$$\therefore\ T\frac{Ds}{Dt} = \frac{De}{Dt} - p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = 0 \quad \Big\}\ \text{as seen by particle}$$

$$\therefore \frac{Ds}{Dt} = 0 \quad \cdots (4) \quad \rightarrow \text{replaces } (3)$$

for the particle $ds = 0$

$$de = (\gamma-1)\, p\, d\left(\frac{1}{\rho}\right)$$

for ideal gas : $de = \frac{1}{\gamma-1}\left[p\, d\left(\frac{1}{\rho}\right) + \left(\frac{1}{\rho}\right)dp\right]$

$$\frac{dp}{p} = \gamma\frac{d\rho}{\rho} \Rightarrow p = \text{"Const"}\cdot\rho^{\gamma} \quad \cdots (5)$$

Final equations ... (1), (2), (5), state eqn.

Everything still valid for unsteady flow

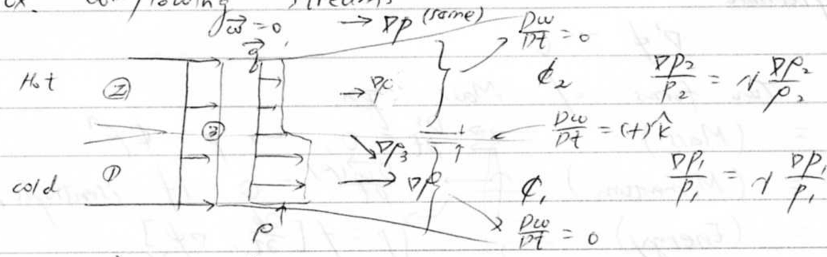
"Const" ... each particle has its own "const"

for steady flow, each streamline has its own "Const"

In general, $p = e^{(\gamma-1)s/R}\rho^{\gamma}$

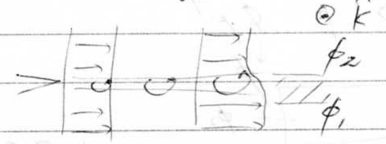
$$\nabla\times(2) \rightarrow \frac{D}{Dt}\left(\frac{\vec{\omega}}{\rho}\right) = \frac{\vec{\omega}}{\rho}\cdot\nabla\vec{q} + \frac{1}{\rho^2}\nabla\rho\times\nabla p$$

Ex. Co-flowing streams



$$\vec{\omega} \equiv \nabla\times\vec{q} \quad (\text{"vorticity"})$$

$\odot\ \hat{k} \quad +\vec{\omega}$



$\frac{\vec{\omega}}{\rho}\cdot\nabla\vec{q} = 0$ in 2-D , not zero in 3-D

- Potential Flow Relations

$\vec{\omega} = 0$ in pieces of domain

$\rightarrow \vec{q} = \nabla\phi$ in pieces of domain

momentum eqn.

$$\frac{\partial}{\partial t}(\nabla\phi) + \underbrace{\nabla\phi\cdot\nabla(\nabla\phi)}_{\hookrightarrow \frac{1}{2}\nabla(\nabla\phi\cdot\nabla\phi) - \nabla\phi\times\nabla\times(\nabla\phi)\ \rightarrow 0} = -\frac{\nabla p}{\rho}$$

$$\nabla\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}q^2\right] + \frac{\nabla p}{\rho} = 0$$

p_{ref}, q_{ref} --- typical at inlet or at ∞

$$\nabla\left[\frac{\partial\phi}{\partial t}+\frac{1}{2}q^2\right]+\frac{\nabla p}{\rho}=0$$ --- Bernoulli's Eqn. for pressure

Must still ensure mass conservation

$$\frac{\partial\rho}{\partial t}+\vec{q}\cdot\nabla\rho+\rho\nabla\cdot\vec{q}=0$$

$$p=\phi\rho^{\gamma}$$

$$\frac{\nabla p}{p}=\gamma\frac{\nabla\rho}{\rho}$$

$$\frac{1}{p}\frac{\partial p}{\partial t}=\frac{\gamma}{\rho}\frac{\partial\rho}{\partial t}$$

$$\frac{\partial\rho}{\partial t}+\vec{q}\cdot\nabla\rho+\rho\nabla\cdot\vec{q}=0$$

$$\underbrace{\overbrace{\frac{\rho}{\gamma p}}^{\frac{1}{a^2}}\frac{\partial p}{\partial t}}_{\frac{\partial}{\partial t}(\text{Bernoulli})} \qquad \underbrace{\overbrace{\frac{\rho}{\gamma p}}^{\frac{1}{a^2}}\nabla p}_{\text{from original momentum eqn.}} \qquad \nabla^2 p$$

$$\nabla^2\phi=\frac{1}{a^2}\left[\frac{\partial^2\phi}{\partial t^2}+\frac{\partial}{\partial t}(q^2)+\vec{q}\cdot\nabla\left(\frac{1}{2}q^2\right)\right]$$ --- Ideal Gas

Incompressible

$$\nabla^2\phi=0$$

$\Rightarrow$ Two forms of Mass eqn.

