_{s_1}, ..., \sigma_N I_{s_N}) \text{ and } \Sigma^2 = \Lambda$$

This realization with $\overline{W_c} = \overline{W_o}$ is called "*balanced realization*", and the decreasingly ordered numbers $\sigma_1 > .... > \sigma_N \geq 0$ are called the "*Hankel singular values*".

When (A, B): Controllable, (A, C): Observable, define $T \in \mathbb{C}^{n\times n}$ using the following steps:

1. Compute R such that $W_c = R^* R$.
2. Diagonalize $RW_oR^* = U\Sigma^2U^*$, U: unitary, Σ : diagonal.
   Then $W_cW_o = R^*RW_o$, so $(R^*)^{-1}W_cW_oR^* = RW_oR^* = U\Sigma^2U^*$.
   evals of $W_cW_o = \sigma_i^2$
3. Let $T^{-1} := R^*U\Sigma^{-\frac{1}{2}}$
   Then $TW_cT^* = \Sigma^{-\frac{1}{2}}U^*(R^*)^{-1}W_c\ (R)^{-1}U(\Sigma^{\frac{1}{2}})^* = \Sigma^{\frac{1}{2}}U^*U(\Sigma^{\frac{1}{2}})^* = \Sigma$
   $(T^*)^{-1}W_oT^{-1} = (\Sigma^{-\frac{1}{2}})^*U^*RW_oR^*U\Sigma^{-\frac{1}{2}} = \Sigma$
   $\therefore$ balanced

More generally, if a realization of a stable system is not minimal, then $\exists$ a transformation such that the controllability and observability gramians for the transformed realization are diagonal and the controllable and observable subsystem is balanced (by the following fact)

Theorem: Let P, Q be positive semi definite, then $\exists$ a nonsingular matrix T such that,

$$TPT^* = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}, \quad (T^{-1})^*QT^{-1} = \begin{bmatrix} \Sigma_1 & & & \\ & 0 & & \\ & & \Sigma_3 & \\ & & & 0 \end{bmatrix}$$

See text for proof.

13.2 Model Reduction (Balanced Truncation)

$\dot{x} = Ax, \quad y = Cx \qquad \Rightarrow y(t) = Ce^{At x_o}$

The 2-norm of the output is

$$\|y\|_2^2 = \int_0^\infty y^*(t)y(t)dt = x_o^*W_ox_o = \overline{x}_o^*\overline{W}_o\overline{x}_o$$

and $\overline{W}_o = \sum = diag[\ \underbrace{\sigma_1, ..., \sigma_n}_{decreasing\ order}\ ]$. So,

$$\|y\|_2^2 = \overline{x}_o^* \sum \overline{x}_o = \sum_{i=1}^{n} \sigma_i \overline{x}_{oi}^2$$

.

If $\sigma_1 >> \sigma_n$, then the effect of $\overline{x}_o = [1\ 0\ 0\ \cdots\ 0]^T$ on $\|y\|_2^2$ is much larger than the effect of $\overline{x}_o = [0\ 0\ 0\ \cdots\ 1]^T$.

Suppose $\sigma_r >> \sigma_{r+1}$ for some r. Then the balanced realization implies that those states corresponding to the singular values of $\sigma_{r+1}....\sigma_N$ are less controllable and less observable than those states corresponding to $\sigma_1....\sigma_r$. Therefore, truncating those less controllable & less observable states will not lose much info about the system.

<u>Theorem:</u> (bounds for H_∞ norm and L_1 norm of a stable system)
Let $G(s) = \left[\begin{array}{c|c} \text{A} & \text{B} \\ \hline \text{C} & 0 \end{array}\right] \in \mathcal{RH}_\infty$ be a bal.real. i.e.

$$\exists\, \Sigma = diag(\sigma_1 I_{s1}, ..., \sigma_N I_{sN}) \;\geq 0 \text{with } \sigma_1 > \sigma_2 >, \cdots > \sigma_N \geq 0$$

such that

$$A\Sigma + \Sigma A^* + BB^* = 0,\; A^*\Sigma + \Sigma A + C^*C = 0.$$

Then

$$\sigma_1 \leq \|G\|_\infty \leq \int_0^\infty \|g(t)\|dt \leq 2\Sigma_{i=0}^N \Sigma_i,$$

where $g(t) = Ce^{At}B$.

Suppose that $G(s) = \left[\begin{array}{c|c} \text{A} & \text{B} \\ \hline \text{C} & \text{D} \end{array}\right]$ is a bal.real. for a stable system i.e. controllability and observability gramians are equal and diagonal $= \Sigma = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}$.

And partition the system accordingly as $G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & \text{D} \end{array}\right]$.

<u>Theorem:</u>
Assume that Σ_1, Σ_2 have no diagonal entries in common. Then both subsystems (A_{ii}, B_i, C_i), i = 1,2 are asymptotically stable.

<u>Theorem:</u>
Assume Σ_1 = diag $(\sigma_1 I_{s_1}, ..., \sigma_r I_{s_r})$, Σ_2 = diag $(\sigma_{r+1} I_{sr+1}, ..., \sigma_N I_{sN})$ and $\sigma_1 > \sigma_2 >, \cdots > \sigma_N$, where σ_i has multiplicity s_i (i = 1, ..., N) and $s_1 + ... + s_N = n$. Then the truncated system $G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & \text{D} \end{array}\right]$ is balanced and asymptotically stable.
Furthermore, $\|G(s) - G_r(s)\|_\infty \leq 2\ (\sigma_{r+1} + \sigma_{r+2} + ... + \sigma_N)$.

Matlab Commands

- $[A_b,\ B_b,\ C_b,\ \sigma,\ T_{inv}]$ = balreal (A,B,C); whereas σ is a vector of Hankel singular values.
- $[G_b, \sigma]$ = sysbal (G), whereas G_b is bal.real and σ is Hankel singular values.
- G_r = strunc (G_b, r) truncate to r^{th} order.

Note that model reduction bound can be loose for systems with Hankel singular values close to each other.

Example:

$$G(s) = \sum_{i=1}^{n} \frac{b_i}{s + a_i}, \qquad a_i, b_i > 0$$

Then

$$\|G(s)\|_\infty = G(0) = \sum_{i=1}^{n} \frac{b_i}{a_i}$$

$$G = \left[\begin{array}{ccccc|c} -a_1 & & & & & \sqrt{b_1} \\ & -a_2 & & & & . \\ & & . & & & . \\ & & & . & & . \\ & & & & -a_n & \sqrt{b_n} \\ \hline \sqrt{b_1} & . & . & . & \sqrt{b_n} & 0 \end{array}\right]$$

$$P = Q = \left[\frac{\sqrt{b_i b_j}}{a_i + a_j}\right]$$

$$\sigma_i = \lambda_i(P) = \lambda_i(Q)$$

$$\sum_{i=1}^{n} \sigma_i = \sum_{i=1}^{n} \lambda_i(P) = \text{trace } (P) = \sum_{i=1}^{n} \frac{b_i}{2a_i} = \frac{1}{2} G(0) = \frac{1}{2} \|G\|_\infty$$

.

If $a_i = b_i = \alpha^{2i}$ then
$P = Q \to \frac{1}{2} I_n$ (i.e. $\sigma_j \to \frac{1}{2}$ as $\alpha \to \infty$), thus $\frac{1}{2}\|G\|_\infty = \frac{n}{2}$.

Therefore, even when the Hankel singular values are extremely close they may not be considered as repeated singular values.