$\frac{D\rho}{Dt}$ = (Mass) $\longrightarrow$ $\frac{Ds}{Dt}=0$, $\rightarrow p=\phi\rho^{\gamma}$

$\frac{D\vec{q}}{Dt}$ = (Momentum) $\longrightarrow$ $\frac{D\vec{\omega}(\rho)}{Dt}=0$ if isentropic $\rightarrow \vec{q}=\nabla\phi$

$\frac{De}{Dt}$ = (Energy)

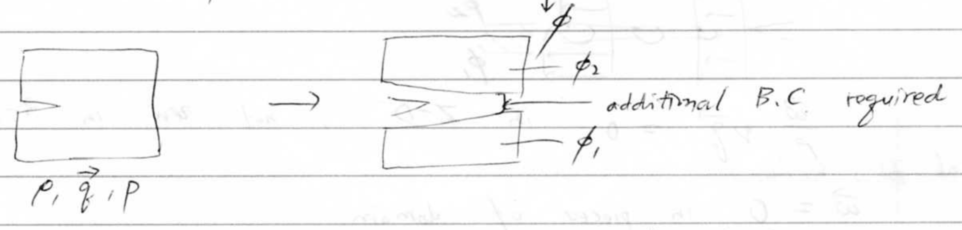
$\downarrow$

ρ, u, v, w, p

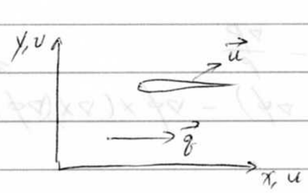
$$\begin{cases} p=f\left[\frac{\partial\phi}{\partial t},\nabla\phi\right] \\ \nabla^2\phi=M^2 f\left[\frac{\partial^2\phi}{\partial t^2},\frac{\partial}{\partial t}(\nabla\phi),\nabla\phi\right] \\ p=\phi\rho^{\gamma} \end{cases}$$

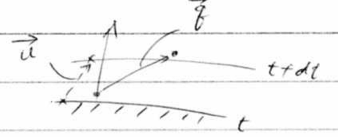
$\downarrow$

ϕ



Solid surface --- flow tangency, $\vec{q}\cdot\hat{n}=0$
(steady flow only)





For particle to stay on surface,
$$\vec{q}\cdot\hat{n} = \vec{u}\cdot\hat{n}$$

Across wakes, $\Delta p = 0$

$$p_1 - p_2 = 0 \rightarrow \frac{\partial \phi_1}{\partial t} + \frac{1}{2}(q_1^2) + \phi_1 = \frac{\partial \phi_2}{\partial t} + \frac{1}{2}(q_2^2) + \phi_2$$

$$\frac{\partial \Delta\phi}{\partial t} + \vec{q}\cdot\nabla(\Delta\phi) + \Delta\phi = 0$$

$$\frac{D(\Delta\phi)}{Dt} = 0$$

↳ usually zero in external flows

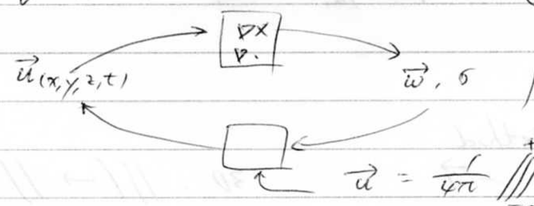
$\Delta\phi$ maintained by moving point. --- 1, 2, or 3-D

- "Practical" Flow field Description

For any vector field $\vec{u}(x,y,z,t)$, there are associated

$$\vec{\omega} = \nabla\times\vec{u}$$
$$\sigma = \nabla\cdot\vec{u}$$

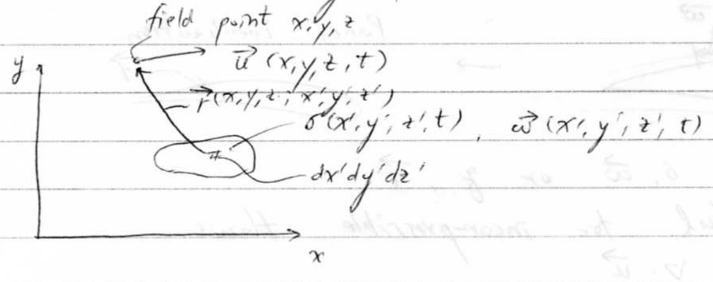
← apply instantaneously at any t



$$\vec{u} = \frac{1}{4\pi}\iiint_{-\infty}^{+\infty} \sigma\frac{\vec{r}}{r^3}\,dx'dy'dz' + \frac{1}{4\pi}\iiint \frac{\vec{\omega}\times\vec{r}}{r^3}\,dx'dy'dz' + \nabla\phi_c$$

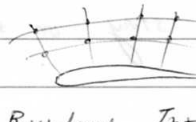
ϕ_c : arbitrary, provided $\nabla^2\phi_c = 0$

ϕ_c determined by B.C.'s



$$\vec{r} = \begin{Bmatrix} x-x' \\ y-y' \\ z-z' \end{Bmatrix}$$

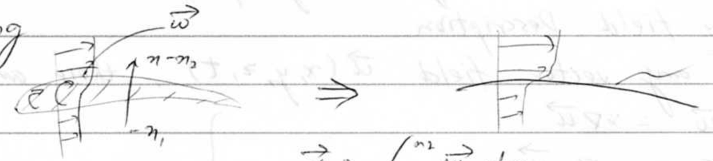
- Grid methods (Field methods)



- Boundary Integral Methods (Panel methods)
  Vortex Lattice Methods (Analytic Singularity)
  --- $\vec{u}$ described via $\vec{\omega}$, σ

Geometry (t) $\vec{q}_\infty$ $\rightarrow$ Grid method $\rightarrow \vec{u} \rightarrow (\nabla\times, \nabla\cdot) \rightarrow \vec{\omega}, \sigma$
$\hookrightarrow$ Panel method $\rightarrow \underbrace{\sigma, \vec{\omega}}_{\text{or } q, \vec{\gamma}} \rightarrow [\iiint \sim] \rightarrow \vec{u}$

- Lumping

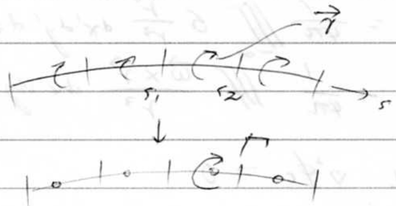


$3D: \iiint \rightarrow \iint$
$2D: \iint \rightarrow \int$

$$\vec{\gamma} = \int_{n_1}^{n_2} \vec{\omega}\, dn$$

--- Simplification
& Idealization

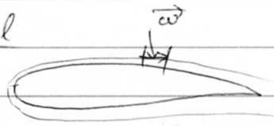
∘ Vortex Lattice Method



$3D: \iiint \rightarrow \iint \rightarrow \int$
$2D: \iint \rightarrow \int \rightarrow \boxed{+}$

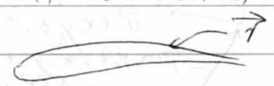
$$\vec{\Gamma} = \int_{s_1}^{s_2} \vec{\gamma}\, ds$$

Real



$\rightarrow$

Panel Idealization



In practice, $\sigma, \vec{\omega}$ or $q, \vec{\gamma}$
only useful for incompressible flows

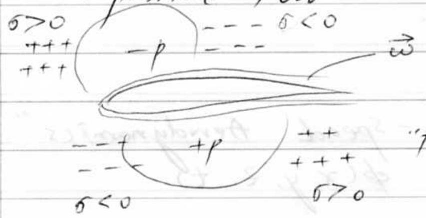
$$\sigma = \nabla\cdot\vec{u}$$

continuity : $\frac{\partial\rho}{\partial t} + \nabla\rho\cdot\vec{u} + \rho\,\underbrace{\nabla\cdot\vec{u}}_{\sigma} = 0$

$$\sigma = -\frac{1}{\rho}\left[\frac{\partial\rho}{\partial t} + \nabla\rho\cdot\vec{u}\right] \neq 0 \text{ in flowfield}$$

$$\vec{\omega} = \nabla\times\vec{u} \neq 0 \text{ only in Boundary Layers}$$

- Compressible flow



"field sources"

$$\vec{u} = \frac{1}{4\pi}\iiint \underbrace{\sigma \frac{\vec{r}}{r^2}}_{?}\, dx'dy'dz'$$

- Incompressible flow

$\sigma = 0$ everywhere

Approximate $\sigma \equiv -1/\rho \left[\frac{\partial \rho}{\partial t} + \underbrace{\vec{u}}_{\hookrightarrow \vec{U}_\infty} \cdot \nabla \rho \right]$

$$\sigma \simeq -\frac{\partial}{\partial \rho}[\ln \rho] - \vec{U}_\infty \cdot \nabla(\ln \rho)$$

"Prandtl – Glauert" approximation

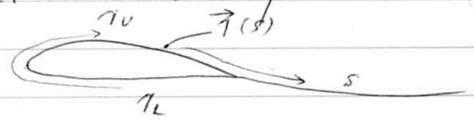
In steady case,

$$[1 - M_\infty^2]\, \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

came from R.H.S

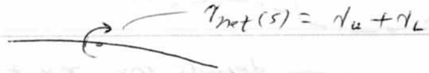
Appropriate only for $\vec{u} \simeq \vec{U}_\infty$ (small disturbance flows)

Exact Inviscid Representation



"lifting-body" theory

⇓



"lifting-surface" theory

⇓

Linearized lifting-surface theory

- Unsteady Panel Methods
  - A. Description
  - B. Implementation

Katz & Plotkin, "Low-Speed Aerodynamics"

Determine $\vec{q}(x, y, z, t)$ or $\phi(x, y, z, t)$

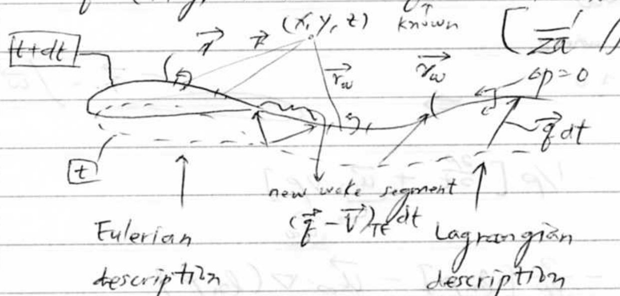
Vorticity + source representation

$$\vec{q}(x,y,z,t) = \underset{\text{known}}{\vec{q}_\infty} + \frac{1}{4\pi}\iint \frac{\vec{\gamma}\times\vec{r}}{r^3}\, dA \quad (3\text{-}D) \leftarrow \text{unknown}$$

$$\left[\frac{1}{2\pi}\iint \frac{\vec{\gamma}\times\vec{r}}{r^2}\, dA \quad (2\text{-}D)\right]$$

$$+ \frac{1}{4\pi}\iint \frac{\vec{\gamma}_w \times \vec{r}_w}{r_w^3}\, dA_w \leftarrow \text{known from history}$$

$$+ \iint \text{sources}$$

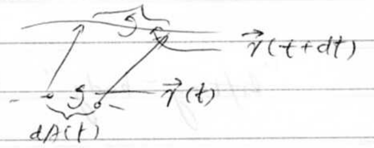


$\vec{V}(t)$ given on every airfoil location

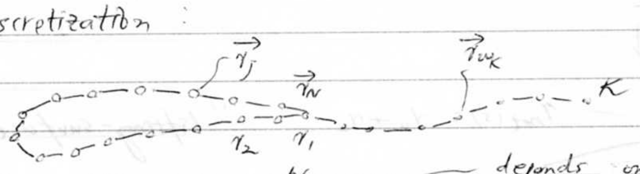
B.C.'s : $\vec{q}\cdot\hat{n} = \vec{V}\cdot\hat{n}$ on airfoil $\rightarrow$ gives $\vec{\gamma}$ on airfoil

$\Delta p = 0 \rightarrow \frac{D}{Dt}(\Delta\phi) = 0$

$dA(t+dt) \rightarrow \vec{\gamma}_w \cdot dA = \text{const in time}$



- Discretization :



$$\hat{n}_i \cdot \vec{q}_i = \Big[\vec{q}_\infty + \sum_{j=1}^{N} a_{ij}\,\gamma_j \;\text{L.H.S}$$

$$\text{R.H.S} + \sum_{K=1}^{K} b_{iK}\,\gamma_{wK} \rightarrow \text{R.H.S}\Big]\cdot\hat{n}$$

depends on x, y, z

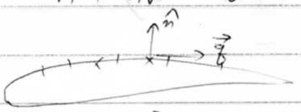
a_{ij}, b_{ij} = "A.I.C"

: Aerodynamic Influence Coefficient

Flow tangency

$$\begin{array}{c} \uparrow \\ N-1 \\ \downarrow \end{array} \begin{bmatrix} & & \\ & a_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_N \end{bmatrix} = \begin{bmatrix} -\vec{V}_\infty \cdot \hat{n} + \Sigma b_{ij} \; w \end{bmatrix}$$

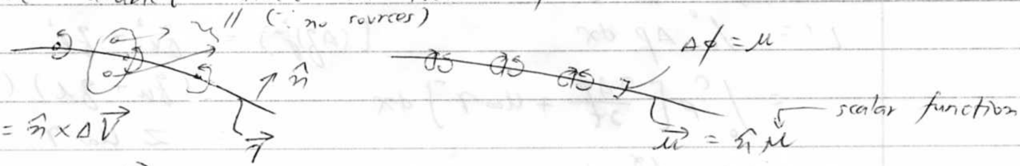
$\gamma_1 + \gamma_N = 0$



In 3-D, $\vec{\gamma} = \gamma_x \hat{i} + \gamma_y \vec{j} + \gamma_z \vec{k}$

must satisfy $\nabla_A \cdot \vec{\gamma} = 0$

Use doublet sheet in lieu of vortex sheet



$\gamma = \hat{n} \times \Delta\vec{V}$ $\qquad$ $\vec{\mu} = \hat{n}\mu$

$\vec{\gamma} = \nabla_{2D}\, \mu$

$\nabla_A \{ \Delta\phi = \mu \}$

$$\phi(x, y, z, t) = \frac{1}{4\pi} \iint \mu \frac{\partial}{\partial n}\left(\frac{1}{r}\right) dA + \frac{1}{4\pi} \iint \mu_w \frac{\partial}{\partial n}\left(\frac{1}{r_w}\right) dA_w + \phi_\infty$$

$(\phi_\infty = u_\infty x + v_\infty y + w_\infty z)$.

on body : $\Sigma a_{ij} \mu_j = \{ -\phi_\infty + \iint_{wake} \}$

B.C. on wake : $\frac{D}{Dt} \Delta\phi = 0$

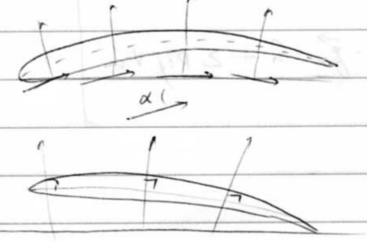
μ_w = const in time for each panel

° Simplifications

i) Thin Airfoil , Small Disturbances

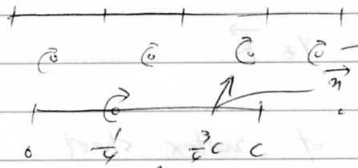
$|\vec{V}| \ll U_\infty$

$v, w \ll U_\infty$



$\alpha\hat{i}$

$$\vec{r} = (x - x')\,\hat{i} + 0\,\hat{j}$$



$$\Gamma_i(t) = \int_{x_i} \gamma(x,t)\,dx$$

$$\vec{n} = (\alpha - \alpha_0)\,\hat{i} + \hat{j}$$

$0 \quad \frac{1}{4} \quad \frac{3}{4}c \quad c$

$$L' = \int_0^c \Delta p\, dx \qquad \left[\ \Delta(q^2) = q_u^2 - q_l^2 \right.$$

$$= \int_0^c \rho\left[\frac{\partial \Delta\phi}{\partial t} + U_\infty \gamma\right] dx \qquad = (q_u + q_l)(q_u - q_l)$$

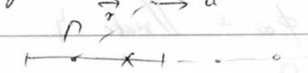
$$\left. = 2\,U_\infty\,\gamma \ \right]$$

$$\Delta\phi(x) = \int_0^x \gamma\, dx$$

For single vortex :

$$L' = \rho U_\infty \Gamma + \underbrace{\rho c \frac{\partial \Gamma}{\partial t}}_{\text{"new"}}$$

$$\vec{u}(x,y,t) = \vec{U}_\infty + \frac{1}{2\pi}(-\Gamma)\frac{\hat{k}\times\vec{r}}{r^2} + \frac{1}{2\pi}\sum(-\Gamma_{w_i})\frac{\hat{k}\times\vec{r}}{r}$$



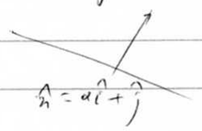
at c.p. $\rightarrow$ $\vec{u}\cdot\hat{n} = 0 \longrightarrow \Gamma$

if $d\Gamma/dt > 0$: $\qquad D > 0$

$d\Gamma/dt < 0$: $\qquad D < 0$

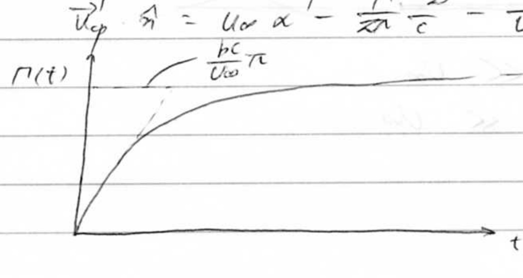
$$v_{c.p.} = -\frac{\Gamma}{2\pi}\frac{1}{c/2} - b\frac{d\Gamma}{dt}\frac{1}{U_\infty}$$

vertical velocity $\qquad b$: some empirical constant

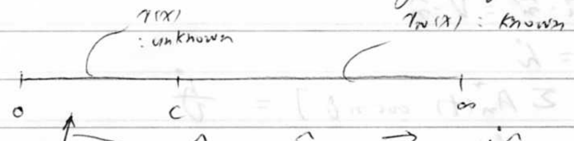


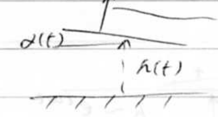
$$\vec{u}_{cp} = U_\infty + v_{cp}$$

$$\vec{u}_{cp}\cdot\hat{n} = U_\infty\alpha - \frac{\Gamma}{2\pi}\frac{2}{c} - \frac{b}{U_\infty}\frac{\partial}{\partial t}\Gamma = 0$$



contains $\gamma(x)$, linearized geometry





$\hat{n} = \alpha \hat{i}$, $\vec{V} = \dot{h}\hat{j}$

B.C. : $\vec{u} \cdot \hat{n} = \vec{V} \cdot \hat{n}$

$$U_\infty \alpha + V = \dot{h}$$

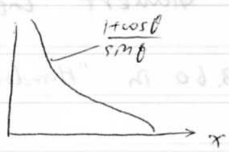
Coordinate change $x \to \theta$ "Glauert coordinate"

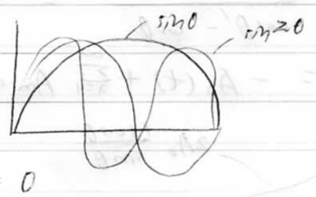
$$x = \frac{c}{2}(1 - \cos\theta) \quad , \quad dx = \frac{c}{2}\sin\theta \, d\theta$$

$0 \to C$, $0 \to \pi$

$\gamma(x) \rightarrow A_0(t), A_1, A_2, \ldots, A_\infty(t)$

$$\gamma(\theta) = 2U(t)\left[A_0(t)\left(\frac{1+\cos\theta}{\sin\theta}\right) + \sum_{n=1}^{\infty} A_n(\sin n\theta)\right]$$



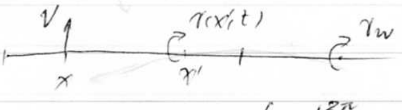


automatically gives $\gamma(c) = 0$

(guaranteed Kutta condition)

$$V(x,t) = \underbrace{\frac{1}{2\pi}\int_0^c \gamma(x',t)\frac{dx'}{x-x'}}_{V^*} + \underbrace{\frac{1}{2\pi}\int_c^\infty \gamma_w \frac{dx'}{x'-x}}_{V_w}$$

$\circlearrowleft$ (+) γ



$$V(\theta,t) = \frac{1}{2\pi}\int_0^{2\pi} \gamma(\theta,t)\frac{\sin\theta' \, d\theta'}{(\cos\theta' - \cos\theta)}$$

$$\int_0^\pi \frac{\sin\theta' \sin n\theta'}{\cos\theta' - \cos\theta} d\theta'$$

$V^*(\theta,t) = U_\infty(t)\left[-A_0(t) + \Sigma A_n(t)\cos(n\theta)\right]$

↳ plug into $\vec{u}\cdot\hat{n} = V\cdot\hat{n}$

$U_\infty\alpha + V^* = \dot{h}$

$\alpha + \left[-A_0^* + \Sigma A_n^*(t)\cos n\theta\right] = \frac{\dot{h}}{U}$

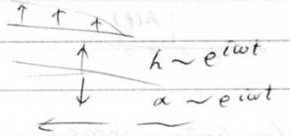
· Thin Airfoil Results (Analytical)

Steady (Glauert)

Unsteady Harmonic (Theodorsen)

Impulsive start (Wagner)

Impulsive Gust (Sears)



- Steady Flow-field Equation $\nabla^2\phi = 0$

+ unsteady B.C's

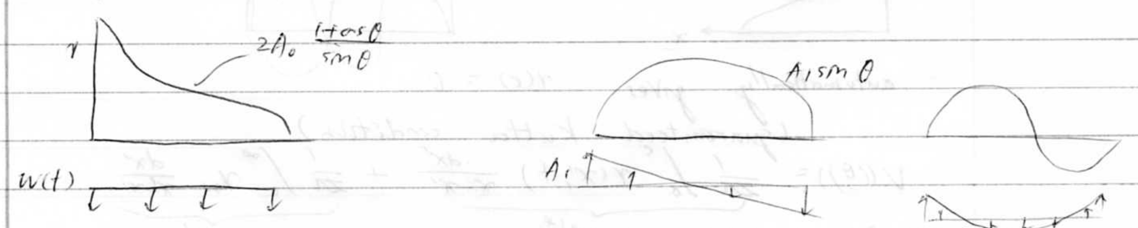
$x \to \theta$, $x = \frac{c}{2}(1-\cos\theta)$

$\gamma(x,t) \to \gamma(\theta,t) = 2U(t)\left[A_0(t)\frac{1+\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} A_n \sin n\theta\right]$

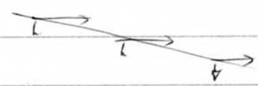
$w(x,t) = \frac{1}{2\pi}\int_0^c \gamma\frac{dx'}{x'-x} = \frac{2U(t)}{\pi}\int_0^\pi\left[A_0\frac{1+\cos\theta'}{\sin\theta'} + \sum_{n=1}^{\infty} A_n\sin n\theta'\right]\frac{\sin\theta' d\theta'}{\cos\theta'-\cos\theta}$

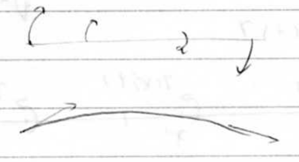
$\int_0^\pi \frac{\cos n\theta' d\theta'}{\cos\theta' - \cos\theta} = \frac{\pi\sin n\theta}{\sin\theta}$: "Glauert Integral"

$\frac{w(\theta,t)}{U(t)} = -A_0(t) + \sum_{n=1}^{\infty} A_n\cos n\theta$ [B.60 in "Handout"]

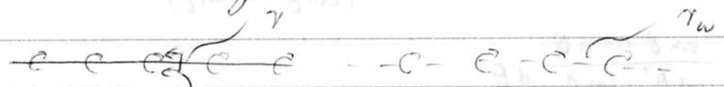


flat-plate wake (α mode)

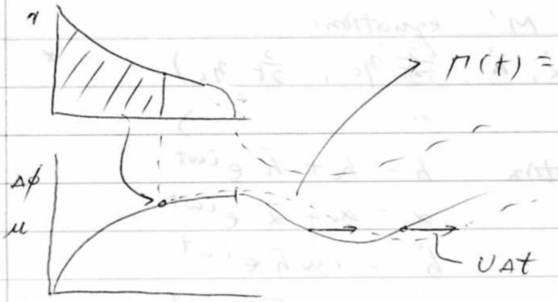




- Assume A_n given



$\Delta\phi(x,t) = \int_0^x \gamma(x',t)\,dx'$



$$\Gamma(t) = \Delta\phi(x_{TE}, t)$$

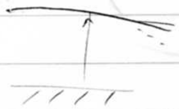
$$L' = \int_0^c \Delta p \, dx' = \rho \int \Big[\underbrace{\tfrac{\partial}{\partial t}(\Delta\phi)}_{A_n} + \underbrace{U \cdot \gamma}_{A_n} \Big] dx'$$

$$L' = \pi \rho c \, [\, A_0 \;\; A_1 \;\; \dot{A}_0 \;\; \dot{A}_1 \;\; \dot{A}_2 \,] \qquad (13.71a)$$

$$M_a' = \pi \rho c \, [\, A_0 \;\; A_1 \;\; A_2 \;\; \dot{A}_0 \;\; \dot{A}_1 \;\; \dot{A}_2 \;\; \dot{A}_3 \,] \qquad (13.72)$$

B.C : $\vec{q} \cdot \hat{n} = \vec{V} \cdot \hat{n}$

$\vec{q} = u\,\hat{i} + w\,\hat{j}$



$y = \eta(x, t)$: 'Dowell' $F(x, y, z, t) = \cdots$

$\eta(x, t) = -\alpha x + h + \underbrace{\eta_c(x, t)}_{\text{oscillating flap}}$

Assume vertical motion : $\vec{V} = \frac{\partial \eta}{\partial t}\hat{j}$

$= [-\dot{\alpha} x + \dot{h} + \frac{\partial}{\partial t}\eta_c]\,\hat{j}$

$\hat{n} = \frac{\partial \eta}{\partial x}\hat{i} + \hat{j} = [-\alpha + \frac{\partial}{\partial x}\eta_c]\,\hat{i} + \hat{j}$

$U(\alpha - \frac{\partial \eta_c}{\partial x}) + w = \dot{h} - \dot{\alpha} x + \frac{\partial}{\partial t}\eta_c$ --- B.C.

→ determines A_n's

$$\underbrace{U(\alpha - \tfrac{\partial}{\partial x}\eta_c) + U\Big[-A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta\Big]}_{\text{L.H.S} \;\leftarrow\; w^*} = \underbrace{\dot{h} - \dot{\alpha} x + \tfrac{\partial \eta_c}{\partial t}}_{\text{R.H.S}}$$

$\int_0^\pi (*) \cos k\theta \, d\theta$, $\quad k = 0, 1, 2, \ldots, K$

$$\begin{array}{l} k=0 \rightarrow \\ k=1 \rightarrow \\ k=2 \rightarrow \\ \\ \\ \end{array} \begin{bmatrix} & & \\ & & \\ & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ \vdots \\ A_K \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} \leftarrow \alpha, \dot{\alpha}, \dot{h}, \tfrac{\partial}{\partial x}\eta_c, \tfrac{\partial}{\partial t}\eta_c$$

$$A_0^*, A_1^*, \cdots = f(\alpha, \dot{\alpha}, \dot{h}, \tfrac{\partial}{\partial x}\eta_c, \tfrac{\partial}{\partial t}\eta_c)$$

→ plug into L', M' equations

$L'^{*} = f(\alpha, \dot{\alpha}, \dot{h}, \frac{\partial}{\partial x}\eta_c, \frac{\partial}{\partial t}\eta_c)$ * quasi-steady

$M'^{*} = f($ " $)$

Assume oscillating motion $h = h_0 + \bar{h}\, e^{i\omega t}$

$\alpha = \alpha_0 + \bar{\alpha}\, e^{i\omega t}$

$\dot{h} = i\omega \bar{h}\, e^{i\omega t}$

$\dot{\alpha} = i\omega \bar{\alpha}\, e^{i\omega t}$

$$w = \underbrace{\frac{1}{2\pi}\int_0^c \gamma \frac{dx'}{x'-x}}_{w^* = f(A_n)} + \underbrace{\int_c^\infty \gamma_w \frac{dx'}{x'-x}}_{w_w \text{ --- case-specific}}$$

(quasi-steady)   → $\gamma_w = \gamma_{TE}(t - x/U)$

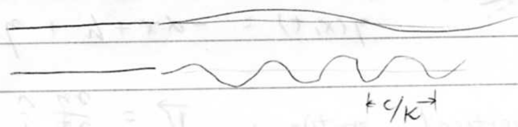
same for all cases

$\bar{L} = (\quad) + (\quad)\, C(k)$, $k = \omega c / 2U$

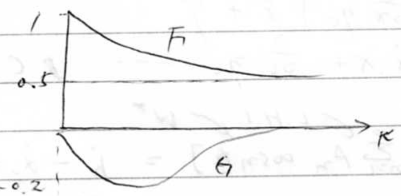
if $C(k) = 1$, $L = (\quad) + (\quad)$ "reduced frequency"

small k :

large k :



$C(k) = F(k) + i\,G(k)$



$L = L_0 + Re[\bar{L}\, e^{i\omega t}]$

$M = M_0 + Re[\bar{M}\, e^{i\omega t}]$